On Strongly k-Extendable Graphs

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Abstract. Let G be a simple connected graph on 2n vertices with a perfect matching. For a positive integer k, $1 \le k \le n - 1$, G is *k-extendable* if for every matching M of size k in G, there is a perfect matching in G containing all the edges of M. For an integer k, $0 \le k \le n - 2$, G is *strongly k-extendable* if $G - \{u, v\}$ is k-extendable for every pair of vertices u and v of G. The problem that arises is that of characterizing k-extendable graphs and strongly k-extendable graphs. The first of these problems has been considered by several authors whilst the latter has been investigated only for the case $k = 0$. In this paper, we focus on the problem of characterizing strongly k-extendable graphs for any k. We present a number of properties of strongly k-extendable graphs including some necessary and sufficient conditions for strongly k-extendable graphs.

1. Introduction

All graphs considered in this paper are finite, connected, loopless and have no multiple edges. For the most part our notation and terminology follows that of Bondy and Murty [4]. Thus G is a graph with vertex set V(G), edge set E(G), $\nu(G)$ vertices, $\varepsilon(G)$ edges, minimum degree $\delta(G)$, connectivity $\kappa(G)$ and independence number $\alpha(G)$. For $V' \subseteq V(G)$, $G[V']$ denotes the subgraph induced by V'. Similarly $G[E']$ denotes the subgraph induced by the edge set E' of G. $N_G(u)$ denotes the neighbour set of u in G and $\overline{N}_G(u)$ the non-neighbours of u. Note that $\overline{N}_G(u) = V(G) \setminus (N_G(u) \cup \{u\})$. The *join* $G \vee H$ of disjoint graphs G and H is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H.

A *matching* M in G is a subset of E(G) in which no two edges have a vertex in common. M is a *maximum matching* if $|M| \ge |M'|$ for any other matching M' in G. A vertex v is *saturated* by M if some edge of M is incident to v; otherwise, v is said to be *unsaturated.* A matching M is *perfect* if it saturates every vertex of the graph. For simplicity we let V(M) denote the vertex set of the subgraph G[M] induced by M.

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Let G be a simple connected graph on 2n vertices with a perfect matching. For a given positive integer k, $1 \le k \le n - 1$, G is *k-extendable* if for every matching M of size k in G, there exists a perfect matching in G containing all the edges of M. For convenience, a graph with a perfect matching is said to be 0-extendable. For an integer k, $0 \le k \le n - 2$, we say that G is *strongly k-extendable* or simply *k*-extendable* if for every pair of vertices u and v of G, G - {u, v} is k-extendable. A graph G is *bicritical* if G - {u, v} has a perfect matching for every pair of vertices u and v. Clearly, 0*-extendable graphs are bicritical and a concept of k*-extendable graphs is a generalization of bicritical graphs.

Observe that the complete graph K_{2n} of order 2n is k*-extendable for all k, $0 \le k \le n - 2$ whilst the complete bipartite graph $K_{n,n}$ with bipartition (X, Y) is k-extendable, $0 \le k \le n - 2$, but not k*-extendable since a deletion of any two distinct vertices of X results in a graph $K_{n-2,n}$ which clearly has no perfect matching. In fact, k*-extendable graphs are not bipartite. Further, since a bipartite graph on 2n vertices with minimum degree at least $\frac{1}{2}(n + k)$ is k-extendable (see Ananchuen and Caccetta [3]), it follows that the classes of k*-extendable graphs and k-extendable graphs do not coincide. Moreover, there exists a k-extendable non-bipartite graph on 2n vertices, $0 \le k \le n - 2$, which is not k*-extendable. Such a graph is $G = G' \vee G''$, where $G' = P_3 \cup (n - k - 2)K_2$, P_3 is a path on 3 vertices, and $G'' = K_{2k+1}$ (see Figure 1.1). Note that in our diagrams a "double line" denotes the join. It is not difficult to show that G is

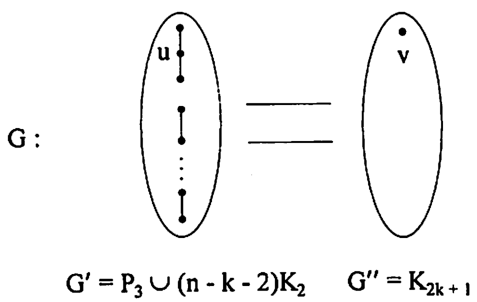

Figure 1.1

k-extendable. Let u be the vertex of P_3 having degree 2 and v any vertex of G″. Consider $G_1 = G - \{u, v\}$. Clearly, G″- v contains a matching M of size k which cannot extend to a perfect matching in G_1, since $G_1 - V(M) = 2K_1 \cup (n - k - 2)K_2$.

A number of authors have studied k-extendable graphs. An excellent survey is the paper of Plummer [9]. Lovasz [5], Lovasz and Plummer [6, 7] and Plummer [8] have studied k*-extendable graphs for k = 0 (bicritical graphs) whilst k*-extendable graphs for $k \ge 1$ have not been previously investigated. In this paper, we focus on the problem of characterizing these graphs. We establish a necessary and sufficient condition for k*-extendable graphs. In fact, we prove that a graph G on 2n vertices is k*-extendable, $0 \le k \le n - 2$, if and only if for all $S \subseteq V(G)$

$$o(G - S) \le \begin{cases} |S| - 2t, & \text{for } |S| \le 2k+1 \\ |S| - 2t - 2, & \text{for } |S| \ge 2k+2 \end{cases}$$

where $t = \min \{ |M(S)|, k\}$ and M(S) denotes a maximum matching in G[S]. We also present a number of sufficient conditions for a graph to be k*-extendable. We establish that a (k+2)-extendable non-bipartite graph on 2n vertices; $0 \le k \le n - 3$, is k*-extendable.

Section 2 contains some preliminary results that we make use of in establishing our results. In Section 3, we establish a number of results on properties of k*-extendable graphs. Some sufficient conditions for k*-extendable graphs are given in Section 4.

