On the Counting of Caterpillar Continua

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ABSTRACT. In this paper we count the number of non-homeomorphic continua in a certain collection of continua. The continua in these collections are trees with certain restrictions on them. We refer to a continuum in one of these collections as a caterpillar continuum.

1 Introduction

In this paper we define what we call caterpillar continua. A caterpillar continuum is a tree with certain restrictions on it. For a given class of caterpillar continua, we count the number of non-homeomorphic caterpillar continua in that class.

2 Notation and Definitions

We use the notation and definitions from continuum theory (see [5]). A *graph* is a continuum that can be written as the union of finitely many arcs, any two of which are either disjoint or intersect in one or both of their endpoints. A *tree* is a graph containing no simple closed curves. If T is a non-degenerate, connected tree then for $x \in T$, $ord(x, T)$ is just the number of components of $T - \{x\}$. The point x is a *non-cut point* of T if $ord(x, T) = 1$. If $ord(x, T) > 1$ then x is a *cut point* of T.

We define a caterpillar continuum as follows:

C is a *caterpillar continuum* provided

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a) C is a non-degenerate connected tree.

b) There is an arc $A(C)$ in C such that

$$V(C) = \{x \mid x \in C, ord(x, C) \geq 3\} \subset A(C)$$

and the two endpoints of $A(C)$ are elements of $V(C)$.

Clearly, $A(C)$ is unique for each caterpillar contimuum, since C is non-cyclic. If $\#V(C) = n$, then we can label the endpoints of $A(C)$ v_1 and v_n. Also, we can label the other points in $V(C)$ such that their labeling corresponds to the natural ordering on $A(C)$, that is

$$v_1 < v_2 < \cdots < v_n.$$

It is clear that $A(C)$ induces two orderings on $V(C)$ depending on which endpoint is labeled v_1. By $C(m, j)$ we denote the following set:

$$\begin{aligned} C(m,j) =& \{C \mid C \text{ is a catepillar continuum, } \#V(C) = j, \\ & \text{there are } m \text{ non-cut points in } C\} \end{aligned}$$

Our objective is to count the number of non-homeomorphic elements in each set $C(m, j)$. We will denote this number by $n(C(m, j))$. Clearly each element in $C(2, 0)$ is an arc, so $n(C(2, 0)) = 1$. Also, for $m \geq 3$, $C(m, 0) = \phi$. It is also clear that each element of $C(m, 1)$ is an m-*odd* for $m \geq 3$, so $n(C(m, 1)) = 1$. It is the case that for $j \geq 1$, $C(m, j)$ is non-empty only when $m \geq j + 2$.

In our counting argument we use the fact that the number of ways to distribute n objects into k positions is given by

$$\binom{n+k-1}{n} \text{ (see [6])} \qquad (*)$$

We also employ Burnside's Theorem (see [1]) which we state below:

Burnside's Theorem: *Suppose a finite group G acts on a finite set S. Let $B_1, \ldots, B_N$ be the different orbits. For each $\alpha \in G$, let $F(\alpha)$ be the set of fixed points of α. Then*

$$N = \frac{1}{|G|} \sum_{\alpha \in G} |F(\alpha)|.$$

3 Main Result

Theorem 1. *Let $C(m, j)$ be the collection of caterpillar continua as defined above. Then the number of non-homeomorphic elements $n(C(m, j))$ in*

$C(m,j)$ for $j \geq 1$ and $m \geq j+2$ is given below:

$$n(C(m,j)) = \begin{cases} \frac{1}{2}\binom{m-3}{m-j-2} & j \text{ even, } m \text{ odd} \\ \frac{1}{2}\left[\binom{m-3}{m-j-2} + \binom{\frac{m-4}{2}}{\frac{m-j-2}{2}}\right] & j \text{ even, } m \text{ even} \\ \frac{1}{2}\left[\binom{m-3}{m-j-2} + \sum_{i=0}^{\frac{m-j-2}{2}} \binom{\frac{m-5-2i}{2}}{\frac{m-j-2-2i}{2}}\right] & j \text{ odd, } m \text{ odd} \\ \frac{1}{2}\left[\binom{m-3}{m-j-2} + \sum_{i=0}^{\frac{m-j-3}{2}} \binom{\frac{m-6-2i}{2}}{\frac{m-j-3-2i}{2}}\right] & j \text{ odd, } m \text{ even.} \end{cases}$$

