5-Cycle Systems of $\lambda(K_v - K_u)$

Darryn Bryant
Centre for Discrete Mathematics and Computing
Department of Mathematics
The University of Queensland
Qld 4072
Australia

Abstract

In this paper, necessary and sufficient conditions for the existence of a 5-cycle system of the λ-fold complete graph of order v with a hole of size u, $\lambda(K_v - K_u)$, are proved.

1 Introduction

An *m-cycle* $(v_0, v_1, \ldots, v_{m-1})$ is a graph with vertex set $\{v_0, v_1, \ldots, v_{m-1}\}$ and edge set $\{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{m-1}, v_0\}\}$. An *m-cycle system* of a multigraph G (undirected and without loops) is an ordered pair (V, C) where V is the vertex set of G and C is a collection of m-cycles, the edges of which partition the edges of G. The complete multigraph of order n and index λ, denoted λK_n, is the multigraph with n vertices and λ edges between each pair of vertices. The multigraph $\lambda(K_v - K_u)$ is defined on the vertex set V with $|V| = v$ as follows. Let $U \subseteq V$, called the *hole*, with $|U| = u$ and join each pair a, b of vertices in V by λ edges if and only if it is not the case that both a and b are in U. In this paper necessary and sufficient conditions for the existence of a 5-cycle system of $\lambda(K_v - K_u)$ are given (see Theorem 5.1). In the case $u = 1$, $\lambda(K_v - K_u) \cong \lambda K_v$ and we

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have the following result which follows easily from [1].

Lemma 1.1 *There exists a 5-cycle system of λK_v if and only if $v \geq 5$ and*

(1) $v \equiv 1, 5$ (mod 10) when $\lambda \equiv 1, 3, 7, 9$ (mod 10);

(2) $v \equiv 0, 1$ (mod 5) when $\lambda \equiv 2, 4, 6, 8$ (mod 10); and

(3) v is odd when $\lambda \equiv 5$ (mod 10).

Much has been done on the problem of finding m-cycle systems of $K_v - K_u$ (the case $\lambda = 1$), particularly under the guise of *embeddings* of m-cycle systems. An m-cycle system (V, C) is said to be *embedded* in the m-cycle system (W, P) if $V \subset W$ and $C \subset P$. Doyen and Wilson solved the embedding problem for 3-cycle systems (Steiner triple systems) in 1973.

Theorem 1.1 ([7]) $S_3(u) = \{v \mid v \equiv 1 \text{ or } 3 \pmod 6,\ v \geq 2u + 1\}$.

The embedding problem for 5-cycle systems was solved by Bryant and Rodger.

Theorem 1.2 ([4]) $S_5(u) = \{v \mid v \equiv 1 \text{ or } 5 (\text{mod } 10), v \geq 3u/2 + 1\}$.

More recently, Bryant and Rodger [5] gave a general approach for embedding an m-cycle system of order u in the case m is odd and u is 1 or m (mod 2m). The embedding problem for m-cycle systems has now been completely settled for all $m \leq 14$ [6].

When an m-cycle system of order u is embedded in an m-cycle system of order v, an m-cycle system of the complete graph of order v with a hole of size u, $K_v - K_u$, is obtained. However, the solution of the embedding problem for m-cycle systems does not completely solve the existence problem for m-cycle systems of $K_v - K_u$; it does not solve the cases where there is no m-cycle system of order u. This more general problem was solved in the case $m = 3$ by Mendelsohn and Rosa [10] and in the case $m = 5$ by Bryant, Hoffman and Rodger [3].

Theorem 1.3 ([10]) *There exists a 3-cycle system of $K_v - K_u$ if and only if $v \geq 2u + 1$ and*

(1) u, $v \equiv 1$ or $3 \pmod 6$, or

(2) $u \equiv v \equiv 5 \pmod 6$.

Theorem 1.4 ([3]) *There exists a 5-cycle system of $K_v - K_u$ if and only if $v \geq 3u/2 + 1$ and*

(1) u, $v \equiv 1$ or $5 \pmod{10}$, or

(2) $u \equiv v \equiv 3 \pmod{10}$, or

(3) u, $v \equiv 7$ or $9 \pmod{10}$.

This problem has also been settled for the cases $m = 4, 6, 7, 8$ and 10 [6]. The existence problem for m-cycle systems of $\lambda(K_v - K_u)$ has been settled only for the case $m = 3$ [8].

