On the character of the matching polynomial and its application to circuit characterization of graphs

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ABSTRACT. In this paper, we show that some graphs are circuit unique by applying new tool, which is the character of the matching polynomial. Some properties of the character of the matching polynomial is also given.

1 Introduction

The graphs considered here are finite and simply graphs [1]. Let G be a graph, $V(G)$ be its vertex set, $E(G)$ be its edge set, $N_G(v) = \{u|u \in E(G), uv \in E(G)\}$ for $v \in V(G)$, $p(G) = |V(G)|, q(G) = |E(G)|$.

Let G be such a graph,we define a circuit with one and two vertices in G to be a vertex and a edge respectively. Circuits with more than two vertices are called proper circuit. A Circuit cover of G is a spanning subgraph of G in which all the components are circuits. Let us associate an indeterminate or weight W_α with each circuit α in G and the monomial $W(S) = \prod W_\alpha$, where product is taken over all the components in cover S. Then the definition of the circuit polynomial of G is as follows.

Definition 1.1. [2]. *Let G be a graph. Then the circuit polynomial of G is*

$$C(G, \mathrm{w}) = \sum W(S)$$

Definition 1.2. [2]. *Let G be a graph .We say that G and H are cocircuit, if $C(G, \mathrm{w}) = C(H, \mathrm{w})$. If $C(G, \mathrm{w}) = C(H, \mathrm{w})$ implies that $G \cong H$, then G is called circuit unique.*

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Definition 1.3. [13]. *Let G be a graph, the matching polynomial of G is*

$$M(G,\mathbf{w}) = \sum_i a_i(G) w_1^{n-2i} w_2^i$$

where $n = p(G)$ and $a_i(G)$ denote the number of with i-matching in G, i.e., the number of subgraphs formed from k vertex-disjoint edge.

We list some classes of graphs (see figure 1):

(1) P_n, C_n denote path and cycle with n vertices.

(2) $T_n(l_1, l_2, l_3)$ or simply $T(l_1, l_2, l_3)$ denotes a tree obtained by extending three paths from one vertex. The lengths of the three paths are l_1, l_2, l_3, respectively.

(3) $Q(s_1, s_2)$ denotes a graph from identifying a vertex of degree 2 in C_{s_1} with a vertex of degree 1 in P_{s_2}.

(4) U_n denotes a graph by identifying any vertex of degree 1 in P_{n-4} with center of $K_{1,2}$.

(5) $T(t_1, t_2)$ will denote graph from identifying a vertex of degree 1 in U_{t_1+4} with a vertex of degree 1 in P_{t_2-1}.

(6) T^5 denotes a tree which has two vertices of degree 3 only.

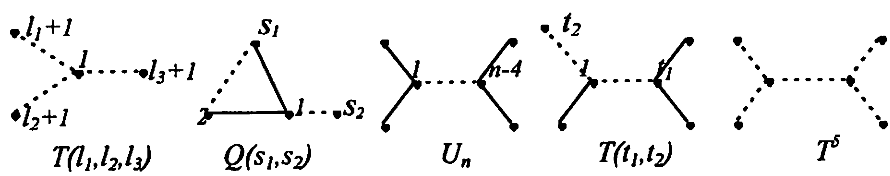

Figure 1

For convenience, we introduce some notations:

(1) For a graph G, let $e = v_1v_2 \in E(G)$, $N_G(e) = N_G(v_1) \bigcup N_G(v_2) - \{v_1, v_2\}$ and $d_G(e) = |N_G(e)|$.

(2) Let $e \in E(G)$, $G - e$ denotes the graph obtained from G by deleting edge e.

(3) $(G)_E$ denotes edge-induced subgraph of G.

(4) For a graph, let $e = ij \in E(G)$, *called* $d_\times(e) = d_i d_j$ *the product degree of the edge* e. $\Pi(G), \Pi_\times(G)$ and $\sum \Pi_\times(G)$ will be used the sequence of degree of G, the sequence of the product of degree of G, and $\sum_{i=1}^{q} d_\times(e_i)$ respectively.

In this paper, a new parameter $R_m(G)$ (*Called the character of the Matching polynomial*) is introduced as tool, and some good properties of the $R_m(G)$ is given. The circuit uniqueness of some graphs is also discussed.

2 Preliminaries

We now give some results which have already been established and which will be useful in the material which follows.