2. Preliminaries

In this section we state a number of results on k-extendable graphs which we make use of in our work. We state only results which we use; for a more detailed account we refer to the paper of Plummer [9]. We begin with some fundamental results of k-extendable graphs proved by Plummer [8]:

Theorem 2.1: Let G be a k-extendable graph on 2n vertices, $1 \le k \le n - 1$. Then

(i) G is (k - 1)-extendable;

(ii) G is (k + 1)-connected. ❑

Theorem 2.2: Let G be a graph on 2n vertices and $1 \le k \le n - 1$. If $\delta(G) \ge n + k$, then G is k-extendable. ❑

Denoting the number of odd components in a graph H by o(H) we can now state Tutte's theorem which gives a necessary and sufficient condition of the existence of a perfect matching in a graph.

Theorem 2.3: Tutte's Theorem (see Bondy and Murty [4] p. 76)

A graph G has a perfect matching if and only if

$$o(G - S) \le |S| \quad \text{for all } S \subset V(G).$$ ❑

Our next result concerns a sufficient condition for a graph to be hamiltonian (see Bondy and Murty [4] p. 54).

Theorem 2.4: If G is a simple graph with $v(G) \geq 3$ and $\delta(G) \geq \frac{1}{2}v(G)$, then G is hamiltonian. ❑

We conclude this section by stating two results proved by Ananchuen and Caccetta [1, 2].

Theorem 2.5: Let G be a graph on $2n \geq 4$ vertices. Then G is $(n - 1)$-extendable if and only if G is K_{2n} or $K_{n,n}$. ❑

Lemma 2.6: Let G be a connected graph on 2n vertices with $\delta(G) \geq n - 1$ having a maximum matching M of size $n - 1$. Then for M-unsaturated vertices u and v of G, $N_G(u) = N_G(v)$. Furthermore, no two vertices of $N_G(u)$ are joined by an edge of M, and the vertices of $V(G) \setminus N_G(u)$ form an independent set. ❑

3. Basic properties of k*-extendable graphs

Our first result concerns a necessary condition of k*-extendable graphs.

Lemma 3.1 : If G is a k*-extendable graph on 2n vertices; $1 \leq k \leq n - 2$, then G is $(k - 1)^*$-extendable.

Proof: Let u, v be vertices of G and $G^* = G - \{u, v\}$. Then G^* is k-extendable, by Theorem 2.1, and so $(k - 1)$-extendable. Thus G is $(k - 1)^*$-extendable as required. ❑

A consequence of Lemma 3.1 is the following corollary:

Corollary 3.2: If G is a k*-extendable graph on 2n vertices; $1 \leq k \leq n - 2$, then for $0 \leq t \leq k$, G is t*-extendable. ❑

The next result establishes a relationship between k*-extendable and k-extendable graphs.

Lemma 3.3 : If G is a k*-extendable graph on 2n vertices; $0 \leq k \leq n - 2$, then G is $(k + 1)$-extendable.

Proof: Let M be a matching of size $k + 1$ in G and uv an edge of M. Since G is k*-extendable, $G - \{u, v\}$ has a perfect matching F containing $M - \{uv\}$. Thus $F \cup \{uv\}$ is a perfect matching containing M. This proves our result. ❑

Theorem 2.1 and Lemma 3.3 imply the following corollary.

Corollary 3.4: If G is a k*-extendable graph on 2n vertices; $0 \leq k \leq n - 2$, then G is t-extendable for $0 \leq t \leq k + 1$. ❑

Note that the converse of Lemma 3.3 is not true. The graphs G_1 and G_2 in Figure 3.1 are both (k + 1)-extendable (see Ananchuen and Caccetta [1]) but not k*-extendable since if we delete vertices u and v which are in diagonally opposite K_{k+1}' s (K_k and K_{k+2}) in the graph G_1 (G_2), then the resulting graph is not k-extendable.

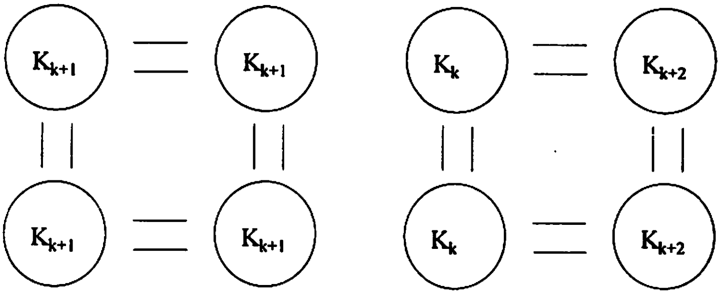

Figure 3.1

We have observed that if G is k*-extendable, then G is not bipartite. The following lemma establishes that G - V(M) is also a non-bipartite graph for every matching M in G of size at most k.

Lemma 3.5: Let G be a k*-extendable graph on 2n vertices, $0 \le k \le n - 2$. If M is a matching of size $t \le k$ in G, then G - V(M) is not a bipartite graph.

Proof : Suppose G′ = G - V(M) is a bipartite graph for some matching M of size $t \le k$ in G. Let (V_1, V_2) be bipartition of G′. Since G is k*-extendable, by Corollary 3.4, G′ has a perfect matching. Thus $|V_1| = |V_2| = n - t \ge n - k \ge 2$. Let x and y be vertices of V_1 and G″ = G - {x, y}. Since G″ - V(M) is a bipartite graph with bipartitioning sets of order $|V_1| - 2$ and $|V_2|$(= $|V_1|$), G″ - V(M) has no perfect matching. Hence, G is not t*-extendable. This contradicts Corollary 3.2 and completes the proof of our lemma. ❑

Our next two theorems yield a necessary and sufficient condition for k*-extendable graphs.

Theorem 3.6: Let G be a graph on 2n vertices. For $0 \le k \le n - 2$, G is k*-extendable if and only if for every matching M in G of size t, $0 \le t \le k$, G - V(M) is (k - t)*-extendable.