Throughout we will use standard graph theoretic terminology which if not defined here can be found in [2]. All graphs are undirected and without loops but may contain multiple edges. Let E_x denote the empty graph on x vertices (that is, the graph with x vertices and no edges). If G and H are two graphs then let $G \cup H$ be the graph with vertex set $V(G \cup H) = V(G) \cup V(H)$ and edge set $E(G \cup H) = E(G) \cup E(H)$. If $V(G) \cap V(H) = \phi$, then let $G \vee_x H$ be the graph with $V(G \vee_x H) = V(G) \cup V(H)$ and $E(G \vee_x H) = E(G) \cup E(H) \cup S$ where S consists of x edges joining g to h for each $g \in V(G)$ and $h \in V(H)\}$. In the case $x = 1$ we drop the subscript and write just $G \vee H$. Note that $\lambda(K_v - K_u) \cong E_u \vee_\lambda \lambda K_w$ where $w = v - u$. If H is a subgraph of G, let $G - H$ be the graph containing those edges of G which are not in H.

We will make much use of differences in constructing 5-cycle systems and we now introduce the necessary notation. Let $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$.

Define $|i-j|_x = \min\{|i-j|, x-|i-j|\}$. If D is a multiset with elements chosen from $\{1,2,\ldots,\lfloor x/2\rfloor\}$, then define the graph $\langle D\rangle_x$ to be the graph with vertex set $\mathbb{Z}_x$ and having vertex i joined to vertex j exactly λ times if and only if $|i-j|_x$ occurs λ times in D. If D_0 and D_1 are multisets with elements chosen from $\{1,2,\ldots,\lfloor x/2\rfloor\}$ and S is a multiset with elements chosen from $\mathbb{Z}_x$, then define the graph $\langle D_0, S, D_1\rangle_x$ to be the graph with the vertex set $\mathbb{Z}_x \times \mathbb{Z}_2$ and having vertex (i,y) joined to vertex (j,y) exactly λ times if and only if $|i-j|_x$ occurs λ times in D_y ($y \in \mathbb{Z}_2$) and vertex $(i,0)$ joined to vertex $(j,1)$ exactly λ times if and only if $(i-j) \pmod{x}$ occurs λ times in S.

There are some obvious necessary conditions for the existence of a of a 5-cycle system of $\lambda(K_v - K_u)$.

Lemma 1.2 *Let $v > u \geq 1$. If there exists a 5-cycle system of $\lambda(K_v - K_u)$ then*

(1) $v \geq 5$;

(2) if λ is odd then $u \equiv v \equiv 1$ (mod 2);

(3) $\lambda(\binom{v}{2} - \binom{u}{2}) \equiv 0$ (mod 5);

(4) $v \geq \frac{3}{2}u + 1$; and

(5) $(u,v) \notin \{(2,5),(3,6)\}$.

Proof: Condition (1) is immediate. Each of the u vertices in the hole of $K_v - K_u$ has degree $\lambda(v-u)$, and each of the remaining $v-u$ vertices has degree $\lambda(v-1)$. Since each m-cycle includes an even number of edges incident with each vertex, (2) follows. Since 5 divides the number of edges in $\lambda(K_v - K_u)$, (3) follows. Since any 5-cycle can have at most 4 edges which are adjacent to a hole vertex, (4) follows. Finally, since the degree of any vertex must be at most twice the number of 5-cycles, (5) follows. □

The following lemma is a particular case of a result of Stern and Lenz.

Lemma 1.3 *[12] Let $D \subseteq \{1, 2, \ldots, \lfloor x/2 \rfloor\}$ and $S \subseteq \mathbb{Z}_x$. If either $|S| \geq 1$ or $x/2 \in D$ then $\langle D, S, D \rangle_x$ has a 1-factorization.*

A 2-factor is said to be even if each component is a cycle of even length. The following result was proved in [4] but only for the case of simple graphs.

Corollary 1.1 *[4] If T is an even 2-factor (containing no C_2 component) then there exists a 5-cycle system of $E_4 \vee T$.*

For this paper we need the following generalisation.

Lemma 1.4 *There is a 5-cycle system of $E_4 \vee H$ where H is any even 2-factor (possibly containing 2-cycles) except $H = C_2$.*

Proof: If H contains no C_2 components then the result follows from Corollary 1.1. If H contains exactly one C_2 component then we write H as $H = (C_2 + C_{2x}) + T$ where $x \geq 2$ and T contains no C_2 components. If H contains more than one C_2 component then we write H as $H = \sum^r C_2 + T$ where $r \geq 2$ and T contains no C_2 components. The result then follows from Corollary 1.1 if there exists a 5-cycle systems of

(i) $E_4 \vee (\sum^r C_2)$ for each $r \geq 2$; and

(ii) $E_4 \vee (C_2 + C_{2x})$ for each $x \geq 2$.