Proof: In order to count the number of non-homeomorphic caterpillar continua in each set $C(m,j)$, we need a simple way to represent the various classes of elements in this set. Let $G \in C(m,j)$, let $V(G) = \{v_1, \ldots, v_j\}$ where v_1 and v_j are the endpoints of A and $v_1 < v_2 < \cdots < v_j$. Let $k_i = ord(v_i) - 2$ for $i = 1, 2, \ldots, j$. Note that $\sum_{i=1}^{j} k_i = m - 2$. We can represent G and all elements of $C(m,j)$ which are homeomorphic to it by an ordered j tuple $(k_1, \ldots, k_j)$. Clearly, $(k_j, k_{j-1}, \ldots, k_1)$ also represents this same set of graphs in $C(m,j)$. We need to determine the number of distinct sets of graphs determined by all such j-tuples.

For example when $j = 3$ and $m = 7$, we obtain the following 3-tuples: $Z_1 = (3,1,1)$, $X_2 = (1,3,1)$, $X_3 = (2,2,1)$, $X_4 = (2,1,2)$, $X_5 = (1,1,3)$ and $X_6 = (1,2,2)$. Let G_i be the set of graphs in $C(7,3)$ (all of which are homeomorphic) which is represented by X_i for each i. Clearly, $G_1 = G_5$ and $G_3 = G_6$. It is also clear that G_1, G_2, G_3 and G_4 are distinct subsets of $C(7,3)$ and that $\bigcup_{i=1}^{4} G_i = C(7,3)$ and $nC(7,3) = 4$.

Clearly, two j-tuples $(a_1, a_2, \ldots, a_j)$ and $(b_1, b_2, \ldots, b_j)$ represent the same subset of $C(m,j)$ if and only if $a_i = b_i$ for $i = 1, 2, \ldots, j$ or $a_i = b_{j-i+1}$ for $i = 1, 2, \ldots, j$. That is, they represent the same subset of $C(m,j)$ when they are either identical or mirror images of each other.

Let S be the set of all possible $j-1$-tuples, the sum of whose digits (all non-zero) is equal to $m-2$. We want to count the number of different orbits of the group $G = \{e, \overline{m}\}$ where e is the identity permutation and $\overline{m}$ is the mirror image permutation on S.

To simplify our calculations, we first consider S to be made up of k-tuples $(d_1, d_2, \ldots, d_k)$, where $\sum_{i=1}^{k} d_i = n$ and d_i could be zero.

By $(*)$ above we note that $|S| = \binom{n+k-1}{n}$. By Burnside's theorem, the number N of different orbits of G is

$$N = \frac{1}{|G|} \sum_{\alpha \in G} |F(\alpha)|.$$

where $F(\alpha) = \{x \in S \colon ax = x\}$. That is we have

$$N = \frac{1}{2}(|F(e)| + |F(\overline{m})|.$$

Since $ex = x$ for all $x \in S$, then

$$|F(e)| = |S| = \binom{n+k-1}{n}.$$

Now if k is even and n is odd, then obviously it is impossible for any of the k-tuples to be mirror images of each other. Therefore, $F(\overline{m}) = \phi$,and $|F(\overline{m})| = 0$. Hence, we have

$$k \text{ even, } n \text{ odd } \Rightarrow N = \frac{1}{2}\binom{n+k-1}{n}. \tag{1}$$

Now if k is even and n is even, then any elements of S which will be fixed by $\overline{m}$ must be symmetric about it's center. Thus, since k is even, the first $\frac{k}{2}$ digits must be the same as the last $\frac{k}{2}$ digits (in reverse order). Thus, the $\frac{k}{2}$ digits must add up to $\frac{n}{2}$. The number of combinations of $\frac{k}{2}$ digits summing to $\frac{n}{2}$ is given by

$$\binom{n/2+k/2-1}{n/2}.$$

Since no other element of S remains fixed by $\overline{m}$, we have

$$|F(\overline{m})| = \binom{n/2+k/2-1}{n/2}.$$

Hence, we have

$$k \text{ even, } n \text{ even } \Rightarrow N = \frac{1}{2}\left[\binom{n+k-1}{n} + \binom{n/2+k/2-1}{n/2}\right]. \tag{2}$$

