Tenacity-Maximum Graphs

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Abstract. Tenacity is a recently introduced parameter to measure vulnerability of networks and graphs. We characterize graphs having maximum number of edges among all graphs with given number of vertices and tenacity.

1. Introduction

The parameter tenacity of a graph was introduced by Cozzens, Moazzami and Stueckle [3] as an alternative to connectivity, integrity and toughness, to measure the vulnerability of interconnection networks. The tenacity is a better vulnerability parameter as it measures both (i) the cost involved in disrupting the network (like connectivity), **and** (ii) the extent to which the network is disrupted. For a comparison of various vulnerability parameters, see [3]. For a graph G, let $p(G)$, $e(G)$, $\omega(G)$ and $\mu(G)$ respectively denote the number of vertices, edges, components, and the order of the maximum component. All our graphs are finite, undirected and simple. We refer to [1] for graph theoretic terminology. If $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are two vertex disjoint graphs, then (i) the **union** $G_1 \cup G_2$ is the graph with the vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$, and (ii) the **join** $G_1 + G_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$. For any graph G, rG denotes the union of r copies of G. If A, B are disjoint vertex sets of G then $[A, B]$ is the set of all edges in G with one end in A and the other end in B and $[A]$ is the subgraph of G induced by A.

For a graph $G(V, E)$, the **score** $sc(S; G)$ of a set $S \subseteq V$, and the **tenacity** $T(G)$ are defined by

$$sc(S;G) = \frac{|S| + \mu(G-S)}{\omega(G-S)}, \text{ and } T(G) = \min\{sc(S) : S \subseteq V(G)\}.$$

A subset S is called a T-set if $sc(S; G) = T(G)$. Clearly, if H is a spanning subgraph of G, then $T(H) \leq T(G)$. A graph G is called a ***T*-maximal** graph if $T(G+x) > T(G)$, for every edge $x \in G^c$, and it is called a (p, T)**-maximum** graph if it has maximum number of edges among all T-maximal graphs on p vertices and tenacity T. In this paper, we characterize (p, T)-maximum graphs. While several extremal problems connected with the parameters toughness and integrity are studied in [4], the tough maximum graphs are characterized in [2].

2. Tenacity-Maximal Graphs

As with many other parameters, it is easier to characterize (p, T)-maximal graphs than (p, T)-maximum graphs.

Theorem 1: *A graph G is T-maximal if and only if (i) $G = K_p$, or (ii) $G = K_{p_1} \cup K_{p_2} \cup \cdots \cup K_{p_n}$, where $p_1 = p_2 \geq p_3 \geq \cdots \geq p_n$, or (iii) $G = K_s + \{K_{p_1} \cup K_{p_2} \cup \cdots \cup K_{p_n}\}$, where $p_1 = p_2 \geq p_3 \geq \cdots \geq p_n$.*

Proof: Let S be a T-set, and $D_1, D_2, \ldots, D_n$ be the components of $G - S$. Clearly each D_i is complete and $[S, \bigcup_1^n V(D_i)]$ is complete; otherwise by joining a pair of non-adjacent vertices, we get a graph G' with more number of edges and $T(G') \leq sc(S; G') = sc(S; G) = T(G) \leq T(G')$. Thus G is of the form $K_s + (K_{p_1} \cup K_{p_2} \cup \cdots \cup K_{p_n})$, where $|S| = s \geq 0$, $n \geq 1$ and $p_1 \geq p_2 \geq \cdots \geq p_n$. If $p_1 > p_2$, then again by joining a vertex x of K_{p_1} with a vertex y of K_{p_2}, we get a graph G' with $e(G') = e(G) + 1$ such that $T(G') \leq sc(S \cup \{x\}; G') \leq sc(S; G) = T(G) \leq T(G')$. Hence $p_1 = p_2$. We conclude that G is as described in (i) or (ii) or (iii) according as $(s \geq 0,\ n = 1)$ or $(s = 0, n \geq 2)$ or $(s \geq 1, n \geq 2)$. □

3. Tenacity-Maximum Graphs

The characterization of tenacity-maximum graphs is complicated, since a graph G can achieve tenacity $\frac{m}{n}$, either by the existence of a T-set S_1 such that $| S_1 | + \mu(G - S_1) = m$ and $\omega(G - S_1) = n$, or by the existence of a T-set S_2 such that $| S_2 | + \mu(G - S_2) = lm$, and $\omega(G - S_2) = ln$. In view of these situations we first characterize pairs $(p, T = \frac{m}{n})$ for which there exist a graph G with p vertices and tenacity T, and then show that if G_1 and G_2 are p-vertex graphs with T-sets S_1 and S_2 respectively, as above and $l > 1$, then $e(G_1) < e(G_2)$.

We separately identify connected and disconnected T-maximum graphs.

3.1. Connected Tenacity-Maximum Graphs

Clearly, for any connected graph G, $1 \leq T(G) \leq p$, and moreover, $T(G) = p$ if and only if $G \simeq K_p$; so K_p is the unique (p, p)-tenacity-maximum graph. Hence, in the following we consider only connected, incomplete graphs; consequently, whenever $G = K_s + \{K_{p_1} \cup K_{p_2} \cup \cdots \cup K_{p_n}\}$ is T-maximal, we assume that $s \geq 1$ and $n \geq 2$. For convenience, we introduce the following classes of graphs.

- $\mathcal{M} = \{G : G$ is connected, incomplete and T-maximal$\}$.