To prove (i), it is sufficient to show that there exists a 5-cycle system of $E_4 \vee (C_2 + C_2)$ and of $E_4 \vee (C_2 + C_2 + C_2)$. For $E_4 \vee (C_2 + C_2)$, let $V(E_4) = \{\infty_1, \infty_2, \infty_3, \infty_4\}$ and let $C_2 + C_2 = (1, 2) + (3, 4)$. Then $\{(\infty_1, 1, 2, \infty_2, 3), (\infty_1, 2, 1, \infty_2, 4), (\infty_3, 3, 4, \infty_4, 1), (\infty_3, 4, 3, \infty_4, 2)\}$ is the required 5-cycle system. For $E_4 \vee (C_2 + C_2 + C_2)$, let $V(E_4) = \{\infty_1, \infty_2, \infty_3, \infty_4\}$ and let $C_2 + C_2 + C_2 = (1, 2) + (3, 4) + (5, 6)$. Then $\{(\infty_1, 1, 2, \infty_2,$

3), $(\infty_1, 2, 1, \infty_3, 5), (\infty_1, 6, 5, \infty_4, 4), (\infty_2, 5, 6, \infty_4, 1), (\infty_2, 4, 3, \infty_3, 6)$, $(\infty_3, 4, 3, \infty_4, 2)\}$ is the required 5-cycle system.

We prove (ii), by induction on x. We have just seen that there is a 5-cycle system of $E_4 \vee (C_2 + C_2)$ so assume that there exists a 5-cycle system of $E_4 \vee (C_2 + C_{2x})$, the 4 vertices in E_4 being $\infty_1, \infty_2, \infty_3$ and ∞_4, the cycle C_{2x} being $(v_0, v_1, \ldots, v_{2x-1})$. Let the three 5-cycles containing v_{2x-1} be $(v_0, v_{2x-1}, \infty_1, a_1, a_2), (v_{2x-2}, v_{2x-1}, \infty_2, a_3, a_4)$ and $(\infty_3, v_{2x-1}, \infty_4, a_5, a_6)$, where $\{a_1, \ldots, a_6\} \subseteq V(E_4 \cup C_{2x})$. We add two new vertices v_{2x} and v_{2x+1} and replace these three 5-cycles with $(v_0, v_{2x+1}, \infty_1, a_1, a_2), (v_{2x-2}, v_{2x-1}, \infty_2, a_3, a_4), (\infty_3, v_{2x}, \infty_4, a_5, a_6)$, $(\infty_1, v_{2x}, v_{2x+1}, \infty_4, v_{2x-1})$ and $(\infty_2, v_{2x}, v_{2x-1}, \infty_3, v_{2x+1})$ to obtain a 5-cycle system of $E_4 \vee (C_2 + C_{2x+2})$, the cycle C_{2x+2} being $(v_0, v_1, \ldots, v_{2x+1})$. □

2 $\lambda = 2$

Lemma 2.1 *If d_1, d_2, d_3 are any three differences in $2K_w$ then there exists a 5-cycle system of $K_1 \vee_2 \langle d_1, d_2, d_3 \rangle_w$.*

Proof: Since $d_1, d_2, d_3 \leq w/2$ are differences in $2K_w$, we can assume $d_2 < d_3$ and let $C_i = (\infty, 0, d_1, d_1 + d_3, d_1 + d_3 - d_2) + (0, i, i, i, i)$ for $i \in \mathbb{Z}_w$. so that the collection $\{C_i | i \in \mathbb{Z}_w\}$ is the required 5-cycle system. □

Lemma 2.2 *If d is any difference in λK_w then there exists a 5-cycle system of $E_2 \vee_2 \langle d \rangle_w$.*

Proof: Let $C_i = (\infty_1, 0, d, \infty_2, d + 1) + (0, i, i, 0, i)$ for $i \in \mathbb{Z}_w$. Then the collection $\{C_i | i \in \mathbb{Z}_w\}$ is the required 5-cycle system. □

Lemma 2.3 *There is a 5-cycle system of $E_3 \vee_2 2C_5$.*

Proof: Let $V(E_3) = \{1, 2, 3\}$ and let $C_5 = (4, 5, 6, 7, 8)$. Then the required 5-cycle system is $\{(1, 7, 8, 2, 4), (1, 8, 4, 2, 5), (1, 5, 6, 2, 7), (1, 4, 5, 3, 6), (1, 6,$

