On two conjectures concerning (a, d)-antimagic labellings of antiprisms

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Abstract

A connected graph $G = (V, E)$ is *(a,d)-antimagic* if there exist positive integers a, d and a bijection $g : E \longrightarrow \{1, 2, \ldots, |E|\}$ such that the induced mapping
$f_g = \Sigma\{g(u, v) : (u, v) \in E(G)\}$ is injective and
$f_g(V) = \{a, a + d, a + 2d, \ldots, a + (|V| - 1)d\}$.
In this paper we prove two conjectures of Baca concerning (a, d)-antimagic labellings of antiprisms.

1 Introduction

In this paper all graphs are undirected, connected and simple. Let G be such a graph with the vertex set $V(G)$ and the edge set $E(G)$, where $|V(G)|$ and $|E(G)|$ are the number of vertices and edges of G.

The concept of an antimagic graph was introduced by Hartsfield and Ringel [3]. An *antimagic graph* is a graph whose edges can be labeled with the integers $1, 2, \ldots, |E(G)|$ so that the sum of the labels at any given vertex is different from the sum of the labels at any other vertex, that is, no two vertices have the same sum. Hartsfield and Ringel conjecture

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that every tree other than K_2 is antimagic, and, more strongly, that every connected graph other than K_2 is antimagic. Bodendiek and Walther [2] defined the concept of an (a,d)-antimagic graph as a special case of an antimagic graph as follows.

A connected graph $G = (V, E)$ is said to be *(a,d)-antimagic* if there exist positive integers a, d and a bijection $g : E \longrightarrow \{1, 2, \ldots, |E|\}$ such that the induced mapping f_g is also a bijection, where $W = \{w(v) : v \in V(G)\} = \{a, a+d, a+2d, \ldots, a+(|V|-1)d\}$.

The *antiprism* Q_n, $n \geq 3$ is a regular graph of degree $r = 4$ (*Archimedean convex polytope*). In particular, Q_3 is the *octahedron*.

Baca [1] showed that (a,d)-antimagic labellings of antiprisms do not exist for all values of (a,d) other than $(6n+3,2)$, $(4n+4,4)$ and $(2n+5,6)$. In the same paper he also gave the following labellings of Q_n.

- $(6n+3,2)$-antimagic labelling for $n \geq 3$, $n \not\equiv 2 \bmod 4$ and
- $(4n+4,4)$-antimagic labelling for $n \geq 3$, $n \not\equiv 2 \bmod 4$.

In this paper we give $(6n+3,2)$-antimagic labelling and $(4n+4,4)$-antimagic labelling of Q_n for every even n, thereby proving two of Baca's three conjectures listed in [1].

2 New results

Let $I = \{1, 2, \ldots, n\}$ be an index set. We will label the vertices of Q_n by $x_{i,j}$ where $i \in \{1,2\}$ and $j \in I$. For simplicity, we use the convention that $x_{i,n+1} = x_{i,1}$ and $x_{i,0} = x_{i,n}$.

The *weight* $w(v)$ of a vertex $v \in V(G)$ under an edge labelling g is the sum of values $g(e)$ assigned to all the edges incident to a given vertex v.

Theorem 1 *If $n \equiv 2 \bmod 4$, $n \geq 6$, then the antiprism Q_n has a $(6n+3,2)$-antimagic labelling.*

Proof. We assume that $n \equiv 2 \bmod 4$, $n \geq 6$. The desired $(6n+3,2)$-antimagic labelling of Q_n can be described by the following formulae.

$$g_1(x_{1,i}x_{1,i+1}) = (2i-1)\delta(i) + [2(n+i)-3]\delta(i+1)$$

$$g_1(x_{2,i}x_{2,i+1}) = 2i$$

$$g_1(x_{1,i}x_{2,i}) = 3\lambda(1,i,1) + [4(n-i)+7]\lambda(2,i,n)$$

$$g_1(x_{1,i}x_{2,i-1}) = (2n+2)\lambda(1,i,1) + 2(2n-i+2)\lambda(2,i,n)$$

for $i \in I$, where

$$\lambda(x,y,z) = \begin{array}{ll} 0 & \text{if } y < x \\ 1 & \text{if } x \le y \le z \\ 0 & \text{if } y > z \end{array}$$

$$\delta(x) = \begin{array}{ll} 0 & \text{if } x \equiv 0 \bmod 2 \\ 1 & \text{if } x \equiv 1 \bmod 2 \end{array}$$

It is simple to verify that the labelling g_1 is a bijection from the edge set $E(Q_n)$ onto the set $\{1, 2, \ldots, |E(Q_n)|\}$.