Lemma 2.1. [3]. *Let G be a graph and $uv \in G$. Then*

$$C(G,\mathbf{w}) = C(G-uv,\mathbf{w}) + w_2 C(G-u-v,\mathbf{w}) + C(G^*,\mathbf{w})$$

where G^* is the restriction graph G, which uv must be part of a proper cycle.

Lemma 2.2. [3,13]. *Let G be a graph consisting of components $G_1, G_2, \cdots, G_k$. Then*

$$C(G,\mathbf{w}) = \prod_{i=1}^{k} C(G_i,\mathbf{w}), \qquad M(G,\mathbf{w}) = \prod_{i=1}^{k} M(G_i,\mathbf{w})$$

Lemma 2.3. [9]. *Let G be a graph. Then*

$$C(G,\mathbf{w}) = M(G,\mathbf{w}) + C(G^*,\mathbf{w})$$

where G^* is a polynomial over $\mathbf{w}$, containing all the monomials corresponding to circuit covers of G with at least one proper cycle.

If G has no proper cycles, then $C(G^*,\mathbf{w}) = 0$. In particular, let T be a tree, then $C(T,\mathbf{w}) = M(T,\mathbf{w})$.

From Definition 1.2 and Definition 1.3, the coefficient of $w_1^p, w_1^{p-2}w_2$, $w_1^{p-4}w_2^2$ and $w_1^{p-6}w_2^3$ in $C(G,\mathbf{w})$ equal $a_0(G), a_1(G), a_2(G)$ and $a_3(G)$, respectively. Hence we have

Lemma 2.4. [10]. *Let G be a graph, a_0, a_1, a_2, and a_3 be the coefficient of $w_1^p, w_1^{p-2}w_2$, $w_1^{p-4}w_2^2$ and $w_1^{p-6}w_2^3$ of the matching polynomial or the circuit polynomial respectively. Then*

(1) $\partial(C(G,\mathbf{w})) = p$

(2) $a_0 = 1$

(3) $a_1 = q$

(4) $a_2 = \binom{q}{2} - \sum_{i=1}^{p} \binom{d_i}{2}$

(5) $a_3 = \binom{q}{3} - (q-2)\sum_{i=1}^{p}\binom{d_i}{2} + 2\sum_{i=1}^{p}\binom{d_i}{3} + \sum_{ij}(d_i - 1)$
$(d_j - 1) - N_T$,
where N_T denotes the number of K_3 in G.

Since $\sum_{i=1}^{p}\binom{d_i}{2} = \frac{1}{2}\sum_{i=1}^{p} d_i^2 - q$, hence,

$$a_2 = \binom{q+1}{2} - \frac{1}{2}\sum_{i=1}^{p} d_i^2$$

Similarly ,a_3 can be written as follows.

$$a_3 = \frac{1}{6}q(q^2+3q+4) - \frac{q+2}{2}\sum_{i=1}^{p} d_i^2 + \frac{1}{3}\sum_{i=1}^{p} d_i^3 + \sum_{ij} d_i d_j - N_T$$

The following Lemma 2.5 is easly proved by definition of the circuit polynomial.

Lemma 2.5. *The coefficient of $w_1^{p-k}w_k$ equal $N_k(G)$ where $N_k(G)$ denotes the number of cycle of length k.*

3 The character of the matching polynomial and graph classification

Definition 3.1. *The character of the matching polynomial of a graph, denoted by $R_m(G)$, is defined as follows:*

$$R_m(G) = \begin{cases} 0 & \text{if } q(G) = 0 \\ a_2(G) - \binom{a_1(G) - 1}{2} + a_0(G) & \text{if } q(G) > 0 \end{cases}$$

where $a_0(G)$, $a_1(G)$, and $a_2(G)$ are the coefficients of w_1^p, $w_1^{p-2}w_2$ and $w_1^{p-4}w_2^2$ of the $M(G, \mathrm{w})$ respectively.

Theorem 3.2. *Let G be a graph consisting of components $G_1, G_2, G_3, \cdots, G_k$. Then*

$$R_m(G) = \sum_{i=1}^{k} R_m(G_i)$$

Proof: It suffices to prove the cases $k = 2$.

Let $G = G_1 \bigcup G_2$, $|V(G_i)| = p_i (i = 1, 2)$, and $V(G_1) \bigcap = V(G_2) = \phi$.