- $\mathcal{G}(p,T) = \{G \in \mathcal{M} : p(G) = p$ and $T(G) = T\}$.
  If $\mathcal{G}(p,T) \neq \phi$, (p,T) is called a **c-valid pair.**
- $\mathcal{G}^*(p,T) = \{G \in \mathcal{G}(p,T) : e(G) \geq e(H)$, for all $H \in \mathcal{G}(p,T)\}$.
- $\mathcal{G}(p,m,n) = \{G \in \mathcal{M} : p(G) = p$, and for some T-set S of G,
  $| S | + \mu(G-S) = m$ and $\omega(G-S) = n\}$.
  If $\mathcal{G}(p,m,n) \neq \phi$, (p,m,n) is called a **c-valid triple.**
- $\mathcal{G}^*(p,m,n) = \{G \in \mathcal{G}(p,m,n) : e(G) \geq e(H)$, for all $H \in \mathcal{G}(p,m,n)\}$.
- $\mathcal{G}(p,m,n,t) = \{G \in \mathcal{M} : p(G) = p$, and for some T-set S of G,
  $\mu(G-S) = t$, $| S | = m-t$ and $\omega(G-S) = n\}$.
- $\mathcal{G}^*(p,m,n,t) = \{G \in \mathcal{G}(p,m,n,t) : e(G) \geq e(H)$,
  for all $H \in \mathcal{G}(p,m,n,t)\}$.

Remark: Let $T = \frac{m}{n}$, where $gcd(m,n) = 1$. If (p,T) is a c-valid pair, then (p,m,n) need not be a c-valid triple, but (p,km,kn) is a c-valid triple for some k.

Theorem 2: *If $G = K_s + \{D_1 \cup D_2 \cup \cdots \cup D_n\}$ is a T-maximal graph with $|D_1| = \mu(G-S)$ and S is a minimal T-set, then $S = V(K_s)$, and hence* $T(G) = \frac{| S | + | D_1 |}{n}$.

Proof: (i) Clearly, $S \supseteq V(K_s)$; otherwise $T(G) = p$.
(ii) $S \cap D_i = \phi$, for every i; on the contrary, if $x \in S \cap D_i$, for some i, $sc(S - \{x\}) \leq sc(S)$. Hence, $S = V(K_s)$. □

Theorem 3: *An integral triple (p,m,n) is c-valid if and only if*
$$m+n-1 \leq p \leq mn-n+1.$$

Proof: Let (p,m,n) be c-valid and G be a connected graph with p vertices having a T-set S with $| S | + \mu(G-S) = m$ and $\omega(G-S) = n$. Let $D_1, D_2, \ldots, D_n$ be the components of $G-S$, where $| D_1 | = \mu(G-S)$. We have $p = m + \sum_{i=2}^{n} D_i \geq m+n-1$. Next, since G is connected, $|S| \geq 1$ and consequently $\mu(G-S) \leq m-1$. Hence,

$$p = m + \sum_{i=2}^{n} | D_i | \leq m + (n-1)(m-1) = mn - n + 1.$$

Conversely, for any triple (p,m,n) with $m+n-1 \leq p \leq mn-n+1$, we construct a graph $G(p,m,n) \in \mathcal{G}(p,m,n)$. Let $k = p-m-n+1$, and u be the smallest integer such that $k \leq u(n-1)$. Let $k = uq + r$, where

$0 \leq r < u$, and $q = 0$ if $u = 0$. By our assumptions on p, m, n, we have $k \geq 0$, $u \leq m - 2$ and $n \geq q + 1$. Let $G^* = G(p, m, n)$ be the graph

$$K_{m-u-1} + (D_1 \cup D_2 \cup \cdots \cup D_n),$$

where $D_1 = D_2 = \cdots = D_{q+1} = K_{u+1}$, $D_{q+2} = K_{r+1}$ and $D_i = K_1$, for every i, $q + 2 < i \leq n$.

By Theorem 2, $S = V(K_{m-u-1})$ is a T-set of G^*. Moreover, $| S | + \mu(G^* - S) = m$ and $\omega(G^* - S) = n$. Hence, $G(p, m, n) \in \mathcal{G}(p, m, n, u+1) \subseteq \mathcal{G}(p, m, n)$. □

- In the paper, we will often refer to the graph $G^* = G(p, m, n)$ defined above whenever (p, m, n) is c-valid; the integers k, u, q, r which are functions of m, n will have the above meaning.

Combining Theorem 3 and Remark 1, we get the following characterization of tenacity-c-valid pairs.

Corollary 3.1: *Let $T = \frac{m}{n}$, where $gcd(m, n) = 1$. Then (p, T) is c-valid if and only if for some positive integer k, (p, km, kn) is c-valid, that is,*

$$km + kn - 1 \leq p \leq k^2mn - kn + 1.$$ □

A simple arithmetic yields the following result.

Corollary 3.2: *If (p, km, kn), (p, lm, ln) are c-valid and $k < l$, then $(p, (l-1)m, (l-1)n)$ is c-valid.* □

A component D of a graph G is called **trivial** or **non-trivial** according as $| D | = 1$ or $| D | > 1$.

Theorem 4: *Let $G \in \mathcal{G}^*(p, m, n)$ and let S be a T-set of G such that $| S | + \mu(G - S) = m$ and $\omega(G - S) = n$. Then $G - S$ has the following properties:*

(4.1) There are at least two non-trivial components in $G - S$ if and only if $p > m + n - 1$.

(4.2) There are either zero or at least two non-trivial components in $G - S$.