$7, 3, 8), (2, 5, 4, 3, 6), (2, 8, 4, 3, 7), (3, 4, 5, 6, 7)\}$. □

Lemma 2.4 *Let u, v be positive integers with $u \geq 2$, $v \geq 5$, $(u, v) \equiv (0, 1), (1, 0), (2, 4), (4, 2)$ or $(3, 3)$ (mod 5) and $\frac{3}{2}u + 1 \leq v \leq 4u + 1$. Then there exists a 5-cycle system of $2(K_v - K_u)$.*

Proof: Let $w = v - u$ and let $y = (2(w-1) - u)/5$. Notice that for the given congruence classes of u and v, y is an integer. Also notice that since $v \leq 4u + 1, y \leq u$ and since $\frac{3}{2}u + 1 \leq v, y \geq 0$. Now $2K_w = \langle d_1, d_2, \ldots, d_{w-1} \rangle_w$ so (since $w - 1 = (5y + u)/2$) we can partition these differences into y 3-sets and $(u - y)/2$ 1-sets. Similarly, we partition the graph E_u into y copies of K_1 and $(u - y)/2$ copies of E_2. The required 5-cycle system can now be constructed by using Lemma 2.1 (y times) and Lemma 2.2 ($(u - y)/2$ times). □

Lemma 2.5 *For $r = 0, 1, 2, 4$, let $K(r) = 5, 13, 6, -3$ respectively and let u, v be positive integers with $u \equiv v \equiv r$ (mod 5), $u \geq 3$ and $\frac{3}{2}u + 1 \leq v \leq 4u - K(r)$. Then there exists a 5-cycle system of $2(K_v - K_u)$.*

Proof: Let $w = v - u$, let $(s, t) = (3, 3), (2, 4), (1, 2), (4, 2)$ when $r = 0, 1, 2, 4$ respectively, and let $y = (2(w - 1) - u - s)/5$. Now $2K_w = \langle d_1, d_2, \ldots, d_{w-1} \rangle_w$ so (since $w - 1 = (5y + u + s)/2$) we can partition these differences into y 3-sets, $(u - y + s)/2 - t$ 1-sets and the t-set $T_r = \{w/5, w/5, 2w/5\}, \{w/5, w/5, 2w/5, 2w/5\}, \{w/5, w/5\}, \{w/5, w/5\}$ when $r = 0, 1, 2, 4$ respectively. Notice that in all cases, $w - 1 \geq 3$ and $w - 1 \geq t$. We need to verify that y and $(u - y + s)/2 - t$ are non-negative integers.

Firstly, since $u \equiv v \equiv r$ (mod 5) and $s \equiv 3 - r$ (mod 5), it is easy to check that y is an integer. To show that $y \geq 0$, we let $u = 5x + r$ and $w = v - u = 5z$ where x and z are non-negative integers. Since $v \geq \frac{3}{2}u + 1$, we have $2w/5 \geq x + (r + 2)/5$ and hence for $r = 0, 1$ and 2, $2w/5 \geq x = 1$ and for $r = 4$, $2w/5 \geq x + 2$. Putting $x = (u - r)/5$ and solving for v

we have $v \geq \frac{3}{2}u + (5-r)/2$ for $r = 0, 1$ and 2 and $v \geq \frac{3}{2}u + (10-r)/2$ for $r = 4$. Now, for $r = 0, 1$ and 2, $(5-r)/2 = s/2 + 1$ and for $r = 4$, $(10-r)/2 = s/2 + 1$. Hence for $r = 0, 1, 2$ and 4, $v \geq \frac{3}{2}u + s/2 + 1$ which is equivalent to $y \geq 0$. Secondly, we consider $(u - y + s)/2 - t$. From what we have said (with regard to partitioning the differences) and since $y, w-1$ and t are integers, it follows that $(u-y+s)/2-t$ is an integer. It remains to show that $(u-y+s)/2-t \geq 0$. Substituting $y = (2(v-u-1)-u-s)/5$ we see that $(u-y+s)/2-t \geq 0$ if and only if $4u - v + 3s - 5t + 1 \geq 0$. But we know that $v \leq 4u - K(r)$ and one can check that $K(r) = 5t - 3s - 1$ so it follows that $4u - v + 3s - 5t + 1 \geq 0$ and hence $(u - y + s)/2 - t \geq 0$.

