On Tree Partitions

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Abstract

A splitting partition for a graph $G = (V, E)$ is a partition of V into sets R, B, and U so that the subgraphs induced by $V - R$ and $V - B$ are isomorphic. The splitting number $\mu(G)$ is the size of $|R|$ for any splitting partition which maximizes $|R|$. This paper determines $\mu(G)$ for trees of maximum degree at most three and exactly one degree two vertex and for trees all of whose vertices have degree three or one.

1 Introduction

A *splitting partition* for a graph $G = (V, E)$ is a partition of V into sets R (red), B (blue), and U (uncolored) so that the subgraphs $<V - R>$ and $<V - B>$ induced by $V - R$ and $V - B$, respectively, are isomorphic. The *splitting number* $\mu(G)$ is the size of $|R|$ (and $|B|$) for any splitting partition which maximizes $|R|$ (minimizes $|U|$). The authors have determined $\mu(G)$ when G is a path, cycle, complete bipartite graph, Fibonacci tree, and spider [1, 2].

A related problem is that of determining *even* graphs. A graph is even if its edges can be colored R and B in such a way that the subgraph induced by the R edges is isomorphic to the subgraph induced by the B edges. Knisely, Wallis, and Domke [4] have proven that Fibonacci trees and binary heaps with an even number of edges are even. Heinrich and Horak [3] have studied trees with maximum degree three and have characterized even trees of this type in which vertices of degrees one and three are not adjacent.

This paper provides results which have the flavor of Heinrich and Horak's work. In particular, $\mu(G)$ is found for trees with maximum degree at most three and one degree two vertex and for trees all of whose vertices have degree three or one. Two values of $\mu(G)$ are possible in the latter case and trees with each value are characterized.

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2 Binary Trees With One Degree Two Vertex

This section demonstrates that every binary tree T on p vertices with exactly one degree two vertex has $\mu(T) = (p-1)/2$. This is the best possible since such trees have an odd number of vertices. The collection of all trees of this type will be designated B_1. We assume trees in B_1 are rooted at the degree two vertex. For any $T \in B_1$ and any vertex $x \in T$, define S_x to be the subtree rooted at vertex x, and denote $S_x - x$ by S'_x. Observe that S_x is either a single vertex or is a member of B_1. A tree in B_1 is *complete* if all leaves are on the same level. The first result shows that we may assume S_x has one of only four possible forms.

Lemma 1 *Every tree $T \in B_1$ is either complete or has a vertex x where S_x is isomorphic to one of the subtrees in Figure 1.*

Proof: The trees in B_1 with at most five vertices are either complete or isomorphic to T_2. Suppose T has $p \geq 7$ vertices, is not complete, and that the result is true for all trees in B_1 having fewer vertices. Consider T_L, the larger of the two subtrees rooted at the children of T's root. It has at least five and no more than $p-2$ vertices. If T_L is complete with 15 or more vertices, it has a vertex x with S_x isomorphic to T_1. Otherwise we conclude from the inductive hypothesis that T_L is either complete with exactly seven vertices, or it, and thus T, possesses a vertex x where S_x is isomorphic to one of T_1 to T_4. Thus we need only consider T_L being complete with exactly seven vertices. Let T_S be the smaller subtree of a child of the root. It has at most seven vertices, and if complete, T is itself an instance of T_1, T_3, or T_4. Otherwise, T_S has five or seven vertices and is not complete. Thus T_S, and therefore T, has a vertex x where S_x is isomorphic to T_2. □

We are now ready to show that $\mu(T)$ assumes the maximum possible value for any $T \in B_1$. It is convenient to prove a stronger result.

Theorem 2 *For any $T = (V, E) \in B_1$ with p vertices, $\mu(T) = (p-1)/2$. Furthermore, there is a splitting partition for which (1) no leaf vertex is in U, and (2) $\langle V - R \rangle$ (and $\langle V - B \rangle$) have maximum degree at most two.*

Proof: Observe that any complete graph can be partitioned as shown for T_1 in Figure 1, where the root is placed in U (indicated by a small u), one child of the root is placed in R (small r), and the other child is in B (small b). Then successive levels are placed alternately in R and B. This partitioning satisfies the condition of the theorem.