(4.3) There is at most one component D in $G - S$ such that $1 < | D | < \mu(G - S)$.

(4.4) If $k = p - (m + n - 1)$ and u is the smallest integer such that $k \leq u(n-1)$, then $\mu(G - S) \geq u + 1$.

Proof: Let $D_1, D_2, \ldots, D_n$ be the components of $G - S$, where $| D_1 | \geq | D_2 | \geq \cdots \geq | D_n |$. Since G is T-maximal, D_i's are complete.

(4.1) Since, $p = | S | + \sum_{i=1}^{n} | D_i | = m + \sum_{i=2}^{n} | D_i |$, the statement follows.

(4.2) On the contrary, assume that D_1 is the only non-trivial component in $G - S$. Let $x \in V(D_1)$ and G' be the graph with $V(G') = V(G)$, $E(G') = E(G) \cup [V(D_1) - \{x\}, \bigcup_{i=2}^{n} V(D_i)]$. If $S' = S \cup (V(D_1) - \{x\})$, then $T(G) \leq T(G') \leq sc(S'; G') = sc(S; G) = T(G)$. Moreover, $| S' | + \mu(G' - S') = m$ and $\omega(G' - S') = n$. So, $G' \in \mathcal{G}(p, m, n)$, a contradiction to the maximality of $e(G)$.

(4.3) On the contrary, suppose there are two components D_i and D_j such that $1 < | D_i | \leq | D_j | < \mu(G - S)$. Let $x \in V(D_i)$ and G' be the graph with $V(G') = V(G)$, $E(G') = E(G) - \{(x, y) : y \in V(D_i)\} \cup \{(x, z) : z \in V(D_j)\}$. Then, $e(G') > e(G)$. By Theorem 2, $T(G') = T(G)$ and S is a T-set of G' such that $| S | + \mu(G' - S) = m$ and $\omega(G' - S) = n$. So, $G' \in \mathcal{G}(p, m, n)$ - again a contradiction to the maximality of $e(G)$.

(4.4) By the definitions of $G \in \mathcal{G}^*(p, m, n)$, k and u, we have

$$\begin{aligned}(u-1)(n-1) < k &= p - (m + n - 1) \\ &= p - (| S | + \mu(G - S) + \omega(G - S) - 1) \\ &= \sum_{i=2}^{n} (| D_i | - 1) \leq (\mu(G - S) - 1)(n - 1).\end{aligned}$$

Hence $\mu(G - S) > u$. □

Corollary 4.1: *If $G \in \mathcal{G}^*(p, m, n, t)$, then $G = K_{m-t} + (D_1 \cup D_2 \cup \cdots \cup D_n)$, where $D_1 = D_2 = \cdots = D_{q+1} = K_t$, $D_{q+2} = K_{r+1}$, $D_i = K_1$, for every i, $q+2 < i \leq n$, and q, r are defined by $k = p - (m+n-1) = q(t-1) + r$, $0 \leq r < t - 1$. Hence, $\mathcal{G}^*(p, m, n, t)$ has at most one element.* □

Theorem 5: *If for some positive integers t and x, $\mathcal{G}(p, m, n, t) \neq \phi$ and $\mathcal{G}(p, m, n, t + x) \neq \phi$, then:*
(5.1) $\mathcal{G}(p, m, n, t + x - 1) \neq \phi$.
(5.2) If $G_t \in \mathcal{G}^*(p, m, n, t)$ and $G_{t+x} \in \mathcal{G}^*(p, m, n, t + x)$, then $e(G_t) > e(G_{t+x})$.

Proof: **(5.1)** Let $G_t \in \mathcal{G}^*(p, m, n, t)$ and $G_{t+x} \in \mathcal{G}^*(p, m, n, t + x)$. By Corollary 4.1, we have:

$$G_t := K_{m-t} + \{D_1 \cup D_2 \cup \cdots \cup D_n\}$$

where $D_1 = D_2 = \cdots = D_{q_1+1} = K_t$, $D_{q_1+2} = K_{r_1+1}$, $D_i = K_1$, for every i, $q_1 + 2 < i \leq n$ and q_1, r_1 are defined by $p - m - n + 1 = k = q_1(t-1) + r_1$, $0 \leq r_1 < t - 1$.

$$G_{t+x} := K_{m-t-x} + \{D'_1 \cup D'_2 \cup \cdots \cup D'_n\}$$

where $D'_1 = D'_2 = \cdots = D'_{q_2+1} = K_{t+x}$, $D'_{q_2+2} = K_{r_2+1}$, $D'_i = K_1$, for every i, $q_2+2 < i \leq n$ and q_2, r_2 are defined by $k = q_2(t+x-1)+r_2$, $0 \leq r_2 < t+x-1$.

In two steps, we construct a graph $G_0 \in \mathcal{G}^*(p,m,n,t+x-1)$, from G_{t+x} as follows:

(1) We delete a set A of q_2+r_2+1 vertices consisting of one vertex from each D'_i, $1 \leq i \leq q_2+1$ and r_2 vertices from D'_{q_2+2}. Next, add one of the vertices, say a, of A to $V(K_{m-t-x})$ and form $K_{m-t-x+1}$. The resulting graph is

$$G'_0 = K_{m-t-x+1} + \{(q_2+1)K_{t+x-1} \cup (\bigcup_{i=1}^{n-q_2-1} [B_i])\}$$

where $[B_i] = K_1$, $1 \leq i \leq n-q_2-1$.

