A New Lemma in Ramsey Theory*

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Abstract

The third author proved earlier [8] that if a Euclidean space is colored with red and blue so that the distance one is forbidden for blue, and translates of some k-point configuration are forbidden for red, then the unit-distance chromatic number of the space is no greater than k. Here we give a generalization.

1 Introduction

We shall freely employ the jargon wherein "...the set S is forbidden for the color C..." means that not all points of S are colored C, which simply means that $S \nsubseteq C$. Of course, it is rare to speak of forbidding a single set; normally one says "the sets...[with a certain property]...are forbidden for C," meaning that each one of them is forbidden for C. A special case of this is "the distance s is forbidden for C," in which the forbidden sets are the two-element sets consisting of pairs of points a distance s apart.

There have been brave attempts at unified approaches to Euclidean Ramsey problems [2], but, in practice, it seems to have been just too hard to work on any very general level in this area. Here are two particular kinds of Euclidean Ramsey problems that have attracted attention, separately, over the past forty years.

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(1) Given $S \subseteq \mathbb{R}^n$, determine $\chi(S) = \chi(S, 1)$, the smallest number of colors needed to color S so that the distance 1 is forbidden for all colors. [In other words, $\chi(S)$ is the chromatic number of the "unit-distance graph on S", the graph with set of vertices S and two vertices adjacent when and only when they are a distance 1 apart. Of course, there is nothing sacred about the distance 1, nor about $\mathbb{R}^n$—see [1], for instance—but let us not digress into maddening generality.]

As anyone the least bit familiar with Euclidean Ramsey problems is aware, there has been a massive amount of work on this kind of problem over the nearly 50 years since Nelson posed the problem of finding $\chi(\mathbb{R}^2)$ (see [7] for some history). Of the great many often spectacular results that have been achieved in this area, we will need to refer here only to the following.

(i) $4 \leq \chi(\mathbb{R}^2) \leq 7$; this is due to Hadwiger, Isbell, and Nelson—see [7] for a clarification.

(ii) If $n \geq 2, \chi(\mathbb{R}^n) \geq n+2$; see [6] and [9]. (This result has been eclipsed in the minds of many by the lower bound of Frankl and Wilson [3], $\chi(\mathbb{R}^n) \geq (1+o(1))(1.2)^n$, but for small $n \geq 2, n+2$ is still king of the lower bounds for $\chi(\mathbb{R}^n)$.)

(iii) $\chi(Q^4) = 4$; see [1], [4], and [10].

(2) Given $S \subseteq \mathbb{R}^n$, consider two-colorings of S, say with red and blue, such that the distance one is forbidden for blue. What can be forbidden for red, in such a coloring?

For short, let us refer to such a two-coloring (in which the distance one is forbidden for blue) as a *rather red* coloring.

There is one great result on rather red colorings, due to R. Juhász [5]: *in any rather red coloring of the plane, the red set contains a congruent copy of each four-point set in the plane.*

In the same paper, Juhász gives an example of a twelve-point set in the plane, and a rather red coloring that forbids congruent copies of the set for red. And there the affair sits.

One could ask, what about forbidding *translates* of four-point sets for red, in a rather red coloring of the plane? While looking into this and some related questions, the third author discovered the following relation between problem types (1) and (2), stated here in slightly greater generality than in [8], but with the same proof.

Theorem A *Suppose that $S \subseteq \mathbb{R}^n$ is closed under addition, and there is a rather red coloring of S which forbids all translates of some k-element subset of S for red. Then $\chi(S) \leq k$.*

It follows, in view of results quoted above, that

(i) in every rather red coloring of $\mathbb{R}^2$, the red set contains a translate of each three-point set in the plane;

(ii) more generally, in every rather red coloring of $\mathbb{R}^n$, for each $n \geq 2$, the red set contains a translate of each $(n+1)$-point set in $\mathbb{R}^n$; and

(iii) in every rather red coloring of Q^4, the red set contains a translate of each 3-point set in Q^4.