Let the matching polynomial of G_1 and G_2 be as follows:

$M(G_1,\mathbf{w}) = w_1^{p_1} + a_1 w_1^{p_1-2} w_2 + a_2 w_1^{p_1-4} w_2^2 + \cdots$
$M(G_2,\mathbf{w}) = w_1^{p_2} + b_1 w_1^{p_2-2} w_2 + b_2 w_1^{p_2-4} w_2^2 + \cdots$
By Definition 3.1:

$$R_m(G_1) = a_2 - \binom{a_1 - 1}{2} + 1,$$

$$R_m(G_2) = b_2 - \binom{b_1 - 1}{2} + 1$$

From Lemma 2.2, $M(G,\mathbf{w}) = M(G_1,\mathbf{w})M(G_2,\mathbf{w}) = w_1^{p_1+p_2} + (a_1 + b_1)w_1^{p_1+p_2-2}w_2 + (a_2 + b_2 + a_1 b_1)w_1^{p_1+p_2-4}w_2^2 + \cdots\cdots$
By Definition 3.1:

$$\begin{aligned} R_m(G) &= a_2 + b_2 + a_1 b_1 - \binom{a_1 + b_1 - 1}{2} + 1 \\ &= a_2 - \binom{a_1 - 1}{2} + 1 + b_2 - \binom{b_1 - 1}{2} + 1 \\ &= R_m(G_1) + R_m(G_2) \end{aligned}$$

Theorem 3.3. *Let G be a graph and $e \in E(G)$. Then $R_m(G) = R_m(G - e) - d_G(e) - A_e + 1$*
where A_e denotes the number of the triangle with e in G.

Proof: By Lemma 2.4, $a_1(G - uv) = a_1(G) - 1$,

$$\begin{aligned} a_2(G - uv) &= \binom{q(G - uv) + 1}{2} - \frac{1}{2}\sum_{i=1}^{p} d_i^2(G - uv) \\ &= \binom{q(G)}{2} - \frac{1}{2}\left(\sum_{i=1}^{p-2} d_i^2(G - uv) + d_u^2(G - uv) + d_v^2(G - uv)\right) \end{aligned}$$

We note that $d_u(G - uv) = d_u(G) - 1, d_v(G - uv) = d_v(G) - 1$ and $d_G(uv) = d_u(G) + d_u(G) - A_{uv} - 2$, hence

$$\begin{aligned} a_2(G - uv) &= \binom{q(G) + 1}{2} - q - \frac{1}{2}\left(\sum_{i=1}^{p} d_i^2(G) - 2(d_u(G) + d_v(G)) + 2\right) \\ &= a_2(G) - a_1(G) + d_G(uv) + A_{uv} + 1 \end{aligned}$$

$$\begin{aligned} R_m(G - uv) &= a_2(G) - a_1(G) + d_G(uv) + A_{uv} + 1 \\ &\quad \binom{a_1(G) - 2}{2} + 1 \\ &= a_2(G) - \binom{a_1(G) - 1}{2} + 1 + d_G(uv) + A_{uv} - 1 \end{aligned}$$

$$R_m(G - uv) = R_m(G) + d_G(uv) + A_{uv} - 1$$

Namely, $R_m(G) = R_m(G-e) - d_G(e) - A_e + 1$.

Theorem 3.4. *Let G be a connected graph. Then:*

(1) $R_m(G) \leq 1$, *and the equality holds if and only if* $G \in \{P_n\}$

(2) $R_m(G) = 0$ *if and only if* $G \in \{C_n, T(l_1, l_2, l_3), K_1\}$

(3) $R_m(G) = -1$ *if and only if* $G \in \{Q(s_1, s_2), T^5\}$.

Proof: (1) By induction on $q(G)$

Since $R_m(K_1) = 0$, and $R_m(K_2) = 1$. Hence (1) holds when $q(G) \leq 1$.

Suppose $q(G) \geq 2$. Choose $e \in E(G)$ such that $(G-e)_E$ is connected. Clearly, $d_G(e) \geq 1, R_m(G-e) = R_m((G-e)_E)$. By the induction hypothesis, $R_m(G-e) \leq 1$. From Theorem 3.3,

$$\begin{aligned} R_m(G) &= R_m(G-e) - d_G(e) - A_e + 1 \\ &\leq R_m(G-e) - A_e \\ &\leq 1 \end{aligned}$$

If $R_m(G) = 1$, then $d_G(e) = 1, A_e = 0$ and $R_m(G-e) = 1$. According to the induction hypothesis, $(G-e)_E \in \{P_n\}(n \geq 2)$. Therefore $G \in \{P_n\}$.