(2) Let $q_2+r_2 = Q(t+x-2)+R$, where $0 \leq R < t+x-2$. Let $A-\{a\} = A_1 \cup A_2 \cup \cdots \cup A_Q \cup A_{Q+1}$ be a partition, where for $1 \leq i \leq Q$, $|A_i| = t+x-2$, $|A_{Q+1}| = R$. We next construct the required G_0 from G'_0.

$$G_0 := K_{m-t-x+1} + \{(q_2+1)K_{t+x-1} \cup (\bigcup_{i=1}^{Q} [B_i \cup A_i]) \cup [B_{Q+1} \cup A_{Q+1}] \cup (\bigcup_{i=Q+2}^{n-q_2-1} [B_i])\},$$

where $[B_i \cup A_i]$ is a complete graph for every i, $1 \leq i \leq Q+1$. Hence,

$$G_0 = K_{m-t-x+1} + \{(q_2+Q+1)K_{t+x-1} \cup K_{R+1} \cup (n-q_2-Q-2)K_1\}.$$

This construction of G_0, of course, can be carried out provided $Q \leq n-q_2-1$, if $R=0$; and $Q+1 \leq n-q_2-1$, if $R>0$. So, we prove these inequalities. Using the definitions of q_1, r_1, q_2, r_2, Q and R, we have

$$\begin{aligned} q_1(t-1)+r_1 = k &= q_2(t+x-1)+r_2 \\ &= q_2(t+x-2)+Q(t+x-2)+R \\ &= (q_2+Q)(t-1)+(q_2+Q)(x-1)+R. \end{aligned}$$

So, $q_1 \geq q_2+Q$.

If $R=0$, then $n \geq q_1+1 \geq q_2+Q+1$, as required. Next suppose $R>0$. If $q_1 > q_2+Q$, then again $n \geq q_1+1 > q_2+Q+1$. So, assume that $q_1 = q_2+Q$. In this case, since $q_1(t-1)+r_1 = k = (q_2+Q)(t-1)+(q_2+Q)(x-1)+R$, we have $r_1 = (q_2+Q)(x-1)+R > 0$.

Hence, $n \geq q_1 + 2 \geq q_2 + Q + 2$, and so G_0 is well defined.

By Theorem 2, $S = V(K_{m-t-x+1})$ is a T-set of G_0 and $T(G_0) = \frac{m}{n}$. Moreover, $|S| + \mu(G_0 - S) = m$, $\omega(G_0 - S) = n$ and $\mu(G_0 - S) = t + x - 1$. Hence, $G_0 \in \mathcal{G}(p, m, n, t + x - 1)$, proving our first assertion.

(5.2) In view of (5.1), it is enough to show that if $G_t \in \mathcal{G}^*(p, m, n, t)$ and $G_{t+1} \in \mathcal{G}^*(p, m, n, t+1)$, then $e(G_t) > e(G_{t+1})$. Let G_t be defined as in the proof of (5.1). Let

$$G_{t+1} = K_{m-(t+1)} + \{D'_1 \cup D'_2 \cup \cdots \cup D'_n\}$$

where $D'_1 = \cdots = D'_{q_2+1} = K_{t+1}$, $D'_{q_2+2} = K_{r_2+1}$, $D'_i = K_1$, for every i, $q_2 + 2 < i \leq n$, and q_2, r_2 are defined by $k = q_2 t + r_2$, $0 \leq r_2 < t$. We have to show that $e(G_t) > e(G_{t+1})$, that is,

$$\binom{m-t}{2} + (m-t)(p-m+t) + (q_1+1)\binom{t}{2} + \binom{r_1+1}{2}$$
$$> \binom{m-t-1}{2} + (m-t-1)(p-m+t+1) + (q_2+1)\binom{t+1}{2} + \binom{r_2+1}{2}.$$

Since $q_1 = \lfloor \frac{k}{t-1} \rfloor \geq \lfloor \frac{k}{t} \rfloor = q_2$, substituting $q_1 = q_2 + h$, using the identity $\binom{a+b}{2} = \binom{a}{2} + \binom{b}{2} + ab$, and cancelling common terms on both sides, we are left to show:

$$(p-m) + h\binom{t}{2} + \binom{r_1+1}{2} > q_2 t + \binom{r_2+1}{2}.$$

Since, by the structure of G_{t+1}, we have

$$p - m = \sum_{i=2}^{n} |D'_i| \geq q_2(t+1) + r_2 + 1 > q_2 t,$$

it is enough to show:

(5.3) $$h\binom{t}{2} + \binom{r_1+1}{2} \geq \binom{r_2+1}{2}.$$

If $h = 0$, then $r_1 = k - q_1(t-1) = q_2 t + r_2 - q_1(t-1) = q_1 t + r_2 - q_1(t-1) \geq r_2$.

If $h > 0$, then $h\binom{t}{2} \geq \binom{t}{2} \geq \binom{r_2+1}{2}$.