Proof : Suppose G is k*-extendable. For a matching M, in G, of size t, $0 \le t \le k$, let G′ = G - V(M). Further, let a, b ∈ V(G′) and consider G″ = G′ - {a, b}. For a matching M″, in G″, of size k - t, M ∪ M″ is a matching, in G - {a, b}, of size t + (k - t) = k. Since G is k*-extendable, there exists a perfect matching F in

$G - \{a, b\}$ containing $M \cup M''$. Thus $F \setminus M$ is a perfect matching, in G'', containing M''. Hence, $G - V(M)$ is $(k - t)^*$-extendable.

Conversely, let x, y be a pair of vertices of G and M_1 a matching of size k in $G - \{x, y\}$. By our hypothesis, $G - V(M_1)$ is 0^*-extendable. Then $G - (V(M_1) \cup \{x, y\})$ contains a perfect matching F_1. Consequently, $F_1 \cup M_1$ is a perfect matching in $G - \{x, y\}$ containing M_1. Hence, G is k^*-extendable. This completes the proof of our theorem. □

Denoting a maximum matching in G[S] by M(S) for any $S \subseteq V(G)$ we can now establish another theorem giving a necessary and sufficient condition for k^*-extendable graphs.

Theorem 3.7: Let G be a graph on 2n vertices. For $0 \le k \le n - 2$, G is k^*-extendable if and only if for all $S \subseteq V(G)$

$$o(G - S) \le \begin{cases} |S| - 2t, & \text{for } |S| \le 2k+1 \\ |S| - 2t - 2, & \text{for } |S| \ge 2k+2 \end{cases}$$

where $t = \min \{|M(S)|, k\}$.

Proof: Suppose G is k^*-extendable. Let $S \subseteq V(G)$ and $t = \min \{|M(S)|, k\}$. If $|S| \le 2k + 1$, $|M(S)| \le k$. Thus $t = |M(S)|$. Since G is k^*-extendable, by Corollary 3.4, $G - V(M(S))$ has a perfect matching. By Theorem 2.3,

$$o(G - S) = o((G - V(M(S))) - (S \setminus V(M(S)))) \le |S \setminus V(M(S))| = |S| - 2t,$$

as required.

Next we consider the case $|S| \ge 2k + 2$. For this case we distinguish two subcases according to $|M(S)|$.

Case 1: $|M(S)| \le k$. Then $t = |M(S)|$. Let $x, y \in S \setminus V(M(S))$ and put

$$G' = G - (V(M(S)) \cup \{x, y\})$$

and

$$S' = S \setminus (V(M(S)) \cup \{x, y\}).$$

Since G is k^*-extendable, Corollary 3.2 implies that G' has a perfect matching. By Theorem 2.3,

$$o(G' - S') \le |S'|.$$

Thus

$$o(G - S) = o(G' - S') \le |S'| = |S| - 2t - 2.$$

Case 2: $|M(S)| \ge k + 1$. Then $t = k$. Let M' be a subset of M(S) with $|M'| = k$ and $x, y \in S \setminus V(M')$. Put

$$G'' = G - (V(M') \cup \{x, y\})$$

and

$$S'' = S \setminus (V(M') \cup \{x, y\}).$$

By the same argument as in the proof of Case 1, we have

$$o(G - S) = o(G'' - S'') \le |S''| = |S| - 2k - 2 = |S| - 2t - 2.$$

This proves sufficiency.

Conversely, suppose that for all $S \subseteq V(G)$

$$o(G - S) \leq \begin{cases} |S| - 2t, & \text{for } |S| \leq 2k+1 \\ |S| - 2t - 2, & \text{for } |S| \geq 2k+2 \end{cases}$$

where $t = \min \{ |M(S)|, k\}$. Let x, y be vertices of G and M a matching of size k in G - {x, y}. Put

$$G' = G - (V(M) \cup \{x, y\}).$$

Let $S' \subseteq V(G')$ and $S = S' \cup (V(M) \cup \{x, y\})$. Clearly,

$$|S| = |S'| + 2k + 2 \geq 2k + 2$$

and

$$o(G' - S') = o(G - S).$$

By our hypothesis, $o(G - S) \leq |S| - 2k - 2 = |S'|$. Thus $o(G' - S') \leq |S'|$. By Theorem 2.3, G′ has a perfect matching. This proves that G is k*-extendable and completes the proof of our theorem. ❑

Theorem 3.7 implies a following corollary which was also proved by Lovasz [5].

Corollary 3.8: Let G be a graph on 2n vertices. Then G is bicritical if and only if for every $S \subseteq V(G)$, $|S| \geq 2$, G - S has at most $|S| - 2$ odd components. ❑

4. Some sufficient conditions for k*-extendable graphs

In this section we establish a number of sufficient conditions for a graph to be k*-extendable. We start with a following result:

Lemma 4.1: Let G be a graph on 2n vertices and $0 \leq k \leq n - 2$. If $\delta(G) \geq n + k + 1$, then G is k*-extendable. Further, the bound is sharp.

Proof: Let u and v be vertices of G and G′ = G - {u, v}. Since $\delta(G) \geq n + k + 1$, $\delta(G') \geq (n + k + 1) - 2 = (n - 1) + k$. By Theorem 2.2, G′ is k-extendable. Hence, G is k*-extendable as required.

To see that the bound is sharp, let $G_1 = K_{n+k}$, $G_2 = \overline{K}_{n-k}$ and $G = G_1 \vee G_2$. Clearly, $\delta(G) = n + k$. Let x and y be vertices of G_1 and M a matching of size k in G_1 - {x, y}. But then M does not extend to a perfect matching in G - {x, y} since $G - (V(M) \cup \{x, y\}) = K_{n-k-2} \vee \overline{K}_{n-k}$. Thus G is not k*-extendable. ❑

Remark 4.1: There exists a graph on 2n vertices with minimum degree n + k + 1, $0 \leq k \leq n - 2$. Such a graph is $K_1 \vee K_{n+k+1} \vee K_{n-k-2}$ which is k*-extendable by Lemma 4.1.