We employ induction on the number of vertices, with the complete graph on three vertices and T_2, partitioned as shown in Figure 1, giving the solution for all trees in B_1 with at most five vertices. The approach for all

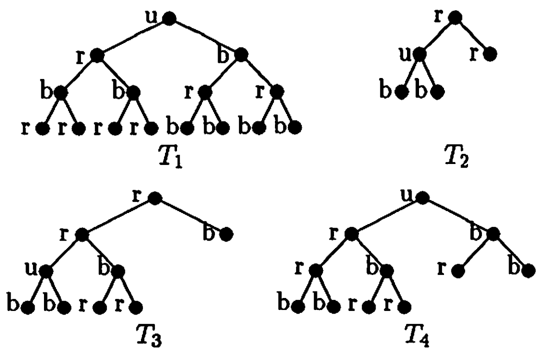


Figure 1: Special trees in B_1

larger trees is to remove one or more subtrees, partition the remaining tree by the inductive hypothesis, and then extend the partition to the original tree. Figures will illustrate this, and the straightforward validation of the correctness of the partitions given on the figures will not be made explicitly.

Thus we may assume $T \in B_1$ has $p \geq 7$ vertices and is not complete. Then, from Lemma 1, T has a vertex x for which S_x is isomorphic to one of T_1 through T_4. We examine each possibility.

Case 1. $S_x \cong T_2$. Let $\hat{T} = T - S'_x$. By the inductive hypothesis, $\mu(\hat{T}) = (p-5)/2$ and, without loss of generality, there is an optimum splitting partition with $x \in R$. Figure 2(a) illustrates how to extend the partition to T and hence shows $\mu(T) = \mu(\hat{T}) + 2 = (p-1)/2$.

Case 2. $S_x \cong T_3$. Letting $\hat{T} = T - S'_x$, the inductive hypothesis ensures $\mu(\hat{T}) = (p-9)/2$ with a splitting partition placing $x \in R$. Figure 2(b) illustrates how to extend the partition to T and hence shows $\mu(T) = \mu(\hat{T}) + 4 = (p-1)/2$.

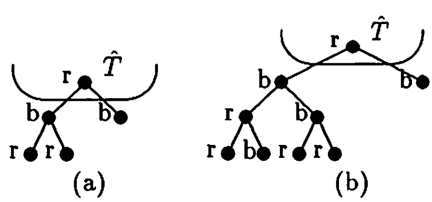


Figure 2: Extensions when S_x is isomorphic to T_2 or T_3

Case 3. $S_x \cong T_1$. Let v and w be the left and right children, respectively, of x and define $\hat{T}$ to be $T - (S'_v \cup S'_w)$. By the inductive hypothesis, $\mu(\hat{T}) = (p-13)/2$. We achieve this with a splitting partition which assigns no leaf to U and, without loss of generality, one of the following colorings to x, v, and w.

1. $v \in R$, $w \in B$, and x arbitrary. The extension to T is shown in Figure 3(a).

2. $v, w \in R$, $x \in B$. The extension to T is given by Figure 3(b) where the isolated vertex in $V - B$ induced by $w \in R$ in $\hat{T}$ is now supplied by any of the singleton leaves of T which are in R.

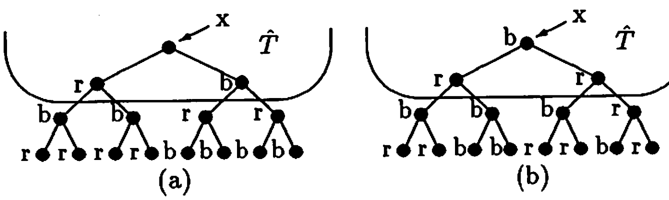


Figure 3: Extensions for Cases 1 and 2 when S_x is isomorphic to T_1

3. $v, w, x \in R$. Since we may assume a splitting partition for which $<V - B>$ has maximum degree at most two, the parent of x is in B. The extension is shown in Figure 4(a). Vertex v, assigned to R in $\hat{T}$, is reassigned to B. Furthermore, the red path $<v, x, w>$ in $\hat{T}$ is replaced by $<x, w, z>$ where z is the child of w shown to be in R in Figure 4(a).