Conversely, by $R_m(P_2) = 1$ and $R_m(G) = R_m(G-e) - d_G(e) - A_e + 1$, $R_m(P_n) = R_m(P_{n-1}) = \cdots = R_m(P_2) = 1$.

(2) By induction on $q(G)$.

Since $R_m(K_1) = 0$, and $R_m(K_2) = 1$. Hence (2) holds if $q(G) \leq 1$.

Suppose $q(G) \geq 2$. Choose $e \in E(G)$ such that $(G-e)_E$ is connected. Then $R_m(G-e) \leq 1$. Since $R_m(G) = 0$, $R_m(G-e) = d_G(e) + A_e - 1$ by Theorem 3.3. Hence $1 \leq d_G(e) \leq 2 - A_e$ and $0 \leq R_m(G-e) \leq 1$. We consider only the following two cases:

Case 1. $R_m(G-e) = 1$ and $d_G(e) = 2 - A_e$

When $A_e = 1$ and $d_G(e) = 1$, $(G-e)_E \in \{P_n\}$ by the induction hypothesis. Hence $G \cong K_3$. When $A_e = 0$ and $d_G(e) = 2$, $(G-e)_E \in \{P_n\}$ by the induction hypothesis. Therefore $G \in \{C_n(n \geq 4), T(l_1, l_2, l_3)\}$.

Case 2. $R_m(G-e) = 0$ and $d_G(e) = 1 - A_e$

Since $d_G(e) \geq 1$, hence $d_G(e) = 1, A_e = 0$. By the induction hypothesis, $(G-e)_E \in \{C_n(n \geq 3), T(l_1, l_2, l_3)\}$. Therefore $G \in \{T(l_1, l_2, l_3)\}$.

Conversely, Since $R_m(P_2) = 1, R_m(K_1) = R_m(T(1,1,1)) = 0$ and $R(C_3) = 0$. From Theorem 3.3, $R_m(C_n) = R_m(P_n) - 1 = 0$ if $n \geq 4$ and $R_m(T(l_1, l_2, l_3)) = R_m(T(1,1,1)) = 0$.

Therefore $R_m(G) = 0$ if $G \in \{C_n(n \geq 3), T(l_1, l_2, l_3), K_1\}$.

(3). We can also prove (3) by the induction on $q(G)$.

Since $R_m(K_1) = 0$, and $R_m(K_2) = 1$. Hence (3) holds if $q(G) \le 1$.

Supposed $q(G) \ge 2$. Choose $e \in E(G)$ such that $(G-e)_E$ is connected. Then $R_m(G-e) \le 1$. Since $R_m(G) = -1$, $R_m(G-e) = d_G(e) + A_e - 2$ by Theorem 3.3. Hence $1 \le d_G(e) \le 3 - A_e$ and $-1 \le R_m(G-e) \le 1$. We consider only the following three cases:

Case 1. $R_m(G-e) = 1$ and $d_G(e) = 3 - A_e$

If $A_e = 2$, then $d_G(e) = 1$, $(G-e)_E \in \{P_n\}$ by the induction hypothesis. Since $P_n \bigcup K_1 + e$ and $P_n + e$ have one triangle at most, hence $A_e \ne 2$. If $A_e = 1$, then $d_G(e) = 2$. By the induction hypothesis, $(G-e)_E \in \{P_n\}$. Hence $G \cong Q(3, s_2)$. If $A_e = 0$, then $d_G(e) = 3$, by the induction hypothesis, $(G-e)_E \in \{P_n\}$. Therefore $G \cong Q(s_1, s_2)(s_1 \ge 4)$.

Case 2. $R_m(G-e) = 0$ and $d_G(e) = 2 - A_e$

When $A_e = 1$ and $d_G(e) = 1$, $(G-e)_E \in \{C_n(n \ge 3), T(l_1, l_2, l_3), K_1\}$ by the induction hypothesis. Hence $G \cong Q(3, s_2)$. When $A_e = 0$ and $d_G(e) = 2$, $(G-e)E\{C_n(n \ge 3), T(l_1, l_2, l_3), K_1\}$ by the induction hypothesis. Therefore $G \in \{Q(s_1, s_2)(s_1 \ge 4), T^5\}$.