Similarly we partition the graph E_u into y copies of K_1, $(u-y+s)/2-t$ copies of E_2 and one copy of E_{2t-s} (note that in all cases, $2t - s \leq u$). By applying Lemma 2.1 (y times) and Lemma 2.2 ($(u - y + s)/2 - t$ times), it remains to find a 5-cycle system of $E_{2t-s} \vee_2 \langle T_r \rangle_w$. So when $r = 0, 1, 2, 4$ we need a 5-cycle systems of $E_3 \vee_2 \langle \{w/5, w/5, 2w/5\} \rangle_w$, $E_6 \vee_2 \langle \{w/5, w/5, 2w/5, 2w/5\} \rangle_w$, $E_3 \vee_2 \langle \{w/5, w/5\} \rangle_w$, $\langle \{w/5, w/5\} \rangle_w$ respectively. Since $\langle \{w/5, w/5\} \rangle_w$ and $\langle \{2w/5, 2w/5\} \rangle_w$ are each a collection of vertex disjoint copies of $2C_5$ (recall that $w \equiv 0 \pmod 5$), the required 5-cycle system can be obtained by applying Lemma 2.3 (1,2,1,0 times for r=0,1,2,4 respectively). □

In order to simplify the proof of Lemma 2.7, we first prove the existence of 5-cycle systems of some small graphs.

Lemma 2.6 *There is a 5-cycle system of* $2(K_v - K_u)$ *for* $(u, v) \in \{(2,7), (2,12), (2,14), (2,17), (2,19), (3,18), (5,20), (6,16), (6,21)\}$.

Proof: First consider the case $(u, v) = (2, 7)$. The graph $2(K_7 - K_2)$ can be written as the edge disjoint union of $K_2 \vee_2 C_5$ and $3C_5$ and clearly there is a 5-cycle system of each of these graphs. For the remaining values of (u, v) we find an integer s such that there is a 5-cycle system of $2(K_v - K_s)$

and of $2(K_s - K_u)$ and hence also a 5-cycle system of $2(K_v - K_u)$. When $u = 2$ we let $s = 7$, when $u = 3$ we let $s = 8$ and when $u = 5$ or 6, we let $s = 10$. □

Lemma 2.7 *There exists a 5-cycle system of* $2(K_v - K_u)$ *if* $v \geq 5$, $v \geq \frac{3}{2}u + 1$ *and*

(1) $u, v \equiv 0$ *or* 1 *(mod 5), or*

(2) $u \equiv v \equiv 3$ *(mod 5), or*

(3) $u, v \equiv 2$ *or* 4 *(mod 5).*

Proof: For the case $u = 1$ see Lemma 1.1. The proof for $u \geq 2$ is by induction on v. For each $u \geq 2$, there exists a $v > u$ such that there is a 5-cycle system of $2(K_v - K_u)$ by Lemma 2.4 so assume that for all v' satisfying the conditions of the lemma and with $v' < v$, there is a 5-cycle system of $2(K_{v'} - K_u)$. If $v \leq 20$ or $v < 9u/4 + 9\frac{1}{2}$, then there is a 5-cycle system of $2(K_v - K_u)$ by Lemma 2.6, Lemma 2.4 or Lemma2.5. Hence we can assume $v \geq 21$ and $v \geq 9u/4 + 9\frac{1}{2}$. Let s be the largest integer $s \equiv u$ or v (mod 5) such that $\frac{3}{2}s + 1 \leq v$. We will show that there is a 5-cycle system of $2(K_v - K_s)$ and a 5-cycle system of $2(K_s - K_u)$ thereby showing that there is a 5-cycle system of $2(K_v - K_u)$. By the maximality of s, it follows that $s \geq 2(v-1)/3 - \frac{14}{3}$, that is $3s \geq 2v - 16$. Since $3s \geq 2v - 16$ and $v \geq 21$, it follows that $v \leq 4s - 14$ and so by Lemma 2.4 or Lemma 2.5, there is a 5-cycle system of $2(K_v - K_s)$. Also, since $v \geq 9u/4 + 9\frac{1}{2}$ and $3s \geq 2v - 16$, it follows that $s \geq \frac{3}{2}u + 1$ (and $s \geq 5$) and so by the induction hypothesis, there is a 5-cycle system of $2(K_s - K_u)$. □

3 $\lambda = 10$

Lemma 3.1 *Let u, v be positive integers with $u \geq 2$, $(u, v) \neq (2,4), (2,5), (3,6)$ and $\frac{3}{2}u + 1 \leq v \leq 4u + 1$. Then there exists a 5-cycle system of $10(K_v - K_u)$.*