4. $v, w, \in R$, $x \in U$. Again, the parent of x must be in B. The extension to T is shown in Figure 4(b). Notice that x has its assignment changed from U to B and the new vertex assigned to U as in the figure allows replacement for the original path $<v, x, w>$ of $\hat{T}$.

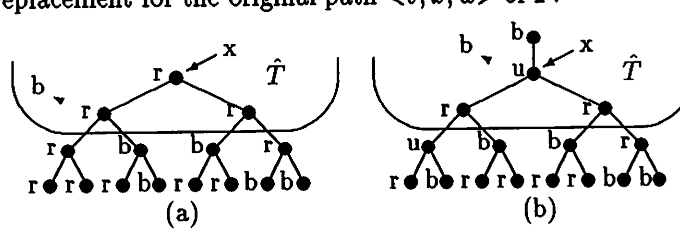


Figure 4: Extensions for Cases 3 and 4 when S_x is isomorphic to T_1

Case 4. $S_x \cong T_4$. First suppose there is a second vertex y where $S_y \cong T_4$. Let $\hat{T} = T - (S'_x \cup S'_y)$. Then, by the inductive hypothesis, $\mu(\hat{T}) = (p - 21)/2$ and we may assume either $x \in R$ and $y \in B$ or $x, y \in R$. The appropriate extensions are shown in Figures 5 (a) and (b), respectively.

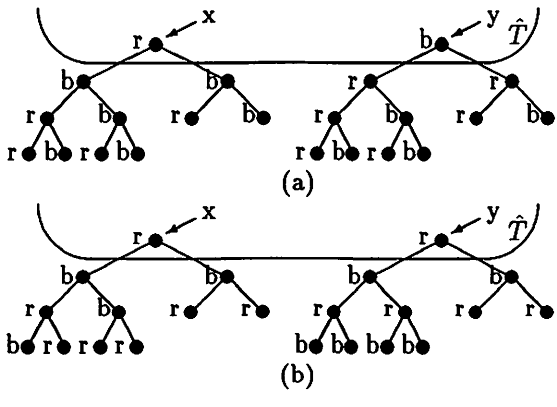


Figure 5: Extensions when S_x and S_y are isomorphic to T_4

Now consider the case when there is no second vertex y such that $S_y \cong T_4$. Also, we may conclude that no S_y is an instance of T_1, T_2, or T_3 for otherwise we can revert to an earlier case. Let w be the parent of x and z be its other child. S_z must be complete on one, three, or seven vertices, and we treat each separately.

1. S_z is complete with one vertex. Figure 6(b) gives a splitting partition if $S_w \cong T$. Otherwise let $\hat{T} \cong T - S'_w$, so $\mu(\hat{T}) = (p-13)/2$ and vertex w can be assumed to be in R. An extension is shown in Figure 6(a).

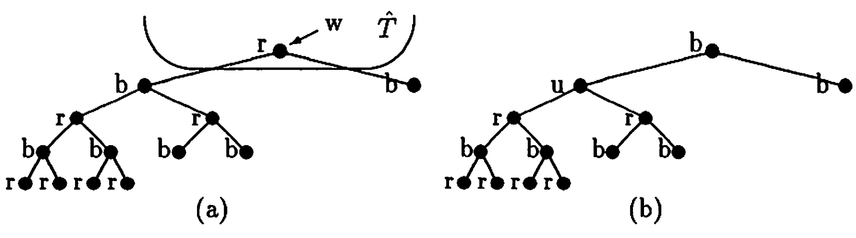


Figure 6: Extension when S_z is complete with one vertex

2. S_z is complete with three vertices. For $\hat{T} = T - (S'_x \cup S'_z)$ we have $\mu(\hat{T}) = (p-13)/2$ and leaves x and z of $\hat{T}$ may be assumed to be both in R or one in R and the other in B. Extensions for each are shown in Figure 7, and part (b) of the figure gives a valid partition when $S_w \cong T$ if the root w is placed in U.

3. S_z is complete with seven vertices. Again let $\hat{T} = T - (S'_x \cup S'_z)$ and observe $\mu(\hat{T}) = (p-17)/2$. Figure 8 gives extensions for all cases.