As a corollary we have:

Corollary 4.2: Let G be a graph on $2n \geq 4$ vertices. If $\delta(G) \geq n + 1$, then G is bicritical. ❑

Theorem 4.3: Let G be a $(k + 1)$-extendable non-bipartite graph on 2n vertices; $0 \leq k \leq n - 2$, with $\delta(G) = n + k$. If $n - k - 1$ is even or $\kappa(G) \geq 2k + 3$, then G is k^*-extendable.

Proof: The case $k = n - 2$ follows directly from Theorem 2.5. So we only need to prove the remaining case $0 \leq k \leq n - 3$.

Let u, v be vertices of G and M a matching of size k in $G - \{u, v\}$. Put $G' = G - (\{u, v\} \cup V(M))$. We need to show that G' contains a perfect matching. First we assume that $\kappa(G) \geq 2k + 3$. Then G' is connected. Suppose G' has no perfect matching. Clearly $uv \notin E(G)$. Further, since $v(G') = 2n - 2k - 2$ it follows from Theorem 2.4 that $\delta(G') = n - k - 2$.

Let M' be a maximum matching in G'. Then $|M'| \leq n - k - 2$. If $|M'| \leq n - k - 3$, then M cannot extend to a perfect matching in G since $G - V(M)$ contains at least 2 independent vertices, a contradiction. Thus $|M'| = n - k - 2$. Let x and y be the M'-unsaturated vertices of G'. Since $v(G') = 2n - 2k - 2$ and $\delta(G') = n - k - 2$, it follows from Lemma 2.6 that $N_{G'}(x) = N_{G'}(y)$. Further, no two vertices of $N_{G'}(x)$ are joined by an edge of M' and $A = V(G') \setminus N_{G'}(x)$ is an independent set. Consequently, $|N_{G'}(x)| = n - k - 2$ and $|A| = n - k$.

Let $x' \in N_{G'}(x)$. If $ux' \in E(G)$, then $M_1 = M \cup \{ux'\}$ is a matching of size $k + 1$ in G which does not extend to a perfect matching since $G - V(M_1)$ contains A as an independent set of order $n - k$ and $v(G - V(M_1)) = 2n - 2k - 2$. Hence, $ux' \notin E(G)$ for all $x' \in N_{G'}(x)$. Similarly, $vx' \notin E(G)$ for all $x' \in N_{G'}(x)$.

Suppose $1 \leq k \leq n - 3$. Since $\delta(G) = n + k$, , there exists an edge ab of M such that $ua, vb \in E(G)$. But then $M_2 = (M \setminus \{ab\}) \cup \{ua, vb\}$ is a matching of size $k + 1$ which does not extend to a perfect matching in G since $G - V(M_2) = G'$, a contradiction. Hence, $k = 0$. If $N_{G'}(x)$ is an independent set , then G is a bipartite graph with bipartitioning sets A and $N_{G'}(x) \cup \{u, v\}$, contradicting the hypothesis of our theorem. Thus there exists an edge x_1x_2 of G with $x_1, x_2 \in N_{G'}(x)$. But then $\{x_1x_2\}$ does not extend to a perfect matching in $G - \{x_1, x_2\}$ since $G - \{x_1, x_2\}$ contains A as an independent set of order $n - k = n$ and $v(G - \{x_1, x_2\}) = 2n - 2$, contradicting the extendability of G. This proves that G' has a perfect matching.

Next we suppose that $n - k - 1$ is even. If G' is connected, then by applying a similar argument as above, G' has a perfect matching. Hence we may assume that G' is disconnected. Since $v(G') = 2n - 2k - 2$ and $\delta(G') \geq n - k - 2$, G' contains exactly 2 components, H_1 and H_2 say. Further, $v((H_1) = v((H_2) =$

$n - k - 1$. Then H_1 and H_2 are complete. Consequently, G' has a perfect matching since $n - k - 1$ is even. This completes the proof of our theorem. ❑

Theorem 4.3 is best possible in the sense that there exists a $(k + 1)$-extendable non-bipartite graph G on 2n vertices with $\delta(G) = n + k$ and $\kappa(G) = 2k + 2$ but G is not k^*-extendable when $n - k - 1$ is odd. Let $G = (\overline{K}_k \vee \overline{K}_{k+2}) \vee 2K_{n-k-1}$ (see Figure 4.1). For $n \geq 2k + 3$ it is not difficult to verify that G is

G :

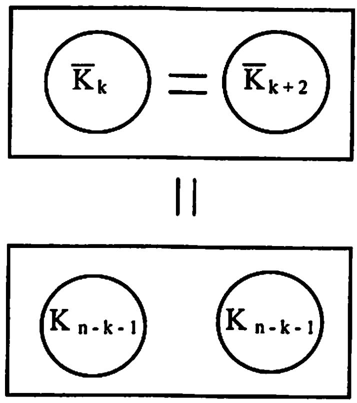

Figure 4.1

$(k + 1)$-extendable with $\delta(G) = n + k$ and $\kappa(G) = 2k + 2$. But G is not k^*-extendable when $n - k - 1$ is odd, since $G - (V(\overline{K}_k) \cup V(\overline{K}_{k+2})) = 2K_{n-k-1}$ has no perfect matching where $G[V(\overline{K}_k) \cup V(\overline{K}_{k+2})]$ contains a pair of vertices u and v and a matching M of size k for which $V(M) \cup \{u, v\} = (V(\overline{K}_k) \cup V(\overline{K}_{k+2}))$.

Theorem 4.4: Let G be a graph on 2n vertices with $\delta(G) = n + k$; $0 \leq k \leq n - 2$. If $n - k - 1$ is even and $\alpha(G) \leq n - k - 1$, then G is k^*-extendable.