Hence, in both the cases (5.3) holds. □

Corollary 5.1: *If (p, m, n) is c-valid, then the graph $G^* = G(p, m, n)$, is the unique element of $\mathcal{G}^*(p, m, n)$.*

Proof: By definition, $G^* \in \mathcal{G}^*(p, m, n, u+1)$.
By Theorem 4, $\mathcal{G}^*(p, m, n) = \bigcup_{t \geq u+1} \mathcal{G}^*(p, m, n, t)$, and by Theorem 5, $e(G^*) > e(G)$, for every $G \in \bigcup_{t > u+1} \mathcal{G}^*(p, m, n, t)$. So, $\mathcal{G}^*(p, m, n) = \mathcal{G}^*(p, m, n, u+1)$.
By Corollary 4.1, $\mathcal{G}^*(p, m, n, u+1) = \{G^*\}$. □

Theorem 6: *If $G_1 \in \mathcal{G}^*(p, lm, ln)$ and $G_2 \in \mathcal{G}^*(p, (l+1)m, (l+1)n)$, then $e(G_1) < e(G_2)$.*

Proof: Let $k_1 = p - (lm + ln - 1)$, and u_1 be the smallest integer such that $k_1 \leq u_1(ln - 1)$. Let q_1 and r_1 be defined by the equation $k_1 = q_1 u_1 + r_1$, $0 \leq r_1 < u_1$. Likewise, let $k_2 = p - ((l+1)m + (l+1)n - 1)$, and u_2 be the smallest integer such that $k_2 \leq u_2((l+1)n - 1)$. Let q_2 and r_2 be defined by the equation $k_2 = q_2 u_2 + r_2$, $0 \leq r_2 < u_2$. Let $\mu_1 = u_1 + 1$ and $\mu_2 = u_2 + 1$. Since $k_2 = p - (lm + ln - 1) - m - n < k_1 \leq u_1(ln - 1) < u_1(ln + n - 1)$, we have $u_2 \leq u_1$, by the definition of u_2. So, $\mu_2 \leq \mu_1$. By Corollary 5.1,

$$G_1 \ : \ K_{lm-\mu_1} + \{D_1 \cup D_2 \cup \cdots \cup D_{ln}\}$$

where $D_1 = D_2 = \cdots = D_{q_1+1} = K_{\mu_1}$, $D_{q_1+2} = K_{r_1+1}$, $D_i = K_1$, for every i, $q_1 + 2 < i \leq ln$; and

$$G_2 \ : \ K_{lm+m-\mu_2} + \{D'_1 \cup D'_2 \cup \cdots \cup D'_{ln+n}\}$$

where $D'_1 = D'_2 = \cdots = D'_{q_2+1} = K_{\mu_2}$, $D'_{q_2+2} = K_{r_2+1}$, $D'_i = K_1$, for every i, $q_2 + 2 < i \leq ln + n$.

From the descriptions of G_1 and G_2, it is clear that G_2 can be obtained from G_1 by

(i) shifting a set $D \subseteq \bigcup_{i=1}^{ln} V(D_i)$ of $(m + \mu_1 - \mu_2)$ vertices to $V(K_{lm-\mu_1})$ to form $K_{lm+m-\mu_2}$, and then (ii) regrouping the vertices of $\bigcup_{i=1}^{ln} V(D_i) - D$ to form $(q_2 + 1)$ K_{μ_2}- components, one K_{r_2+1}- component and $(ln + n - q_2 - 2)$ K_1- components. We distinguish three cases and apply the operations (i) and (ii).

Case 1: $q_1 = q_2$.

Subcase 1.1: $r_1 \leq r_2$.

We partition each $V(D_i)$, $1 \leq i \leq q_1 + 1$ into two subsets A_i and B_i such that $| A_i |= \mu_2$, $| B_i |= \mu_1 - \mu_2$, and delete the set of edges
$\mathcal{B} = \bigcup_{i=1}^{q_1+1} ([A_i, B_i] \cup [B_i]) \cup E(D_{q_1+2})$ to obtain the graph

$$\begin{aligned} G_1 - \mathcal{B} &= \underline{K_{lm-\mu_1}} + \underline{\{(q_1+1)K_{\mu_2}} \cup \\ &\qquad \underline{((q_1+1)(\mu_1 - \mu_2) + (r_1+1) + (ln - q_1 - 2))K_1\}} \\ &= [X] + \{[Y] \cup [Z]\} \quad \text{(-say, for notational convenience).} \end{aligned}$$

Clearly,

$$| \mathcal{B} |= (q_1+1)\left[\mu_2(\mu_1 - \mu_2) + \binom{\mu_1 - \mu_2}{2}\right] + \binom{r_1+1}{2}.$$

Next, we (i) shift a subset $D \subseteq Z$ of $m + \mu_1 - \mu_2$ vertices to X to form $K_{lm+m-\mu_2}$, (ii) add enough edges to $Z - D$ to form one K_{r_2+1}-component and (iii) add the edges $[D, (Y \cup Z) - D]$. If $\mathcal{A}$ is the set of edges so added to $G_1 - \mathcal{B}$, then $G_2 = G_1 - \mathcal{B} + \mathcal{A}$. Clearly,

$$| \mathcal{A} |= \binom{m + \mu_1 - \mu_2}{2} + (m+\mu_1-\mu_2)(p-(lm-\mu_1)-(m+\mu_1-\mu_2)) + \binom{r_2+1}{2}.$$

Using $\mu_1 \geq \mu_2$ and $p - lm > q_1\mu_1$, we can show that $| \mathcal{A} |>| \mathcal{B} |$, so that $e(G_1) < e(G_2)$.

Subcase 1.2: $r_1 > r_2$.