So far, applications of Theorem A have all been of this form, drawing conclusions about rather red colorings from knowledge about chromatic numbers. But deductions the other way provide a way of attacking chromatic number problems. For instance, if a rather red coloring of $\mathbb{R}^2$ were found which forbids translates of some six-point planar set for red, it would follow that $\chi(\mathbb{R}^2) \leq 6$.

In the next section we give a generalization of Theorem A that we hope will find application in Ramsey problems outside the realm of Euclidean geometry. The result has to do with two-colorings, and chromatic numbers, but we expunge all references to distance and translation, in the usual sense.

2 The Results

A *hypergraph* is a pair $\mathcal{H} = (V, \mathcal{E})$, in which V (the set of *vertices*) is a non-empty set, and $\mathcal{E}$ (the set of *hyperedges*) is a collection of non-empty subsets of V. If $\mathcal{E}$ contains no singletons, the *chromatic number* of $\mathcal{H}$ (sometimes called the weak chromatic number of $\mathcal{H}$), denoted $\chi(\mathcal{H})$, is the smallest number of colors necessary to color V so that no hyperedge is monochromatic; i.e., so that no $E \in \mathcal{E}$ is contained in any color (or color class, as some say).

Lemma *Suppose that U and V are non-empty sets, and R, B partition $U \times V$. Let $\mathcal{E}_U = \{S \subseteq U;$ for each $v \in V, (S \times \{v\}) \cap R \neq \emptyset\}$ and $\mathcal{E}_V = \{S \subseteq V;$ for each $u \in U, (\{u\} \times S) \cap B \neq \emptyset\}$; let $\mathcal{H}_U = (U, \mathcal{E}_U)$ and $\mathcal{H}_V = (V, \mathcal{E}_V)$. Then either $\mathcal{E}_V = \emptyset$ or $\chi(\mathcal{H}_U) \leq \min_{E \in \mathcal{E}_V} |E|$.*

Proof. Supposing $\mathcal{E}_V \neq \emptyset$, let $E \in \mathcal{E}_V$ be a hyperedge of minimum cardinality. We will color U with the elements of E so that no hyperedge of $\mathcal{H}_U$ is monochromatic, thus demonstrating the desired inequality.

For each $u \in U, (\{u\} \times E) \cap B \neq \emptyset$; color u with some $v \in E$ such that $(u, v) \in B$. [If E is uncountable, this step will require the Axiom of Choice.] U is now colored by the elements of E.

If $F \in \mathcal{E}_U$, and if every element of F is colored with the same $v \in E$, then $F \times \{v\} \subseteq B$, contradicting $F \in \mathcal{E}_U$. Thus the coloring is proper, i.e., no $F \in \mathcal{E}_U$ is monochromatic. ∎

Remarks $\mathcal{E}_V = \emptyset$ if and only if $\mathcal{E}_U$ contains a singleton, i.e., if and only if $\{u\} \times V \subseteq R$, for some $u \in U$.

To make better sense of the theorem, for $S \subseteq V, u \in U$, let us call $\{u\} \times S$ the *u-copy of* S (in $U \times V$) and for $S \subseteq U, v \in V$, let us call $S \times \{v\}$ the *v-copy of* S (in $U \times V$). Let us consider the partition of $U \times V$ into R and B to be a two-coloring of $U \times V$ with red and blue. Then $\mathcal{H}_U$ is the hypergraph on U whose hyperedges are those subsets of U whose V-copies are forbidden for blue, and $\mathcal{H}_V$ is the hypergraph on V whose hyperedges are those subsets of V whose U-copies are forbidden for red. The conclusion of the theorem is that, unless some U-copy of V is all red, the chromatic number of $\mathcal{H}_U$ is no greater than the minimum hyperedge cardinality in $\mathcal{H}_V$ (and, of course, a dual statement holds with the roles of U and V and of red and blue reversed).

The proof is so tautologically simple that it might be suspected that the whole business is quite trivial. Perhaps it is, but the following Corollary shows that the theorem may have its uses, since Theorem A does, and Theorem A is a corollary of the Corollary, taking $V = S$ and $*$ to be vector addition in $\mathbb{R}^n$.