Case 3. $R_m(G-e) = -1$ and $d_G(e) = 1 - A_e$

Since $d_G(e) \ge 1$, hence $d_G(e) = 1, A_e = 0$. From the induction hypothesis, $(G-e)_E \in \{Q(s_1, s_2), T^5\}$. Therefore $G \in \{Q(s_1, s_2), T^5\}$.

Conversely, by $R_m(G) = R_m(G-e) - d_G(e) - A_e + 1$ and Theorem 3.4 (1) (2), $R_m(Q(3, s_2)) = R_m(T(1, 1, s_2)) - 1 = -1, R_m(Q(s_1, s_2)) = R_m(T(1, s_1 - 1, s_2)) - 1 = -1$ if $s_1 \ge 4$, $R_m(T^5) = R_m(T(l_1, l_2, l_3)) - 1 = -1$.

Therefore $R_m(G) = -1$ if $G \in \{Q(s_1, s_2), T^5\}$.

4 The circuit uniqueness of some graphs

The circuit uniqueness of some the well known families graph was obtained in [4–9]. We now show that $T(l_1, l_2, l_3), Q(s_1, s_2), U_n$ and $T(t_1, t_2)$ is circuit unique. The Lemma 4.1 follows immediate from Lemma 2.4 and 2.5.

Lemma 4.1. *If* $C(G, \mathbf{w}) = C(H, \mathbf{w})$. *Then*

(1) $p(G) = p(H)$, *and* $q(G) = q(H)$

(2) $N_k(G) = N_k(H)$

(3) $R_m(G) = R_m(H)$

(4) *In particular, if* G *is a tree, then* H *is a tree also,*
where $N_k(G)$ *denotes the number of cycle of length* k *in* G.

If $\mathbf{w} = (w, w, \cdots, w)$, we called $M(G, w), C(G, w)$ *the simply matching polynomial and the simply circuit polynomial of* G respectively. $h(G, x)$

denotes the adjoint polynomial of G, which was introduced by Liu in [11]. The following Lemma have been given in [11] and [12].

Lemma 4.2. [11] *If G is K_3-free, then $h(G,x) = M(G,x)$.*

Lemma 4.3. [12] *Let $l_1 \le l_2 \le l_3, t_1 \le t_2 \le t_3$. If $(l_1,l_2,l_3) \ne (t_1,t_2,t_3)$, then $h(T(l_1,l_2,l_3),x) \ne h(T(t_1,t_2,t_3),x)$.*

From Lemma 2.3, Lemma 4.2 and Lemma 4.3, we have

Lemma 4.4. *Let $l_1 \le l_2 \le l_3, t_1 \le t_2 \le t_3$. If $(l_1,l_2,l_3) \ne (t_1,t_2,t_3)$, then $C(T(l_1,l_2,l_3),w) \ne C(T(t_1,t_2,t_3),w)$.*

Theorem 4.5. *Let $l_1 \le l_2 \le l_3$. Then $T(l_1,l_2,l_3)$ is circuit unique.*

Proof: Let $C(H,\mathrm{w}) = C(T(l_1,l_2,l_3),\mathrm{w})$, By Lemma 4.1, $H \cong T, p(H) = p(T(l_1,l_2,l_3)) = l_1+l_2+l_3+1$, and $R_m(H) = R_m(T(l_1,l_2,l_3)) = 0$

From Theorem 3.4, $H \cong T(t_1,t_2,t_3)$ and $l_1+l_2+l_3 = t_1+t_2+t_3$, $l_1 \le l_2 \le l_3$

By Lemma 4.4, $t_1 = l_1, t_2 = l_2, t_3 = l_3$. Therefore $H \cong T(l_1,l_2,l_3)$.

Lemma 4.6. *Let $a_3(G)$ denote the coefficient of $w_1^{p-6}w_2^3$ of the circuit polynomials. Then*

$$a_3(U_n) < a_3(T(t_1,t_2)) < a_3(T)$$

where $n \ge 7$ and $T \in \{T^5 \backslash \{U_n, T(t_1,t_2)\}\}$, $q(U_n) = q(T) = q(T(t_1,t_2)), t_1 \ge 3, t_2 \ge 3$.