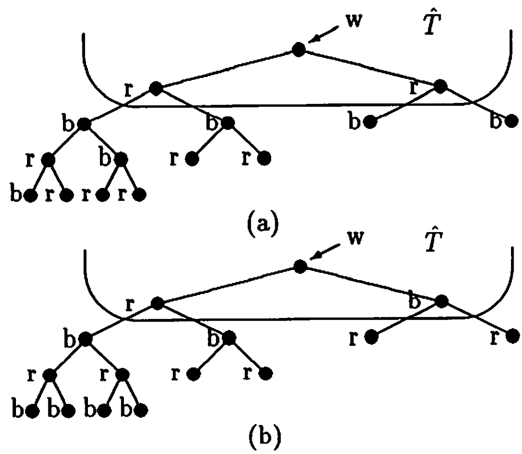


Figure 7: Extension when S_z is complete with three vertices

Also, part (b) gives the partition when $S_w \cong T$ if w is placed in U.

All possibilities have been considered and the proof to the theorem is complete. □

The following corollary gives additional information about the splitting partitions which can exist for trees in B_1 and which will be useful in the next section.

Corollary 3 *Let $T \in B_1$ with root x. Then there is an optimum splitting partition such that either (1) $x \in R$ or (2) $x \in U$ and is the end vertex of two isomorphic paths one of which has all other vertices in R and the other has all other vertices in B.*

Proof: In the proof of Theorem 2, any tree in B_1 which is complete, one of T_1 through T_4, or isomorphic to the graphs in Part (b) of Figures 6, 7, and 8 (where the vertices in R and B are interchanged in Figure 6) has a splitting partition satisfying the claim of the corollary. For every other tree in B_1, a splitting partition was obtained by extending a splitting partition of an appropriately reduced tree $\hat{T}$ which also is in B_1. As in the proof of Theorem 2, we may use an induction argument and assume the splitting partition of $\hat{T}$ satisfies the claim. In each extension, the assignment of the root of $\hat{T}$, and the length of any path containing the root and whose other vertices are either all in R or all in B is unchanged. Therefore the extended splitting partition also satisfies the claim. □

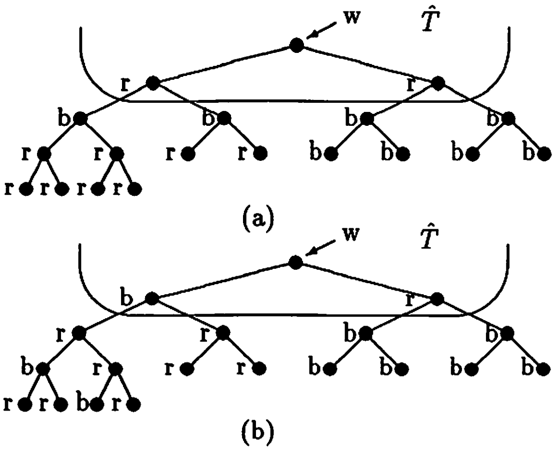


Figure 8: Extension when S_z is complete with seven vertices

3 Trees With Maximum Degree Three and No Degree Two Vertices

Let B_0 be the collection of all trees in which every vertex is either a leaf or has degree three. All such trees have an even number of vertices. The first lemma shows that the splitting number of such a tree with p vertices is either $p/2$ or $(p-2)/2$, that is, the number of vertices in U is zero or two.

Lemma 4 *For any tree $T \in B_0$, there is a splitting partition where the size of U is zero or two and, if a degree one vertex x is in U, then x and a second degree one vertex not in U share a parent.*

Proof: Figure 9 shows all trees in B_0 with at most eight vertices, along with splitting partitions which adhere to the lemma.

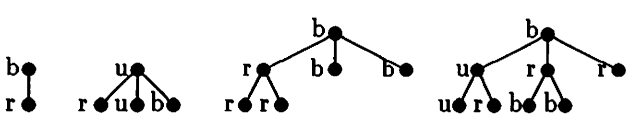


Figure 9: Trees in B_0 with at most eight vertices

Now consider a tree $T \in B_0$ with at least 10 vertices and assume the lemma holds for all trees in B_0 with fewer vertices. We examine two cases.