Proof: Let u and v be vertices of G and M a matching of size k in $G - \{u, v\}$. Put $G' = G - (\{u, v\} \cup V(M))$. Suppose G' is disconnected. Since $\delta(G') \geq n + k - (2k + 2) = n - k - 2$ and $\nu(G') = 2n - 2k - 2$, $G' = 2K_{n-k-1}$. Clearly, G' contains a perfect matching since $n - k - 1$ is even. Next we suppose that G' is connected and has no perfect matching. Let M' be a maximum matching in G'. By a similar argument as that in the proof of Theorem 4.3, there are exactly two M' - unsaturated vertices of G', x and y say. Further, $V(G') \setminus N_{G'}(x)$ is an independent set of order $n - k$. This contradicts the hypothesis that $\alpha(G) \leq n - k - 1$. Thus G' has a perfect matching. This proves that G is k^*-extendable and completes the proof of our theorem. ❑

The condition in Theorem 4.4 is best possible in the sense that there exists a graph G on 2n vertices with minimum degree n + k; $0 \le k \le n - 2$, which is not k*-extendable when n - k - 1 is odd or $\alpha(G) \ge n - k$. Such graphs are $K_{2k+2} \vee 2K_{n-k-1}$ and $K_{2k} \vee (\overline{K}_{n-k} \vee \overline{K}_{n-k})$. Clearly, $K_{2k+2} \vee 2K_{n-k-1}$ is not k*-extendable if n - k - 1 is odd since deleting vertices u and v of K_{2k+2} and a matching of size k in $K_{2k+2} - \{u, v\}$ results in the graph $2K_{n-k-1}$. Further, the graph $K_{2k} \vee (\overline{K}_{n-k} \vee \overline{K}_{n-k})$ which contains an independent set of vertices of order n - k is not k*-extendable since deleting vertices x and y of one of $\overline{K}_{n-k}$'s and a matching of size k in K_{2k} results in a graph $\overline{K}_{n-k-2} \vee \overline{K}_{n-k}$.

We need the following lemmas in establishing our main result in this section.

Lemma 4.5: Suppose G is a (k + 1)-extendable graph on 2n vertices; $0 \le k \le n - 2$ and M is a matching of size $t \le k$ in G. For every non-empty even set $A \subseteq V(G) \setminus V(M)$ with $|A| < 2(n - k)$ there exists an edge e joining a vertex of A to a vertex of $V(G) \setminus (V(M) \cup A)$.

Proof: Suppose to the contrary that there exists a non-empty even set $A \subseteq V(G) \setminus V(M)$ with $|A| < 2(n - k)$ which vertices of A and $B = V(G) \setminus (V(M) \cup A)$ are not adjacent. Since G is (k + 1)-extendable, by Theorem 2.1, G is (k + 2)-connected. So there are at least k + 2 vertices of V(M) which are adjacent to vertices of A. Similarly, there are at least k + 2 vertices of V(M) which are adjacent to vertices of B. Since $|V(M)| = 2t \le 2k$, there must be an edge of M, x_1y_1 say, such that $xx_1, yy_1 \in E(G)$ with $x \in A$ and $y \in B$. Then $(M \setminus \{x_1y_1\}) \cup \{xx_1, yy_1\}$ is a matching of size $t + 1 \le k + 1$ in G which does not extend to a perfect matching in G since $A \setminus \{x\}$ becomes an isolated odd component in $G - (V(M) \cup \{x, y\})$. This contradicts the (k + 1)-extendability of G and completes the proof of our lemma. ❑

Lemma 4.6: Suppose G is a (k + 1)-extendable graph on 2n vertices; $0 \le k \le n - 2$. Let u and v be vertices of G and M a matching of size k in $G - \{u, v\}$. If $S \subseteq V(G_1)$ where $G_1 = G - (V(M) \cup \{u, v\})$ with $o(G_1 - S) \ge |S| + 2$, then $G[V(M) \cup S \cup \{u, v\}]$ contains a maximum matching of size exactly k. Further, $S \cup \{u, v\}$ is an independent set.

Proof: Clearly, $G_2 = G[V(M) \cup S \cup \{u, v\}]$ contains M as a matching of size k. Suppose M_1 is a matching of size k + 1 in G_2. Let

$$S_1 = V(G_2) \setminus V(M_1).$$

Then

$$\begin{aligned}|S_1| &= |V(G_2)| - |V(M_1)| \\ &= (2k + |S| + 2) - (2k + 2) \\ &= |S|.\end{aligned}$$

Since $o((G - V(M_1)) - S_1) = o(G_1 - S) \ge |S| + 2 > |S_1|$, M_1 does not extend to a perfect matching in G, contradicting the (k + 1) - extendablity of G. Thus

$G[V(M) \cup S \cup \{u, v\}]$ contains a maximum matching of size exactly k and hence $S \cup \{u, v\}$ is an independent set, completing the proof of our lemma. ❑

Lemma 4.7: Let G be a (k + 2)-extendable graph on 2n vertices; $0 \leq k \leq n - 3$. Suppose $G_1 = G - (V(M) \cup \{u, v\})$ has no perfect matching for some vertices u and v of G and a matching M of size k in $G - \{u, v\}$. Then there exists a set $S \subseteq V(G_1)$ such that

(i) $o(G_1 - S) = |S| + 2$ and $G_1 - S$ has no even components, and

(ii) each odd component of $G_1 - S$ is a singleton set.

Proof: Since G_1 has no perfect matching, there exists, by Theorem 2.3, a set $S \subseteq V(G_1)$ such that $o(G_1 - S) > |S|$. Because $v(G_1)$ is even, $o(G_1 - S)$ and $|S|$ have the same parity. So $o(G_1 - S) \geq |S| + 2$. Since $o((G - V(M)) - (S \cup \{u, v\})) = o(G_1 - S)$, if $o(G_1 - S) > |S| + 2 = |S \cup \{u, v\}|$, then $G - V(M)$ has no perfect matching. This implies that M does not extend to a perfect matching in G, contradicting the (k + 2) - extendability of G. Hence, $o(G_1 - S) = |S| + 2$.