Proof: According to Lemma 2.4,

$$\begin{aligned} a_3 &= \frac{1}{6}q(q^2+3q+4) - \frac{q+2}{2}\sum_{i=1}^{p} d_i^2 + \frac{1}{3}\sum_{i=1}^{p} d_i^3 + \sum_{ij} d_i d_j - N_T \\ &= \frac{1}{6}q(q^2+3q+4) - \frac{q+2}{2}\sum_{i=1}^{p} d_i^2 + \frac{1}{3}\sum_{i=1}^{p} d_i^3 + \sum \Pi_\times(G) - N_T \end{aligned}$$

We note that any two trees $T_1, T_2 \in T^5$, $q(T_1) = q(T_2), p(T_1) = p(T_2), \Pi(T_1) = \Pi(T_2)$ and $N_{T_i} = 0 (i = 1,2)$. Therefore we consider only $\Pi_\times(T)$.

Let $m \times b$ denote m edges with the product degree b. Since $\Pi_\times(U_n) = \{(q-6)\times 4, 2\times 6, 4\times 3\}, \Pi_\times(T(t_1,t_2)) = \{(q-7)\times 4, 3\times 6, 3\times 3, 1\times 2\}$.

Hence

$$\begin{aligned} a_3(U_n) &= a_3(T(t_1,t_2)) + \sum \Pi_\times(U_n) - \sum \Pi \times (T(t_1,t_2)) \\ &= a_3(T(t_1,t_2)) - 1. \end{aligned}$$

Let $T \in \{T^5 \backslash \{U_n, T(t_1,t_2)\}\}$. Then the product degree sequence of T is the following seven cases:

$\Pi_\times(T_1) = \{(q-6)\times 4, 1\times 9, 1\times 6, 3\times 3, 1\times 2\}$,
$\Pi_\times(T_2) = \{(q-8)\times 4, 4\times 6, 2\times 3, 2\times 2\}$,
$\Pi_\times(T_3) = \{(q-7)\times 4, 1\times 9, 2\times 6, 2\times 3, 2\times 2,\}$,
$\Pi_\times(T_4) = \{(q-8)\times 4, 1\times 9, 3\times 6, 1\times 3, 3\times 2\}$,
$\Pi_\times(T_5) = \{(q-10)\times 4, 6\times 6, 4\times 2\}$,
$\Pi_\times(T_6) = \{(q-9)\times 4, 1\times 9, 4\times 6, 4\times 2\}$,
$\Pi_\times(T_7) = \{(q-9)\times 4, 1\times 3, 5\times 6, 3\times 2\}$
By calculating, $\sum \Pi_\times(T(t_1,t_2)) < \sum \Pi_\times(T_i)(1 \le i \le 7)$. Therefore $a_3(T(t_1,t_2)) < a_3(T_i)$.

Theorem 4.7. *Let $n \ge 6$. Then U_n is circuit unique.*

Proof: Let $C(H,\mathrm{w}) = C(U_n,\mathrm{w})$. By Lemma 4.1, $H \cong T, p(H) = n$ and $R_m(H) = R_m(U_n) = -1$. By Theorem 3.4, $H \in \{T^5\}$. If $n > 6$, since $b_3(H) = b_3(U_n)$, $H \cong U_n$ by Lemma 4.6. If $n = 6$, $p(T) = 6$ and $T \in \{T^5\}$ if and only if $T \cong U_6$.

Lemma 4.8. [10] (1) $C(P_n, w) = \sum_k \binom{p-k}{k} w^{p-k}$

(2) $C(P_n,w) = w(C(P_{n-1},w) + C(P_{n-2},w))$
where $n \ge 2$.

Lemma 4.9. *Let $\{g_i(w)\}_{(i)}$ be the sequence of polynomials with integers coefficients and $g_n(w) = w(g_{n-1}(w) + g_{n-2}(w))$. Then:*

(1) $g_n(w) = g_{n-k}(w)C(P_k,w) + wg_{n-k-1}(w)C(P_{k-1},w)$

(2) *If $t_1 + t_2 = s_1 + s_2$, then $g_{t_1}(w)C(P_{t_2},w) = g_{s_1}(w)C(P_{s_2},w)$ if and only if $g_{t_1-m}(w)C(P_{t_2-m},w) = g_{s_1-m}(w)C(P_{s_2-m},w)$*