We partition each $V(D_i)$, $1 \leq i \leq q_1+1$ as in Subcase 1.1. Furthermore, we partition D_{q_1+2} into two subsets A_{q_1+2} and B_{q_1+2} where $| A_{q_1+2} |= r_2 + 1$, $| B_{q_1+2} |= r_1 - r_2$, and delete the set of edges $\mathcal{B} = \bigcup_{i=1}^{q_1+2} ([A_i, B_i] \cup [B_i])$ to obtain the graph

$$\begin{aligned} G_1 - \mathcal{B} &= \underline{K_{lm-\mu_1}} + \underline{\{(q_1+1)K_{\mu_2}} \cup \underline{K_{r_2+1}} \cup \\ &\qquad \underline{((q_1+1)(\mu_1 - \mu_2) + (r_1 - r_2) + (ln - q_1 - 2))K_1\}} \\ &= [X] + \{[Y] \cup [Z] \cup [W]\} \quad \text{(-say).} \end{aligned}$$

Clearly,

$$| \mathcal{B} |= (q_1+1)\left[\mu_2(\mu_1 - \mu_2) + \binom{\mu_1 - \mu_2}{2}\right] + (r_2+1)(r_1 - r_2) + \binom{r_1 - r_2}{2}.$$

Next, we (i) shift a subset $D \subseteq W$ of $m + \mu_1 - \mu_2$ vertices to X to form $K_{lm+m-\mu_2}$, and (ii) add the edges $[D, (Y \cup Z \cup W) - D]$. If $\mathcal{A}$ is the set of edges so added to $G_1 - \mathcal{B}$, then $G_2 = G_1 - \mathcal{B} + \mathcal{A}$.
Clearly,

$$|\mathcal{A}| = \binom{m+\mu_1-\mu_2}{2} + (m + \mu_1 - \mu_2)(p - lm - m + \mu_2).$$

Using $\frac{r_1 + r_2 + 1}{2} < \mu_1$ and $k_1 - k_2 = m + n$, we can show that $|\mathcal{A}| > |\mathcal{B}|$, so that $e(G_1) < e(G_2)$.

Case 2: $q_1 < q_2$.

Subcase 2.1: $r_1 + 1 \leq \mu_2$.

We partition $V(D_i)$, $1 \leq i \leq q_1 + 1$, as in case 1, and delete the edges $\bigcup_{i=1}^{q_1+1} ([A_i, B_i] \cup [B_i]) \cup E(D_{q_2+2})$ to form the graph

$$\begin{aligned} G_1 - \mathcal{B} &= \underline{K_{lm-\mu_1}} + \underline{\{(q_1+1)K_{\mu_2} \cup} \\ &\qquad \underline{((q_1+1)(\mu_1 - \mu_2) + (r_1 + 1) + (ln - q_1 - 2))K_1\}} \\ &= [X] + \{[Y] \cup [Z]\} \quad \text{(-say)}. \end{aligned}$$

Clearly, $$|\mathcal{B}| = (q_1+1)\left[\mu_2(\mu_1 - \mu_2) + \binom{\mu_1-\mu_2}{2}\right] + \binom{r_1+1}{2}.$$

Next, we (i) shift a subset $D \subseteq Z$ of $m + \mu_1 - \mu_2$ vertices to X to form $K_{lm+m-\mu_2}$, (ii) add enough edges to $Z - D$ to form $(q_2 - q_1)$ K_{μ_2}-components and one K_{r_2+1}-component and (iii) add the edges $[D, (Y \cup Z) - D]$. If $\mathcal{A}$ is the set of edges so added to $G_1 - \mathcal{B}$, then $G_2 = G_1 - \mathcal{B} + \mathcal{A}$. Clearly,

$$|\mathcal{A}| = \binom{m+\mu_1-\mu_2}{2} + (m+\mu_1-\mu_2)(p-lm-m+\mu_2) + (q_2-q_1)\binom{\mu_2}{2} + \binom{r_2+1}{2}.$$

As in the previous case, we can show that $|\mathcal{A}| > |\mathcal{B}|$, so that $e(G_1) < e(G_2)$.

Subcase 2.2: $r_1 + 1 > \mu_2$.

We partition $V(D_i)$, $1 \leq i \leq q_1 + 1$, as in case 1, and partition $V(D_{q_1+2})$ into A_{q_1+2} and B_{q_1+2} such that $|A_{q_1+2}| = \mu_2$ and $|B_{q_1+2}| = r_1 + 1 - \mu_2$, and delete the edges $\bigcup_{i=1}^{q_1+2}([A_i,B_i]\cup[B_i])$ to form the graph

$$\begin{aligned} G_1-\mathcal{B} &= \underline{K_{lm-\mu_1}}+\underline{\{(q_1+2)K_{\mu_2}}\cup \\ &\qquad \underline{((q_1+1)(\mu_1-\mu_2)+(r_1+1-\mu_2)+(ln-q_1-2))K_1\}} \\ &= [X]+\{[Y]\cup[Z]\} \qquad \text{(-say)}. \end{aligned}$$

Clearly,

$$|\mathcal{B}|=(q_1+1)\left[\mu_2(\mu_1-\mu_2)+\binom{\mu_1-\mu_2}{2}\right]+\mu_2(r_1+1-\mu_2)+\binom{r_1+1-\mu_2}{2}.$$

Next, we (i) shift a subset $D\subseteq Z$ of $m+\mu_1-\mu_2$ vertices to X to form $K_{lm+m-\mu_2}$, (ii) add enough edges to $Z-D$ to form (q_2-q_1-1) K_{μ_2}-components and one K_{r_2+1}-component and (iii) add the edges $[D,(Y\cup Z)-D]$. If $\mathcal{A}$ is the set of edges so added to $G_1-\mathcal{B}$, then $G_2=G_1-\mathcal{B}+\mathcal{A}$. Clearly,

$$\begin{aligned} |\mathcal{A}| &= \binom{m+\mu_1-\mu_2}{2}+(m+\mu_1-\mu_2)(p-lm-m+\mu_2)+ \\ &\qquad (q_2-q_1-1)\binom{\mu_2}{2}+\binom{r_2+1}{2} \end{aligned}$$

Using $\frac{r_1+\mu_2}{2}<r_1+1$ and $q_1\mu_1+r_1+1\leq p-lm$, we can show that $|\mathcal{A}|>|\mathcal{B}|$, so that $e(G_1)<e(G_2)$.