Next we will show that $G_1 - S$ has no even components. Suppose to the contrary that H is an even component of $G_1 - S$. Further, let $S' = V(G) \setminus (V(M) \cup V(H))$. By Lemma 4.5, there exists an edge e = xy of G joining a vertex x of H to a vertex y of S′. Then $y \in S \cup \{u, v\}$. But then $M \cup \{e\}$ does not extend to a perfect matching in G since the odd components of $G_1 - S$ together with H - x form at least $|S| + 3$ odd components of $(G - (V(M) \cup \{x, y\})) - ((S \cup \{u, v\}) \setminus \{y\})$ and $|(S \cup \{u, v\}) \setminus \{y\}| = |S| + 1$. This contradicts the fact that G is (k + 2) - extendable. Hence, $G_1 - S$ has no even components. This proves (i).

Now we establish (ii). Suppose to the contrary that $G_1 - S$ contains H_0 as an odd component with $v(H_0) \geq 3$. Consider $E_1 = \{ab \in E(G) \mid a \in S \cup \{u, v\};\ b \in V(H_0)\}$.

Suppose e_1 and e_2 are independent edges of E_1. Then $M_2 = M \cup \{e_1, e_2\}$ is a matching of size k + 2. But then M_2 does not extend to a perfect matching in G since $v(H_0) \geq 3$ and

$$\begin{aligned} |S| + 2 = o(G_1 - S) &= o((G - V(M_2)) - ((S \cup \{u, v\}) \setminus V(M_2))) \\ &> |S| \\ &= |(S \cup \{u, v\}) \setminus V(M_2)|. \end{aligned}$$

This contradicts the fact that G is (k + 2) - extendable. Hence, $G_3 = G[E_1] \cong K_{1,s}$ for some integer $s \geq 1$.

Let (V_1, V_2) be bipartition of $K_{1,s}$ where $V_1 = \{w\}$. Then $w \in V(H_0)$ or $w \in S \cup \{u, v\}$. Suppose $w \in V(H_0)$. Figure 4.2 illustrates the situation with the edges of M drawn in solid lines.

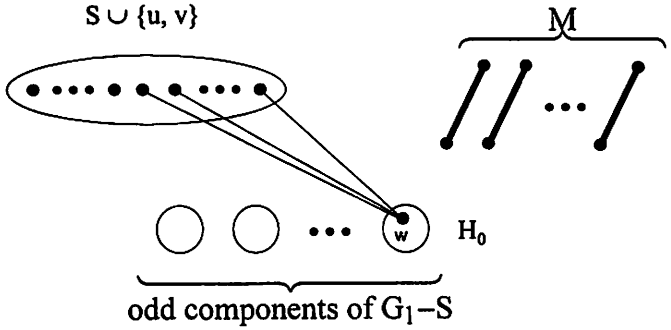

Figure 4.2

Since $v(H_0) \geq 3$, there exists a vertex w′ of H_0 such that $ww' \in E(G)$. Let $M_3 = M \cup \{ww'\}$. Clearly, $|M_3| = k + 1$ and $H_0 - V(M_3)$ becomes an isolated odd component in $G - V(M_3)$. Thus M_3 does not extend to a perfect matching in G, a contradiction to the $(k + 2)$ - extendability of G. Hence, $w \notin V(H_0)$. Consequently, $w \in S \cup \{u, v\}$. Figure 4.3 illustrates the situation.

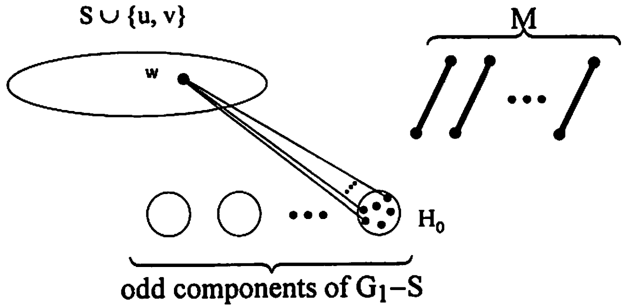

Figure 4.3

We will show that w is not adjacent to any vertex of $V(G) \setminus (V(M) \cup V(H_0))$. Suppose there exists a vertex $w_1 \in V(G) \setminus (V(M) \cup V(H_0))$ such that $ww_1 \in E(G)$. Let $M_4 = M \cup \{ww_1\}$. Clearly, $|M_4| = k + 1$. Since there is no edge joining a vertex of $(S \cup \{u, v\}) \setminus \{w\}$ to a vertex of $V(H_0)$ and $v(H_0)$ is odd, M_4 does not extend to a perfect matching in G, a contradiction. Hence, w is not adjacent to any vertex of $V(G) \setminus (V(M) \cup V(H_0))$. Let

$$A = V(H_0) \cup \{w\}$$

and

$$B = V(G) \setminus (V(M) \cup A).$$

Figure 4.4 depicts the situation.

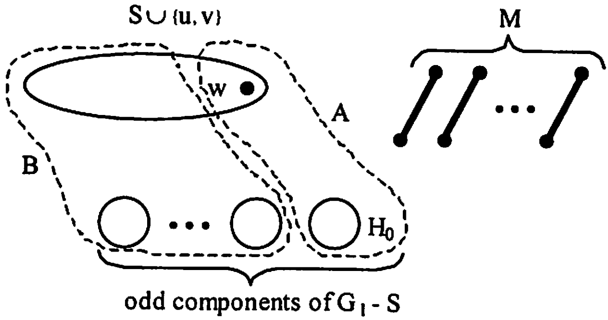

Figure 4.4

Clearly, A ⊆ V(G) \ V(M) which |A| is even and there is no edge joining a vertex of A to a vertex of B, contradicting Lemma 4.5. This proves (ii) and completes the proof of our lemma. ❑

Now we are ready to prove our main result.

Theorem 4.8: If G is a (k + 2)-extendable non-bipartite graph on 2n vertices; $0 \leq k \leq n - 3$, then G is k*-extendable.