Proof: (1) By Lemma 4.8 and $g_n(w) = w(g_{n-1}(w) + g_{n-2}(w))$

$$\begin{aligned} g_n(w) &= w^2 g_{n-2}(w) + w^2 g_{n-3}(w) + wg_{n-2}(w) \\ &= C(P_2,w)g_{n-2}(w) + wC(P_1,w)g_{n-3}(w) \\ &= wC(P_2,w)g_{n-3}(w) + wC(P_2,w)g_{n-4}(w) + wC(P_1,w)g_{n-3}(w) \\ &= C(P_3,w)g_{n-3}(w) + wC(P_2,w)g_{n-4}(w) \\ &= \cdots\cdots \\ &= C(P_k,w)g_{n-k}(w) + wC(P_{k-1},w)g_{n-k-1}(w) \end{aligned}$$

(2) Since $t_1 + t_2 = s_1 + s_2$ and $g_{t_1+t_2}(w) = g_{t_1}(w)C(P_{t_2},w) + g_{t_1-1}(w) C(P_{t_2-1},w)$, $g_{s_1+s_2}(w) = g_{s_1}(w)C(P_{s_2},w) + g_{s_1-1}(w)C(P_{s_2-1},w)$ by Lemma 4.9 (1). Then

$$\begin{aligned} &g_{t_1}(w)C(P_{t_2},w) - g_{s_1}(w)C(P_{s_2},w) \\ =&w(g_{t_1-1}(w)C(P_{t_2-1},w) - g_{s_1-1}(w)C(P_{s_2-1},w)) \end{aligned}$$

Therefore $g_{t_1}(w)C(P_{t_2},w) = g_{s_1}(w)C(P_{s_2},w)$ if and only if $g_{t_1-1}(w) C(P_{t_2-1},w) = g_{s_1-1}(w)C(P_{s_2-1},w)$.

Hence (2) can be proved by repeating the above process.

Lemma 4.10. *Let Z_n denote $T(1,1,n-3)$ and $t_1+t_2 = s_1+s_2$. Then $C(Z_{t_1},w)C(P_{t_2},w) = C(Z_{s_1},w)C(P_{s_2},w)$ if and only if $t_1 = s_1, t_2 = s_2$.*

Proof: We construct a sequences of polynomials $\{g_i(w)\}$, which defined by the following recursive formulas:

$g_1(w) = 2w, g_2(w) = w^2$ and $g_n(w) = w(g_{n-1}(w) + g_{n-2}(w))$

By the recursive formulas, $g_3(w) = w^3+2w^2$ and $g_k(w) = C(Z_k,w)$ when $k \geq 4$.

Let $m = min\{s_1,s_2,t_1,t_2\} - 1$.By Lemma 4.9,

It need only prove $g_{t_1-m}(w)C(P_{t_2-m},w) = g_{s_1-m}(w)C(P_{s_2-m},w)$ if and only if $t_1 = s_1, t_2 = s_2$

Hence it is only necessary to the following two cases:

Case 1. $min\{s_1,s_2,t_1,t_2\} = s_1$ or t_1.

Let $min\{s_1,s_2,t_1,t_2\} = s_1$. Then

$$g_{t_1-m}(w)C(P_{t_2-m},w) = 2wC(P_{s_2-m},w)$$

Since $t_2-m \geq 1, s_2-m \geq 1, t_1-m \geq 1$ and the leading coefficients of $g_{s_1-m}(w), C(P_{s_2-m},w)$ and $C(P_{t_2-m},w)$ is 1 if $t_1-m \geq 2, s_2-m \geq 1, t_2-m \geq 1$. Hence $t_1-m = 1$, and $s_1 = t_1, s_2 = t_2$.

When $min\{s_1,s_2,t_1,t_2\} = t_1$, the proof is entirely similar to the above.

Case 2. $min\{s_1,s_2,t_1,t_2\} = s_2$ or t_2.

Let $min\{s_1,s_2,t_1,t_2\} = s_2$. Then

$$g_{t_1-m}(w)C(P_{t_2-m},w) = wg_{s_1-m}(w)$$

Since $t_1-m \geq 1, t_2-m \geq 1, s_1-m \geq 1$ and the leading coefficients of $g_{t_1-m}(w), C(P_{t_2-m},w)$ and $g_{s_1-m}(w)$ is 1 if $t_1-m \geq 2, t_2-m \geq 1, s_1-m \geq 2$. Hence we have:

If $t_1-m = 1$ (or $s_1-m = 1$), then $s_1-m = 1$ (or $t_1-m = 1$). Therefore $s_1 = t_1, s_2 = t_2$.