Case 1. There exists a vertex x for which S_x is a complete binary tree on seven vertices. Observe that $\hat{T} \cong T - S_x \in B_1$ where $\hat{T}$ is drawn with its degree two vertex z as the root which is the parent of x in T. From Corollary 3, there is a splitting partition of $\hat{T}$ such that $z \in R \cup U$ and, if $z \in U$, z is the end vertex of isomorphic paths where the other vertices of one of the paths are all in R and the other vertices in the other path are all in B. Figure 10 shows the extensions for both set assignments for z.

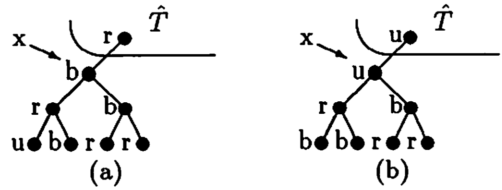


Figure 10: T has complete S_x with seven vertices

Case 2. There is no vertex x such that S_x is a complete binary tree on seven vertices. Then, by Lemma 1, there must be a vertex x for which $S_x \cong T_2$. It follows that $\hat{T} \cong T - S'_x \in B_0$. By the inductive hypothesis, there is a splitting partition of $\hat{T}$ that places either x or its sibling in R. In the latter instance, interchange the assignments of x and its sibling. The extension of the partition shown in Figure 2 (a) is applicable and completes the proof of the lemma. □

The remainder of this section determines which trees in B_0 have splitting number $(p-2)/2$ and which have $p/2$. Two lemmas are needed. For i equal to one and three, let d_i be the number of vertices of degree i for a tree in B_0.

Lemma 5 *If $T \in B_0$ and has p vertices, then $d_1 = p/2 + 1$.*

Proof: Solving for d_1 in $d_1 + d_3 = p$ and $d_1 + 3d_3 = 2(p-1)$ yields the result. □

Lemma 6 *Let $T \in B_0$ with p vertices have a splitting partition such that $<V - R> \cong \frac{p}{2}K_1$. Then $p \equiv 2 \bmod 4$.*

Proof: Since K_2 has such a splitting partition and $K_{1,3}$ does not, the lemma holds for $p \leq 4$. Assume T is a smallest tree in B_0 with $p = 4k \geq 8$ vertices such that T has a splitting partition with $<V - R> \cong \frac{p}{2}K_1$. Notice first that no such T can have a vertex x with S_x being isomorphic to T_2, for otherwise $T - S'_x$ is a smaller tree in B_0 with a splitting partition (inherited from T) for which $<V - R> \cong \frac{p-4}{2}K_1$. By the minimality of T, $p-4 \equiv 2 \bmod$

4 and contradicts the assumption that p is a multiple of four. Furthermore, no such T can have vertices x and y with $S_x \cong S_y$ being paths on three vertices in which the middle vertices, x and y, are in different sets of the partition. Otherwise we again have the contradiction that $T - (S'_x \cup S'_y)$ is a smaller tree satisfying the conditions of the lemma with $p - 4 \equiv 2$ mod 4. Thus T must have a vertex x for which S_x is a complete binary tree on seven vertices and that all sets of leaf pairs having a common neighbor must be in the same set, say, R.

On the other hand, not all leaves can be in R since, by Lemma 5, $d_1 = p/2 + 1$. Let x be a vertex where S_x is a complete binary tree with seven vertices, and let u and v be two leaves in R with a common parent in S_x. Now remove u and v and reattach them to a leaf in B. The resulting tree T^* is in B_0 and has a splitting partition inherited from T and for which $<V^* - R>$ consists of $p/2$ isolated vertices. Now, in T^*, S_x is isomorphic to T_2 which contradicts the fact shown above that these trees can not have such a vertex x. □

The next two theorems characterize trees in B_0 in terms of their splitting number.