Proof: Suppose to the contrary that there exist vertices u and v of G and a matching M of size k in G - {u, v} which does not extend to a perfect matching in G - {u, v}. Let G_1 = G - (V(M) ∪ {u, v}). Since G_1 has no perfect matching, by Lemma 4.7, there exists a set S ⊆ V(G_1) such that G_1 - S contains exactly |S| + 2 odd components, all of them are singletons. Let C be a set of vertices of these components. Clearly, C is an independent set and |C| = |S| + 2. Further, V(G) = V(M) ∪ {u, v} ∪ S ∪ C. Note that, by Lemma 4.6, G[V(M) ∪ S ∪ {u, v}] contains a maximum matching of size exactly k and S ∪ {u, v} is independent. This implies:

Claim 1: For every vertex w of S ∪ {u, v}, if wx ∈ E(G) where xy is an edge of M, then zy ∉ E(G) for every z ∈ (S ∪ {u, v}) \ {w}.

We now establish a number of further claims.

Claim 2: G[V(M) ∪ C] contains a maximum matching of size exactly k. This claim follows immediately from the fact that V(G) = V(M) ∪ {u, v} ∪ S ∪ C, S ∪ {u, v} and C are independent and |C| = |S| + 2.

Claim 3 : Every vertex w of S ∪ {u, v} is adjacent to at most one end vertex of an edge e of M.

Suppose to the contrary that there exist a vertex x' of $S \cup \{u, v\}$ and an edge $e = xy$ of M such that $x'x, x'y \in E(G)$. By Claim 1, $xy', yy' \notin E(G)$ for all $y' \in (S \cup \{u, v\}) \setminus \{x'\}$. Let $M_1 = (M \setminus \{xy\}) \cup \{x'x\}$. Since G is $(k + 2)$ - extendable, there is a perfect matching F containing the edges of M_1. Let $yz \in F$. Clearly, z is a vertex of C. Similarly, there exists a perfect matching F_1 containing the edges of $(M \setminus \{xy\}) \cup \{x'y\}$ and $xz_1 \in F_1$ where $z_1 \in C$. Then $z = z_1$; otherwise, $(M \setminus \{xy\}) \cup \{xz_1, yz\}$ becomes a matching of size $k + 1$ in $G[V(M) \cup C]$, a contradiction to Claim 2. By Claim 2, $xc, yc \notin E(G)$ for all $c \in C \setminus \{z\}$. Further, by similar argument to the one used in the proof of Lemma 4.6, $G[V(M_1) \cup C]$ contains a maximum matching of size exactly k. Thus $x'c \notin E(G)$ for all $c \in C \setminus \{z\}$. Let

$$A_1 = \{x, y, z, x'\}$$

and

$$B_1 = V(G) \setminus (V(M \setminus \{xy\}) \cup A_1).$$

By Lemma 4.5, there is an edge $e = wb$ joining a vertex w of A_1 to a vertex b of B_1. This implies that $w = z$. Then y becomes an isolated vertex of $G - V((M \setminus \{xy\}) \cup \{zb, xx'\})$ since $yy' \notin E(G)$ for all $y' \in (S \cup \{u, v\}) \setminus \{x'\}$ and $yc \notin E(G)$ for all $c \in C \setminus \{z\}$. This implies that $(M \setminus \{xy\}) \cup \{zb, xx'\}$ does not extend to a perfect matching in G, contradicting the $(k + 2)$ - extendability of G. This proves Claim 3.

The above argument can be used to prove:

Claim 4: Every vertex c of C is adjacent to at most one end vertex of an edge e of M.

Claim 5 : If $wx \in E(G)$ for some $w \in S \cup \{u, v\}$ and $xy \in M$, then $xc \notin E(G)$ for all $c \in C$.

Suppose to the contrary that there exist vertices $w_1 \in S \cup \{u, v\}$, $c_1 \in C$ and edge $x_1y_1 \in M$ such that $w_1x_1, x_1c_1 \in E(G)$. Let F_2 be a perfect matching containing the edges of $(M \setminus \{x_1y_1\}) \cup \{w_1x_1\}$. Then $y_1z \in F_2$. Since $G[V(M) \cup S \cup \{u, v\}]$ contains a maximum matching of size exactly k, $z \notin (S \cup \{u, v\} \setminus \{w_1\}$. Then $z \in C$. Since $x_1c_1 \in E(G)$ and c_1 is adjacent to at most one end vertex of an edge of M, $z \neq c_1$. Consequently, $(M \setminus \{x_1y_1\}) \cup \{x_1c_1, y_1z\}$ is a matching of size $k + 1$ in $G[V(M) \cup C]$, contradicting Claim 2. This proves Claim 5.

Claim 6 : For every edge $xy \in M$, if $xw \notin E(G)$ for all $w \in S \cup \{u, v\}$, then $yc \notin E(G)$ for all $c \in C$.

Suppose to the contrary that there exist edge $x_2y_2 \in M$ and a vertex $c_2 \in C$ such that $x_2w \notin E(G)$ for all $w \in S \cup \{u, v\}$ but $y_2c_2 \in E(G)$. Consider $M_2 = (M \setminus \{x_2y_2\}) \cup \{y_2c_2\}$. Clearly, $|M_2| = k$. Since $x_2w \notin E(G)$ for all $w \in S \cup \{u, v\}$, the set $S \cup \{u, v, x_2\}$ is independent. Because $G - V(M_2)$ contains $S \cup \{u, v, x_2\}$

and $C - \{c_2\}$ as independent sets of order $|S| + 3$ and $|S| + 1$ respectively, $G - V(M_2)$ does not have a perfect matching. Thus M_2 does not extend to a perfect matching in G. This contradicts the $(k + 2)$-extendability of G and completes the proof of Claim 6.