Case 3: $q_1>q_2$.

We first observe that $r_2+1\leq u_2=\mu_2-1\leq\mu_1-1$. We partition each $V(D_i)$, $1\leq i\leq q_2+1$ into two subsets A_i and B_i such that $|A_i|=\mu_2, |B_i|=\mu_1-\mu_2$. Furthermore, we partition $V(D_{q_2+2})$ into subsets A_{q_2+2}, B_{q_2+2}, where $|A_{q_2+2}|=r_2+1$, $|B_{q_2+2}|=\mu_1-r_2-1$, and delete the set of edges $\mathcal{B}=\bigcup_{i=1}^{q_2+2}([A_i,B_i]\cup[B_i])\cup(\bigcup_{q_2+3}^{q_1+2}E(D_i))$ to obtain the graph

$$\begin{aligned} G_1-\mathcal{B} &= \underline{K_{lm-\mu_1}}+\underline{\{(q_2+1)K_{\mu_2}}\cup\underline{K_{r_2+1}}\cup\underline{((q_2+1)(\mu_1-\mu_2)+} \\ &\qquad \underline{(\mu_1-r_2-1)+(q_1-q_2-1)\mu_1+(r_1+1)+(ln-q_1-2))K_1\}} \\ &= [X]+\{[Y]\cup[Z]\cup[W]\} \qquad \text{(- say)}. \end{aligned}$$

Clearly,

$$\begin{aligned} |\mathcal{B}| &= (q_2+1)\left[\mu_2(\mu_1-\mu_2)+\binom{\mu_1-\mu_2}{2}\right]+(r_2+1)(\mu_1-r_2-1)+ \\ &\qquad \binom{\mu_1-r_2-1}{2}+(q_1-q_2-1)\binom{\mu_1}{2}+\binom{r_1+1}{2}. \end{aligned}$$

Next, we (i) shift a subset $D \subseteq W$ of $m + \mu_1 - \mu_2$ vertices to X to form $K_{lm+m-\mu_2}$, and (ii) add the set of edges $[D, (Y \cup Z \cup W) - D]$. If $\mathcal{A}$ is the set of edges so added to $G_1 - \mathcal{B}$, then $G_1 - \mathcal{B} + \mathcal{A} = G_2$. Clearly,

$$|\mathcal{A}| = \binom{m + \mu_1 - \mu_2}{2} + (m + \mu_1 - \mu_2)(p - lm - m + \mu_2).$$

Using $r_2 < \mu_1$ and $\frac{r_1+1}{2} < \mu_1$, we can easily show that $|\mathcal{A}| > |\mathcal{B}|$, so that $e(G_1) < e(G_2)$. □

Theorem 6 implies the following characterization of connected (p, T)-maximum graphs.

Theorem 7: *Let $p \geq 1$ be an integer and $T = \frac{m}{n}$, where $gcd(m, n) = 1$. Let L be the largest integer such that (p, Lm, Ln) is c-valid. If a connected graph G is (p, T)-maximum, then it is isomorphic with $G(p, Lm, Ln)$. Conversely, among all connected graphs G such that $p(G) = p$ and $T(G) = T$, $G(p, Lm, Ln)$ has maximum number of edges.* □

3.2. Disconnected Tenacity-Maximum Graphs

In this section, we assume that G is disconnected, T-maximal and hence of the form $K_{p_1} \cup K_{p_2} \cup \cdots \cup K_{p_k}$. The characterization of disconnected T-maximum graphs (Theorem 11) is not so complete as Theorem 7, since Theorem 6 is false for disconnected T-maximum graphs.

Theorem 8: *If $G = K_{p_1} \cup K_{p_2} \cup \cdots \cup K_{p_n}$, where $p_1 = p_2 \geq \cdots \geq p_n$, is a T-maximal graph , then $T(G) = \frac{p_1}{n}$, and moreover empty set is the only T-set of G.*

Proof: While $sc(\phi) = \dfrac{p_1}{n}$, the score of any non-empty set is atleast $\dfrac{p_1 + 1}{n}$.□

Theorem 9: *Let (p, m, n) be a triple of positive integers. There exists a disconnected T-maximal graph G with $p(G) = p$, $\mu(G) = m$, $\omega(G) = n$ and $T(G) = \frac{m}{n}$ if and only if $2m + n - 2 \leq p \leq mn$.*

Proof: If $G = K_{p_1} \cup K_{p_2} \cup \ldots \cup K_{p_n}$, where $m = p_1 = p_2 \geq p_3 \geq \cdots \geq p_n \geq 1$, is a disconnected T-maximal graph on p vertices, with $\mu(G) = m$, $\omega(G) = n$ and $T(G) = \frac{m}{n}$ then clearly, $2m + n - 2 \leq \sum_1^n p_i = p \leq mn$.