Theorem 7 *If $T \in B_0$ and $p \equiv 0$ mod 4, then $\mu(T) = (p-2)/2$.*

Proof: Suppose for some T in B_0 having $4k$ vertices that $\mu(T) = 2k = p/2$. Then, from Lemma 6, one or more edges of T have both end vertices in B and an equal number have both end vertices in R. Construct tree $\hat{T}$ by subdividing once every edge of T with both end vertices in either R or B. For any such edge, let the new vertex be x and append a pendant vertex y from x. Assign x to R (B) and y to B (R) if x subdivides an edge whose end vertices are both in B (R). All vertices of $\hat{T}$ have degrees one and three and hence $\hat{T} \in B_0$. Also, it has a splitting partition such that $<\hat{V} - \hat{R}> \cong \frac{\hat{p}}{2} K_1$ which, by Lemma 6, implies $\hat{p}$, and thus p, cannot be a multiple of four, a contradiction that establishes the result. □

Theorem 8 *If $T \in B_0$ and $p \equiv 2$ mod 4, then $\mu(T) = p/2$ and there is a splitting partition for which $<V - R>$ is a collection of isolated vertices and zero or more P_3's.*

Proof: The first and third trees shown in Figure 9 illustrate appropriate splitting partitions for the only trees in B_0 having $p \equiv 2$ mod 4 for $p \leq 6$. Figure 11 handles all such trees with 10 vertices.

Now suppose $T \in B_0$ with $p = 4k+2 > 10$ vertices and that the theorem holds for smaller trees in B_0. We consider two cases.

Case 1. There is a vertex x for which $S_x \cong T_2$. Then $\hat{T} \cong T - S'_x \in B_0$ and has $\hat{p} = 4(k-1) + 2$ vertices. By the inductive hypothesis, $\mu(\hat{T}) =$

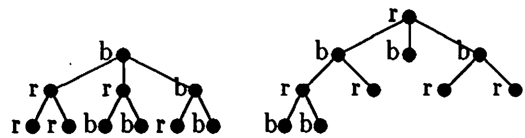


Figure 11: Trees in B_0 with ten vertices

$\hat{p}/2 = (p-4)/2$ and has a splitting partition with $x \in R$ which can be extended to T as in Figure 2. Thus $\mu(T) = \mu(\hat{T}) + 2 = p/2$.

Case 2. There is no vertex x such that $S_x \cong T_2$. Then there must be a vertex z with children x and y such that S_x is a complete binary tree on seven vertices and S_y is a complete binary tree on one, three, or seven vertices. We examine each in turn.

1. S_y has one vertex. Let $\hat{T} \cong T - S'_z$ and observe it is in B_0. Its optimum partition can be extended to one for T as shown in Figure 12.

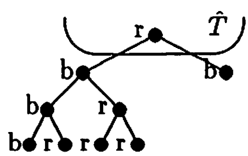
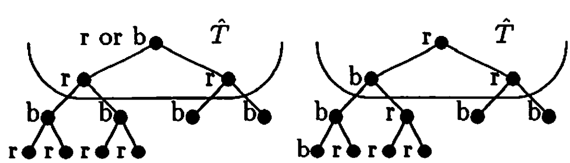


Figure 12: S_y is a complete binary tree with one vertex

2. S_y has three vertices. We extend an optimum splitting partition of $\hat{T} \cong T - (S'_x \cup S'_y)$ to T as shown in Figure 13 which deals with the two situations (a) $x, y \in R$ and (b) $x \in B$ and $y \in R$. In part (b) the role of the isolated vertex in B is taken by one of the isolated vertices in S_y.



(a) $x, y \in R$ (b) $x \in B$, $y \in R$

Figure 13: S_y is a complete binary tree with three vertices

3. S_y has seven vertices. Again let $\hat{T} \cong T - (S'_x \cup S'_y)$ which is in B_0. If, for the splitting partition of $\hat{T}$, $x \in R$ and $y \in B$, place all S'_x vertices

in B and those in S'_y in R. Otherwise suppose $x, y \in R$. Figure 14 shows the extension of the partition of $\hat{T}$ to T for the two cases $z \in R$ and $z \in B$. Notice that in the former the assignment of x has been moved from R to B and in the latter the singleton R vertex x of $\hat{T}$ has its role played by a leaf in R of T.

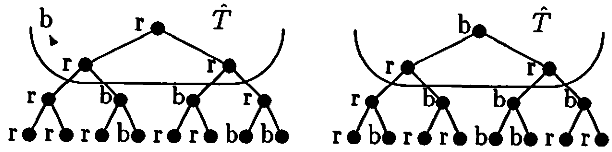


Figure 14: S_y is a complete binary tree with seven vertices

□

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