Now let $M = \{x_1y_1, x_2y_2, \dots, x_ky_k\}$. Consider x_1y_1. If $x_1w \notin E(G)$ for all $w \in S \cup \{u, v\}$, then, by Claim 6, $y_1c \notin E(G)$ for all $c \in C$. Put

$$X_1 = S \cup \{u, v\} \cup \{x_1\}$$

and

$$Y_1 = C \cup \{y_1\}.$$

If $x_1w_1 \in E(G)$ for some $w_1 \in S \cup \{u, v\}$, then, by Claim 5, $x_1c \notin E(G)$ for all $c \in C$. Further, by Lemma 4.6 and Claim 3, $y_1w \notin E(G)$ for all $w \in S \cup \{u, v\}$. Put

$$X_1 = S \cup \{u, v\} \cup \{y_1\}$$

and

$$Y_1 = C \cup \{x_1\}.$$

For each edge $x_iy_i \in M$; $2 \le i \le k$, we can construct sets X_i and Y_i in a similar fashion as we do with the edge x_1y_1. Until the step k, we have

$$X_k = S \cup \{u, v\} \cup \{a_1, a_2, \dots, a_k\}$$

and

$$Y_k = C \cup \{b_1, b_2, \dots, b_k\}$$

where a_i and b_i $(1 \le i \le k)$ are end vertices of edge $a_ib_i = x_iy_i$ of M. Clearly $|X_k| = |Y_k| = |S| + k + 2$. Further, by our construction, there is no edge joining a vertex of $S \cup \{u, v\}$ to a vertex of $\{a_1, a_2, \dots, a_k\}$ and a vertex of C to a vertex of $\{b_1, b_2, \dots, b_k\}$.

Since $S \cup \{u, v\}$ and C are independent sets, to show that (X_k, Y_k) is a bipartition of G it is sufficient to prove that $\{a_1, a_2, \dots, a_k\}$ and $\{b_1, b_2, \dots, b_k\}$ are independent sets. Suppose to the contrary that $\{a_1, a_2, \dots, a_k\}$ is not independent. Without any loss of generality, we may assume that $a_1a_2 \in E(G)$. If $b_1w_1 \in E(G)$ for some $w_1 \in S \cup \{u, v\}$, then $M_3 = \{a_1a_2, b_1w_1\} \cup \{a_ib_i \mid 3 \le i \le k\}$ is a matching of size $(k - 2) + 2 = k$ in G which does not extend to a perfect matching in G since $G - V(M_3)$ contains $(S \cup \{u, v\}) \setminus \{w_1\}$ and $C \cup \{b_2\}$ as independent sets of order $|S| + 1$ and $|S| + 3$ respectively. Thus $b_1w \notin E(G)$ for all $w \in S \cup \{u, v\}$. Similarly, $b_2w \notin E(G)$ for all $w \in S \cup \{u, v\}$. Let

$$A_2 = \{b_1, b_2\}$$

and

$$B_2 = C \cup S \cup \{u, v\}.$$

Figure 4.5 depicts the situation with the edges of M drawn in solid lines.

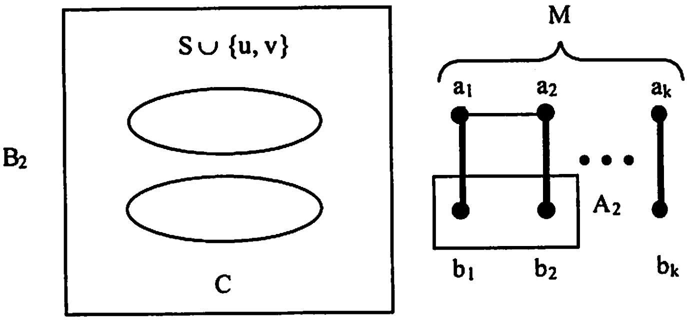

Figure 4.5

Let $M_4 = (M \setminus \{a_1b_1, a_2b_2\}) \cup \{a_1a_2\}$. Clearly, $|M_4| = k - 1$. Notice that $A_2 \subseteq V(G) \setminus V(M_4)$ and $B_2 = V(G) \setminus (V(M_4) \cup A_2)$. By lemma 4.5, there is an edge e joining a vertex of A_2 to a vertex of B_2 which is impossible since b_1 and b_2 are not adjacent to any vertex of $C \cup S \cup \{u, v\}$. This contradiction proves that $\{a_1, a_2, \ldots, a_k\}$ is an independent set. Similarly, $\{b_1, b_2, \ldots, b_k\}$ is an independent set. Hence, G is a bipartite graph with bipartition (X_k, Y_k). This contradicts the hypothesis of our theorem and completes the proof. ❑

An immediate consequence of Theorem 4.8 is the following result of Plummer [8].

Corollary 4.9: If G is a 2-extendable non-bipartite graph on $2n \geq 6$ vertices, then G is bicritical. ❑

A converse of Theorem 4.8 is not true. For integers n, k; $0 \leq k \leq n - 3$, let $G_1 = K_{n+k+1}$, $G_2 = \overline{K}_{n-k-1}$. Clearly, $G = G_1 \vee G_2$ is a graph on 2n vertices with minimum degree n + k + 1. By Lemma 4.1, G is k*-extendable. Let M be a matching of size k + 2 in G_1. Then $G - V(M) = K_{n-k-3} \vee \overline{K}_{n-k-1}$ has no perfect matching. Thus G is not (k + 2)-extendable.

For $1 \leq k \leq n - 1$, let $\mathcal{G}(2n, k)$ denote the class of k-extendable non-bipartite graphs on 2n vertices. Further, for $0 \leq k \leq n - 2$, let $\mathcal{G}^*(2n, k^*)$ denote the class of k*-extendable graphs on 2n vertices. Then Lemma 3.3, Theorems 2.1 and 4.8 imply that these classes are "nested" as follows :

$$\mathcal{G}(2n, 1) \supset \mathcal{G}^*(2n, 0^*) \supset \mathcal{G}(2n, 2) \supset \mathcal{G}^*(2n, 1^*) \supset \ldots \supset \mathcal{G}(2n, n-2) \supset \mathcal{G}^*(2n, (n-3)^*) \supset \mathcal{G}(2n, n-1).$$

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