If $s_1-m = 2$, then

$$w^3 = g_{t_1-m}(w)C(P_{t_2-m},w) \tag{1}$$

The equation (1) holds if and only if $t_1-m = 2$, and $t_2-m = 1$. Hence $s_1 = t_1, s_2 = t_2$.

If $s_1-m = 3$, then

$$w^3(w+2) = g_{t_1-m}(w)C(P_{t_2-m},w) \tag{2}$$

Since $C(P_2,w) = w^2+w, C(P_3,w) = w^3+2w^2, g_1(w) = 2w, \partial(g_{t_1-m}(w)) \geq 1$ and $\partial(C(P_{t_2-m},w)) \geq 4$ if $t_2 - m \geq 4$. Hence the equation (2) holds iff $t_1 - m = 3$, and $t_2 - m = 1$. Therefore $s_1 = t_1, s_2 = t_2$.

If $s_1 - m \geq 4, t_2 - m = 1$, then $s_1 = t_2, s_2 = t_2$.

If $s_1 - m \geq 4, t_2 - m \geq 2$, then

$$wC(Z_{s_1-m},w) = g_{t_1-m}(w)C(P_{t_2-m},w) \tag{3}$$

Since $t_1 - m \geq 2$, $R_m(g_i(w)) \geq 0 (i = 2,3)$ and $R_4(g_i(w)) = R_m(Z_i) = 0$ if $i \geq 4$. According to Theorem 3.2, 3.4, $R_m(Z_{s_1-m}) = 0, R_m(P_{t_2-m}) = 1$. Hence $R_m(Z_{s_1-m}) \neq R_m(g_{t_1-m}) + R_m(P_{t_2-m})$, it is a contradiction.

The sufficiency of Lemma 4.10 is easy proved .

Theorem 4.11. *Let $t_1 \geq 3, t_2 \geq 3$. Then $T(t_1,t_2)$ is circuit unique.*

Proof: Let $C(H,\mathrm{w}) = C(T(t_1,t_2),\mathrm{w})$. By Lemma 4.1, $H \cong T, P(H) = P(T(t_1,t_2))$ and $R_m(H) = R_m(T(t_1,t_2)) = -1$. By Theorem 3.4, $H \in \{T^5\}$. $H \cong T(s_1,s_2)$ and $s_1+s_2 = t_1+t_2$ by Lemma 4.6.

It need only prove $C(T(s_1,s_2),w) = C(T(t_1,t_2),w)$ iff $s_1 = t_1, s_2 = t_2$.

From Lemma 2.1,

$$C(T(s_1,s_2),w) = wC(Z_{s_1+s_2+1},w) + wC(Z_{s_1+1},w)C(P_{s_2-1},w)$$

$$C(T(t_1,t_2),w) = wC(Z_{t_1+t_2+1},w) + wC(Z_{t_1+1},w)C(P_{t_2-1},w)$$

By Lemma 4.10, $C(Z_{s_1+1},w)C(P_{s_2-1},w) = C(Z_{t_1+1},w)C(P_{t_2-1},w)$ iff $s_1 = t_1, s_2 = t_2$

Hence $H \cong T(t_1,t_2)$.

Lemma 4.12. [5] *Let G be circuit unique graph without cycle. Then $G\bigcup(\bigcup_i C_{n_i})$ is circuit unique.*

Corollary 4.13. *Let $G \in \{T(l_1,l_2,l_3), U_n, T(t_1,t_2)\}$. Then $G\bigcup(\bigcup_i C_{n_i})$ is circuit unique.*

Theorem 4.14. [9] *Let $s_1 \geq 2, s_2 \geq 2$. Then $Q(s_1,s_2)$ is circuit unique.*

Proof: Supposed $C(H,\mathrm{w}) = C(Q(s_1,s_2),\mathrm{w})$. Since $p(Q(s_1,s_2)) = q(Q(s_1, s_2))$ and $Q(s_1, s_2)$ has only one cycle. Hence H has only one cycle and it is connected (Otherwise there exist a component H_1 in H such that $q(H_1) \geq p(H_1)+1$ or there exist two components $H_i(i = 1,2)$ in H such that $q(H_i) = p(H_i)$, however H have two cycles at least, it is a contradiction). From Lemma 4.1, $R_m(H) = R_m(Q(s_1,s_2)) = -1$ and $N_k(H) = N_k(Q(s_1,s_2)) = s_1$. By Theorem 3.4, $H \cong Q(s_1,s_2)$.

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