Proof: Let $w = v - u$ and let $y = 2v - 3u - 2$. Notice that since $v \leq 4u + 1, y \leq 5u$, and since $v \geq \frac{3}{2}u + 1, y \geq 0$. Now $10K_w = \langle d_1, d_2, \ldots, d_{5(w-1)} \rangle_w$ so (since $5(w-1) = 5v - 5u - 5$), we can partition these differences into $(5u - y)/2$ 1-sets and y 3-sets, where in each 3-set it is not the case that all 3 differences are equal. Notice that for all permitted values of u and v, except $(u, v) = (4, 7)$ where $y = 0$, $w = v - u \geq 4$ and so there is no difficulty in ensuring that no three differences in a 3-set are equal. Similarly, we can place the vertices of E_u into y 1-sets (K_1's for Lemma 2.1) and $(5u - y)/2$ 2-sets (K_2's for Lemma 2.2) so that each vertex is placed in 10 different sets. The required 5-cycle system can now be constructed using Lemma 2.1 (y times) and Lemma 2.2 ($(5u - y)/2$ times). □

Lemma 3.2 *There exists a 5-cycle system of $10(K_v - K_u)$ if $v \geq 5$, $(u, v) \notin \{(2,5), (3,6)\}$ and $v \geq \frac{3}{2}u + 1$.*

Proof: The proof uses induction on v and is similar to that of Lemma 2.7. For the case $u = 1$ see Lemma 1.1. For each $u \geq 2$, there exists a $v > u$ such that there is a 5-cycle system of $10(K_v - K_u)$ by Lemma 3.1 so assume that for all v' satisfying the conditions of the lemma and with $v' < v$, there is a 5-cycle system of $10(K_{v'} - K_u)$. Let s be the largest integer such that $\frac{3}{2}s + 1 \leq v$. By the maximality of s, it follows that $s \geq 2(v-1)/3 - \frac{2}{3}$, that is $3s \geq 2v - 3$. Hence it follows that $v \leq 4s + 1$ (note that $v \geq 5$) and so by Lemma 3.1 there is a 5-cycle system of $10(K_v - K_s)$. Also, since we can assume $v \geq 4u + 2$ and since $3s \geq 2v - 3$, it follows that $s \geq \frac{3}{2}u + 1$

(and clearly $s \geq 5$ and $(u,s) \notin \{(2,5),(3,6)\}$) and so by the induction hypothesis, there is a 5-cycle system of $10(K_s - K_u)$. □

4 $\lambda = 5$

Lemma 4.1 *Let w be even. If $y \leq 5(w-2)/3$, then there exist y edge-disjoint 3-regular spanning subgraphs $G_1, G_2, \ldots, G_y$ of $5K_w$ such that for $i = 1, 2, \ldots, y$ there exists a 5-cycle system of $K_1 \vee G_i$ and such that $5K_w - (G_1 \cup G_2 \cup \ldots \cup G_y)$ has a 1-factorisation.*

Proof: Let the vertex set of K_1 be $\{\infty\}$, let $x = w/2$, let $5K_w$ have vertex set $\mathbb{Z}_x \times \mathbb{Z}_2$ and let $h = \lfloor (x-1)/2 \rfloor$. Also, let $t = \min\{y, 5h\}$, let $D = \{1,1,1,1,1,2,2,2,2,2,\ldots,5,5,5,5,5\}$ and let $S = \{0,0,0,0,0,1,1,1,1,1, \ldots, x-1, x-1, x-1, x-1, x-1\}$. Select t elements of D, select t elements of S and pair the selected elements of D to the selected elements of S arbitrarily. For $i = 1, 2, \ldots, t$ the graph G_i is defined to be $\langle\{d\},\{s\},\{d\}\rangle$ where d and s make up the ith selected pair and the set $\{(\infty,(0,0),(d,0),(d+s,1),(s,1)) + (j,0) | j \in \mathbb{Z}_x\}$ is the required 5-cycle system of $K_1 \vee G_i$. Partition $3(y-t)$ of the remaining elements of S into $y-t$ sets of size 3. For $t+1 \leq i \leq y$, the graph G_i is defined to be $\langle \emptyset, \{s_1, s_2, s_3\}, \emptyset \rangle$ where s_1, s_2, s_3 make up the ith set of size 3 and the set $\{(\infty,(0,0),(s_1,1),(s_1-s_2,0),(s_1-s_2+s_3,1))+(j,0) | j \in \mathbb{Z}_x\}$ is the required 5-cycle system of $K_1 \vee G_i$. Finally, $5K_w - (G_1 \cup G_2 \cup \ldots \cup G_y) = \langle C, T, C\rangle_x$ where either $x/2$ occurs 5 times in C or $|T| \geq 5$. Hence $5K_w - (G_1 \cup G_2 \cup \ldots \cup G_y)$ has a 1-factorisation by Lemma 1.3. □