Corollary *Suppose that V is a non-empty set, $*$ is a binary operation on V, and R, B partition V. Let $\mathcal{H}_1 = (V, \mathcal{E}_1)$ and $\mathcal{H}_2 = (V, \mathcal{E}_2)$ be defined by $\mathcal{E}_1 = \{E \subseteq V;$ for each $v \in V, (E * v) \cap R \neq \emptyset\}$ and $\mathcal{E}_2 = \{E \subseteq V;$ for each $u \in V, (u * E) \cap B \neq \emptyset\}$. Then either $\mathcal{E}_2 = \emptyset$ or $\chi(\mathcal{H}_1) \leq \min_{E \in \mathcal{E}_2} |E|$.*

Remark $E * v = \{e * v; e \in E\}$, and $u * E$ is defined analogously. Call these the right and left translates of E by v and u, respectively, with respect to $*$. Switch to the language of red-blue colorings and forbidden sets to see that this Corollary is a generalization of Theorem A.

Proof. Apply the theorem with $U = V$, and with $V \times V$ partitioned into $\hat{R} = \{(v_1, v_2); v_1 * v_2 \in R\}$ and $\hat{B} = \{(u_1, u_2); u_1 * u_2 \in B\}$. The details are straightforward. ∎

3 Remarks on Applications

We have hopes that the Corollary, at least, will eventually find application to a variety of Ramsey-type questions involving two-colorings. For instance, suppose one wonders whether it is possible to color the integers $0, \ldots, N-1$

with red and blue so that certain arithmetic sequences are forbidden for red, and certain arithmetic sequences are forbidden for blue. It would seem that the Corollary, with $V = \{0, \ldots, N-1\}$ and $*$ being addition mod N [no harm in forbidding arithmetic sequences mod N, if you want to forbid arithmetic sequences], would come in handy in looking at such problems.

In another major Ramsey-type endeavor, two-coloring the edges of a graph with red and blue so that certain subgraphs are forbidden for red, and certain others for blue, the drawback to application is not only our ignorance of the chromatic numbers of exotic hypergraphs, but, before that, the problem of choosing a binary operation $*$ on the edges of the original graph. But perhaps the Theorem can be applied directly in such situations, letting unordered pairs reappear as pairs of ordered pairs.

We are indebted to Bill Martin for suggesting that the Corollary be applied with $V = S^2$, the unit sphere in $\mathbb{R}^3$, and with $*$ being defined by $u*v =$ the result of applying to u the rotation of S^2 that takes $(1,0,0)$ into v, when $v \neq (-1,0,0)$. When $v = (-1,0,0)$, let $u * v = -u$. Here is one straightforward result that can be deduced.

Clearly for $s > 0$ sufficiently small, the graph with the points of S^2 as vertices, and two points adjacent if and only if they are a distance s apart, has chromatic number at least 4. In a coloring of S^2 with red and blue, with the distance s forbidden for blue, we see that the simple edges of this graph are also hyperedges of the hypergraph H_1 referred to in the Corollary (with R, B standing for the sets of red and blue points, of course). From the Corollary we conclude: for all $s > 0$ sufficiently small, for any coloring of S^2 with red and blue such that the distance s is forbidden for blue, for each three-element set $\{v_1, v_2, v_3\} \subseteq S^2$, there exists $u \in S^2$ such that $u * v_1, u * v_2$, and $u * v_3$ are all red. [We say "are all", but there may just be one or two of them.]

This result is not particularly memorable, and it does not settle any outstanding problem that we know of, but it seems non-trivial. We certainly would never have arrived at this result but for the strange path taken here, starting from Theorem A.

Our thanks to the referee for certain suggestions, and especially for noting that ( ) $*$ $(-1,0,0)$, above, was not well-defined, in the original manuscript. The definition given here is one of many possible.

References

[1] M. Benda and M. Perles, Colorings of metric spaces, famous unpublished manuscript; soon to appear in *Geombinatorics*.