Conversely, if (p, m, n) is a triple such that $2m + n - 2 \leq p \leq mn$, then $G = (q + 2)K_m \cup K_{r+1} \cup (n - q - 3)K_1$, where q and r are defined by the equation $p - (2m + n - 2) = q(m - 1) + r$, $q \geq 0$, $0 \leq r < m - 1$, is a disconnected, T-maximal graph on p vertices. □

- (p, m, n) is called **d-valid** if there exists a graph G as described in Theorem 9.

Let $p > 0$ be an integer and $T = \frac{m}{n}$ where $gcd(m, n) = 1$. There exists a graph G with $p(G) = p$ and $T(G) = T$ if and only if for some integer k, the triple (p, km, kn) is d-valid; that is, if and only if, for some k, $2km + kn - 1 \leq p \leq k^2mn$.

Theorem 10: *Let G be a disconnected T-maximum graph with $p(G) = p$, $\mu(G) = m$, $\omega(G) = n$ and $T(G) = \frac{m}{n}$. Then $G = D(p, m, n) := (q + 2)K_m \cup K_{r+1} \cup (n - q - 3)K_1$, where $p - (2m + n - 2) = q(m - 1) + r$, $q \geq 0$, $0 \leq r < m - 1$.*

Proof: By Theorem 1, G is of the form $K_{p_1} \cup K_{p_2} \cup K_{p_3} \cup \cdots \cup K_{p_n}$ where $m = p_1 = p_2 \geq p_3 \geq \cdots \geq p_n \geq 1$. We next claim that, for at most one i, $1 < p_i < m$. On the contrary, if there are p_i, p_j such that $1 < p_i \leq p_j < m$, let $x \in V(K_{p_i})$. Defining the graph G' with $V(G') = V(G)$ and $E(G') = E(G) - \{(x, y) : y \in V(K_{p_i})\} + \{(x, z) : z \in V(K_{p_j})\}$, we have $p(G') = p$, $\mu(G') = m$, $\omega(G') = n$, $T(G') = \frac{m}{n}$, and $e(G') = e(G) - (p_i - 1) + p_j > e(G)$, a contradiction to the maximality of $e(G)$. So, $G = D(p, m, n)$. □

- Let $D(p, km, kn) := (q + 2)K_{km} \cup K_{r+1} \cup (kn - q - 3)K_1$, where $p - (2km + kn - 2) = q(km - 1) + r$, $0 \leq r < km - 1$. Let $\lambda(p, km, kn) := e(D(p, km, kn))$.

Theorem 11: *Let $p > 0$ be an integer and $T = \frac{m}{n}$, where $gcd(m, n) = 1$. Let z be the largest integer such that $2zm + zn - 2 \leq p \leq z^2mn$. Assume that z exists. Let $\lambda^*(p, m, n) = \max\{\lambda(p, km, kn) : 1 \leq k \leq z\}$. Let $\mathcal{D}^*(p, m, n) = \{D(p, km, kn) : 1 \leq k \leq z$ and $e(D(p, km, kn)) = \lambda^*(p, m, n)\}$. If a disconnected graph G is (p, T)-maximum, then $G \in \mathcal{D}^*(p, m, n)$. Conversely, among all disconnected graphs G such that $p(G) = p$ and $T(G) = T$, the elements of $\mathcal{D}^*$ have the maximum number of edges.*

Proof: The statements follow from the definition of $\mathcal{D}^*(p, m, n)$. □

Finally, we combine Theorems 7 and 11 to obtain a characterization of (p, T)-maximum graphs.

Theorem 12: *Let $p, T, z, \mathcal{D}^*(p, m, n)$ be defined as in Theorem 11 above. Let L and be the largest integer such that (p, Lm, Ln) is c-valid. A graph G is (p, T)-maximum if and only if*

$$G \in \begin{cases} \{G(p, Lm, Ln)\}, & \textit{if } (p, T) \textit{ is c-valid}, \\ \mathcal{D}^*(p, m, n), & \textit{if } (p, T) \textit{ is not c-valid but d-valid}. \end{cases}$$

Proof: Suppose G is (p, T)-maximum. If (p, T) is c-valid and not d-valid, then $G = G(p, Lm, Ln)$, by Theorem 7. If (p, T) is d-valid and not c-valid, then $G \in \mathcal{D}^*(p, m, n)$ by Theorem 11. We next proceed to show that if (p, T) is both c-valid and d-valid, then $G = G(p, Lm, Ln)$. Let l be such that $\lambda^*(p, m, n) = \lambda(p, lm, ln)$. If (p, lm, ln) is c-valid, then $l \leq L$ and so,

$$\begin{aligned} \lambda(p, lm, ln) &< e(G(p, lm, ln)), && \text{(- easy to verify)} \\ &\leq e(G(p, Lm, Ln)), && \text{(by Theorem 6).} \end{aligned}$$

If (p, lm, ln) is not c-valid, then it can be easily verified that $(p, (l+1)m, (l+1)n)$ is c-valid. Moreover, we can obtain $G(p, (l+1)m, (l+1)n)$ from $D(p, lm, ln)$, by using the techniques employed to prove Theorem 6, and show that $\lambda(p, lm, ln) < e(G(p, (l+1)m, (l+1)n))$. Hence,

$$\lambda^*(p, m, n) = \lambda(p, lm, ln) < e(G(p, (l+1)m, (l+1)n)) \leq e(G(p, Lm, Ln)).$$

Conversely, (i) $G(p, Lm, Ln)$ is (p, T)-maximum if (p, T) is c-valid, by Theorem 7, and (ii) every element of $\mathcal{D}^*(p, m, n)$ is (p, T)-maximum if (p, T) is d-valid by the definition of $\mathcal{D}^*$. □

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