Lemma 4.2 *Let u, v be odd integers with $u \geq 5$ and $\frac{3}{2}u + 1 \leq v \leq 4u - 5$. Then there exists a 5-cycle system of $5(K_v - K_u)$.*

Proof: Let $w = v - u$ and let $y = 2v - 3u - 2$. Notice that since $v \leq 4u - 5$, $y \leq 5(w-2)/3$ and since $\frac{3}{2}u + 1 \leq v$, $y \geq 0$. Now, by Lemma

4.1 we can partition the edge set of $5K_w$ into y 3-regular spanning graphs $G_1, G_2, \ldots, G_y$ and a $(5u-y)/2$-regular spanning graph H such that for $i = 1, 2, \ldots, y$ there is a 5-cycle system of $K_1 \vee G_i$ and such that H has a 1-factorisation. Pair up the $(5u-y)/2$ 1-factors in a 1-factorisation of H to form $(5u-y)/4$ even 2-factors. Now, we partition the graph E_u into y copies of K_1 and $(5u-y)/4$ copies of E_4 so that each vertex of E_u is in 5 distinct graphs. The required 5-cycle system is obtained from the y 5-cycle systems of $K_1 \vee G_i$ and by pairing the $(5u-y)/4$ copies of E_4 with the $(5u-y)/4$ even 2-factors and applying Lemma 1.4 ($(5u-y)/4$ times). □

Lemma 4.3 *There exists a 5-cycle system of* $5(K_v - K_u)$ *if* $u, v \geq 5$, *u is odd, v is odd, and* $v \geq \frac{3}{2}u + 1$.

Proof: The proof uses induction on v and is similar to that of Lemma 2.7. For the case $u = 1$ see Lemma 1.1 and for the case $u = 3$ see Lemma 4.5. For each $u \geq 5$, there exists a $v > u$ such that there is a 5-cycle system of $5(K_v - K_u)$ by Lemma 4.2 so assume that for all v' satisfying the conditions of the lemma and with $v' < v$, there is a 5-cycle system of $5(K_{v'} - K_u)$. Let s be the largest odd integer such that $\frac{3}{2}s + 1 \leq v$. By the maximality of s, it follows that $s \geq 2(v-1)/3 - \frac{5}{3}$, that is $3s \geq 2v - 7$. Hence it follows that $v \leq 4s - 4$ (note that $v \geq 17$) and so by Lemma 3.1 there is a 5-cycle system of $5(K_v - K_s)$. Since $3s \geq 2v - 7$, $u \geq 5$ and since we can assume $v \geq 4u - 3$, it follows that $s \geq \frac{3}{2}u + 1$. Hence by the induction hypothesis, there is a 5-cycle system of $5(K_s - K_u)$. □

Lemma 4.4 *There exists a 5-cycle system of* $5(K_v - K_3)$ *for* $v = 7, 9$ *and 11.*

Proof: The following collections of 5-cycles give the required systems. In each case the vertex set is $\{0, 1, \ldots, v-1\}$ and the vertices in the hole are $\{0, 1, 2\}$.

$5(K_7 - K_3)$

(0, 3, 1, 4, 5), (0, 4, 2, 3, 6), (1, 5, 3, 4, 6), (0, 3, 1, 4, 5), (0, 4, 2, 3, 6), (1, 5, 3, 4, 6), (0, 3, 1, 4, 5), (0, 4, 2, 3, 6), (1, 5, 3, 4, 6), (0, 3, 1, 5, 6), (0, 4, 1, 6, 5), (2, 3, 4, 5, 6), (0, 3, 2, 5, 6), (0, 4, 2, 6, 5), (1, 3, 5, 2, 6), (1, 4, 6, 2, 5), (2, 4, 6, 3, 5), (2, 5, 4, 3, 6).