Let us denote the weights (under an edge labelling g) of vertices $x_{1,i}$ and $x_{2,i}$ of Q_n by

- $w_g(x_{1,i}) = g(x_{1,i}x_{1,i+1}) + g(x_{1,i-1}x_{1,i}) + g(x_{1,i}x_{2,i}) + g(x_{1,i}x_{2,i-1})$
- $w_g(x_{2,i}) = g(x_{2,i}x_{2,i+1}) + g(x_{2,i-1}x_{2,i}) + g(x_{1,i}x_{2,i}) + g(x_{1,i+1}x_{2,i})$

for $i \in I$.

The weights of vertices under the labelling g_1 constitute the sets

- $W_1 = \{w_{g_1}(x_{1,i}) : i \in I\} = \{6n+3, 8n+5, 8n+7, \ldots, 10n+1\}$ and
- $W_2 = \{w_{g_1}(x_{2,i}) : i \in I\} = \{6n+5, 6n+7, 6n+9, \ldots, 8n+3\}$

We can see that each vertex of Q_n receives exactly one weight from $W_1 \cup W_2$ and each number from $W_1 \cup W_2$ is used exactly once as a weight. This proves that the edge labelling g_1 is a $(6n+3, 2)$-antimagic labelling. □

Theorem 2 *If $n \equiv 2 \bmod 4$, $n \ge 6$, then the antiprism Q_n has a $(4n+4, 4)$-antimagic labelling.*

Proof. We assume that $n \equiv 2 \bmod 4$, $n \ge 6$. The following formulae, where the functions $\lambda(x, y, z)$ and $\delta(x)$ are defined as in the proof to Theorem 1, yield the desired $(4n+4, 4)$-antimagic labelling of Q_n.

$$g_2(x_{1,i}x_{1,i+1}) = (2n+8i+1)\lambda(1, i, \lfloor \tfrac{\frac{n}{2}-1}{2} \rfloor) + (6n-8i-3)\lambda(\lfloor \tfrac{\frac{n}{2}-1}{2} \rfloor + 1, i, \tfrac{n}{2}) +$$
$$+ (4i-2n-3)\lambda(\tfrac{n}{2}+1, i, n-1) + (2n+1)\lambda(n, i, n)$$

$$g_2(x_{2,i}x_{2,i+1}) = [(2n+2i+1)\delta(i) + (2i+3)\delta(i+1)]\lambda(1, i, n-1) + 3\lambda(n, i, n)$$

$$g_2(x_{1,i}x_{2,i}) = 4i$$

$$g_2(x_{1,i}x_{2,i-1}) = (2n+2)\lambda(1, i, 1) + (2n-4i+6)\lambda(2, i, \tfrac{n}{2}+1) + (6n-4i+6)\lambda(\tfrac{n}{2}+2, i, n)$$

for $i \in I$

It is not difficult to check that the values of g_2 are $1, 2, \ldots, 4n$. By direct computation we obtain that the weights of vertices under the edge labelling g_2 comprise the sets W_3 and W_4, where $w_g(x_{1,i})$ and $w_g(x_{2,i})$ are defined as in the proof of Theorem 1.

- $W_3 = \{w_{g_2}(x_{1,i}) : i \in I\} = \{4n + 4, 6n + 8, 6n + 12, \ldots, 10n\}$ and
- $W_4 = \{w_{g_2}(x_{2,i}) : i \in I\} = \{4n + 8, 4n + 12, 4n + 16, \ldots, 12n\}$

We see that the set $W_3 \cup W_4 = \{a, a + d, a + 2d, \ldots, a + (2n - 1)d\}$, where $a = 4n + 4$ and $d = 4$, is the set of weights of all the vertices of Q_n. Moreover, it can be seen that the induced mapping $f_{g_2} : V(Q_n) \longrightarrow W_3 \cup W_4$ is a bijection. □

Note that the functions g_1 and g_2 used in the proofs of Theorems 1 and 2 define a $(2n + 5, 6)$-antimagic, respectively $(4n + 4, 4)$-antimagic labelling of antiprisms Q_n for all even n, not only $n \equiv 2 \mod 4$.

In this paper we have proved two conjectures of Baca [1], namely that every Q_n is $(6n + 3, 2)$-antimagic and $(4n + 4, 4)$-antimagic.

However, Baca also considered $(2n + 5, 6)$-antimagic labellings of antiprisms and showed that Q_3 does not have a $(2n + 5, 6) = (11, 6)$-antimagic labelling. He gave $(2n + 5, 6)$-antimagic labellings of Q_4 and Q_7, and he conjectured that every Q_n other than Q_3 is $(2n + 5, 6)$-antimagic. This conjecture is still open. We list it below for completeness' sake.

Conjecture 1 *Every antiprism Q_n, $n > 3$ is $(2n + 5, 6)$-antimagic.*

References

[1] M. Baca, Antimagic labelings of antiprisms, to appear.

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[3] N. Hartsfield and G. Ringel, *Pearls of Graph Theory*, Academic Press, Boston-San Diego-New York-London, 1990.