$5(K_9 - K_3)$

(0, 3, 1, 4, 5), (0, 4, 2, 3, 6), (0, 7, 1, 5, 8), (1, 6, 2, 7, 8), (2, 5, 3, 4, 8), (3, 7, 4, 6, 8), (0, 3, 1, 4, 5), (0, 4, 2, 3, 6), (0, 7, 1, 5, 8), (1, 6, 2, 7, 8), (2, 5, 3, 4, 8), (3, 7, 4, 6, 8), (0, 3, 1, 4, 5), (0, 4, 2, 3, 6), (0, 7, 1, 5, 8), (1, 6, 2, 7, 8), (2, 5, 3, 4, 8), (3, 7, 4, 6, 8), (0, 3, 1, 4, 5), (0, 4, 2, 3, 6), (0, 7, 1, 5, 8), (1, 6, 2, 7, 8), (2, 5, 3, 4, 8), (3, 7, 4, 6, 8), (0, 3, 1, 4, 8), (0, 4, 5, 6, 7), (0, 5, 1, 7, 6), (1, 6, 5, 7, 8), (2, 3, 5, 6, 7), (2, 4, 6, 7, 5), (2, 6, 7, 5, 8), (3, 4, 7, 5, 6), (3, 7, 5, 6, 8).

$5(K_{11} - K_3)$

(0, 3, 1, 4, 5), (0, 4, 2, 3, 6), (0, 7, 1, 5, 8), (0, 9, 1, 6, 10), (1, 8, 2, 5, 10), (2, 6, 4, 3, 7), (2, 9, 3, 8, 10), (3, 5, 6, 7, 10), (4, 7, 5, 9, 8), (0, 3, 1, 4, 5), (0, 4, 2, 3, 6), (0, 7, 1, 5, 8), (0, 9, 1, 6, 10), (1, 8, 2, 5, 10), (2, 6, 4, 3, 7), (2, 9, 3, 8, 10), (3, 5, 6, 9, 10), (4, 7, 5, 9, 10), (4, 8, 6, 7, 9), (4, 9, 8, 7, 10), (0, 3, 1, 4, 5), (0, 4, 2, 3, 6), (0, 7, 1, 5, 8), (0, 9, 1, 6, 10), (1, 8, 2, 5, 10), (2, 6, 4, 3, 7), (2, 9, 3, 8, 10), (3, 5, 6, 7, 10), (4, 7, 8, 6, 9), (4, 8, 6, 9, 10), (0, 3, 1, 4, 5), (0, 4, 2, 3, 6), (0, 7, 1, 5, 8), (0, 9, 1, 6, 10), (1, 8, 2, 5, 10), (2, 6, 4, 3, 7), (2, 9, 3, 8, 10), (3, 5, 6, 7, 10), (4, 7, 5, 9, 10), (4, 8, 7, 5, 9), (0, 3, 1, 4, 5), (0, 4, 2, 3, 6), (0, 7, 1, 5, 8), (0, 9, 1, 6, 10), (1, 8, 2, 5, 10), (2, 6, 4, 3, 9), (2, 7, 6, 9, 10), (3, 5, 7, 9, 8), (3, 7, 9, 8, 10), (4, 7, 8, 6, 9), (4, 8, 9, 7, 10), (5, 6, 8, 7, 9).

□

Lemma 4.5 *There exists a 5-cycle system of* $5(K_v - K_3)$ *if* $v \geq 7$ *is odd.*

Proof: By Lemma 4.4, the result holds for $v \leq 11$ and so we need only consider $v \geq 13$. But by Lemma 4.3, there is a 5-cycle system of $5(K_v - K_7)$ for all odd $v \geq 13$. Hence, since there is a 5-cycle system of $5(K_7 - K_3)$, we have a 5-cycle system of $5(K_v - K_3)$. □

5 Main Theorem

Theorem 5.1 *Let u, v and λ be positive integers. There exists a 5-cycle system of $\lambda(K_v - K_u)$ if and only if $(u, v) \notin \{(2,5),(3,6)\}$, $v \geq \frac{3}{2}u + 1$ and*

(1) if $\lambda \equiv 1, 3, 7, 9$ (mod 10) then

(1a) $u, v \equiv 1$ or 5 (mod 10); or

(1b) $u, v \equiv 7$ or 9 (mod 10); or

(1c) $u, v \equiv 3$ (mod 10); and

(2) if $\lambda \equiv 2, 4, 6, 8$ (mod 10) then

(2a) $u, v \equiv 0$ or 1 (mod 5); or

(2b) $u, v \equiv 2$ or 4 (mod 5); or

(2c) $u, v \equiv 3$ (mod 5); and

(3) if $\lambda \equiv 5$ (mod 10) then u and v are odd.

Proof: The necessity of the conditions follows from Lemma 1.2. Clearly if there exists a 5-cycle system of $\mu(K_v - K_u)$, then there exists a 5-cycle system of $\lambda(K_v - K_u)$ for all multiples λ of μ. Hence, since sufficiency is proved for $\lambda = 1, 2, 5$ and 10 (see Lemmas 2.7,4.3,4.5,3.2 and [3]) the result follows immediately. □

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