Now if k is odd and n is odd, then any element of S which will be fixed by $\overline{m}$, must have an odd digit d_c in the center. That leaves $\frac{k-1}{2}$ digits on each side of the center digit d_c. To be fixed by $\overline{m}$, these $\frac{k-1}{2}$ digits must be the same on each side. They must also sum to $\frac{n-d_c}{2}$. Thus, the number of elements in S with an odd digit d_c in the center which are fixed by $\overline{m}$ is given by:

$$\binom{(n-d_c)/2+(k-1)/2-1}{(n-d_c)/2}.$$

Since d_c can be any odd number less than or equal to n, we see that

$$|F(\overline{m})| = \sum_{i=0}^{(n-1)/2} \binom{[n-(2i+1)]/2+(k-1)/2-1}{(n-(2i+1))/2}.$$

Hence, we have

$$k \text{ odd}, n \text{ odd } \Rightarrow N = \frac{1}{2}\left[\binom{n+k-1}{n} + \sum_{i=0}^{(n-1)/2} \binom{[(n-(2i+1)]/2+(k-1)/2-1}{(n-(2i+1))/2}\right]. \quad (3)$$

Now if k is odd and n is even, then any element of S which will be fixed by $\overline{m}$ must have an even digit d_c in the center. That leaves $\frac{k-1}{2}$ digits on each side of the center digit d_c. To be fixed by $\overline{m}$, these $\frac{k-1}{2}$ digits must be the same on each side. They must also sum to $\frac{n-d_c}{2}$. Thus, the number of elements in S with an even digit d_c in the center which are fixed by $\overline{m}$ is given by

$$\binom{(n-d_c)/2+(k-1)/2-1}{(n-d_c)/2}.$$

Since d_c can be any even number less than or equal to n, we see that

$$|F(\overline{m})| = \sum_{i=0}^{n/2} \binom{(n-2i)/2+(k-1)/2-1}{(n-2i)/2}.$$

Hence, we have

$$k \text{ odd}, n \text{ even } \Rightarrow N = \frac{1}{2}\left[\binom{n+k-1}{n} + \sum_{i=0}^{n/2} \binom{(n-2i)/2+(k-1)/2-1}{(n-2i)/2}\right]. \quad (4)$$

Equations (1), (2), (3) and (4) were derived using k-tuples. Also we allowed the digits d_i to be zero and $\sum_{i=1}^{k} d_i = n$. If we do not permit digits to be zero instead of formula

$$\binom{n+k-1}{n} \quad (*)$$

we would use formula

$$\binom{(n-k)+k-1}{n-k} = \binom{n-1}{n-k}. \quad (**)$$

If the sum of the digits is $m-2$ instead of n and we have j positions instead of k positions, we would adjust formula (**) in the following manner:

$$\binom{m-2-j+j-1}{m-2-j} = \binom{m-3}{m-j-2}. \quad (***)$$

Making the corresponding adjustments in equations (1), (2), (3) and (4) we get

$$n(C(m,j)) = \begin{cases} \frac{1}{2}\binom{m-3}{m-j-2} & j \text{ even, } m \text{ odd} \\ \frac{1}{2}\left[\binom{m-3}{m-j-2} + \binom{\frac{m-4}{2}}{\frac{m-j-2}{2}}\right] & j \text{ even, } m \text{ even} \\ \frac{1}{2}\left[\binom{m-3}{m-j-2} + \sum_{i=0}^{\frac{m-j-2}{2}} \binom{\frac{m-5-2i}{2}}{\frac{m-j-2-2i}{2}}\right] & j \text{ odd, } m \text{ odd} \\ \frac{1}{2}\left[\binom{m-3}{m-j-2} + \sum_{i=0}^{\frac{m-j-3}{2}} \binom{\frac{m-6-2i}{2}}{\frac{m-j-3-2i}{2}}\right] & j \text{ odd, } m \text{ even.} \end{cases}$$

□

For m where $3 \leq m \leq 11$, we arrange the values of $n(C(m,j))$ in a triangular array similar to Pascal's triangle (see display 1). The $m-2$ row contains $n(C(m,1))$ through $n(C(m,m-2))$ in order from left to right.

3	1								
4	1	1							
5	1	1	1						
6	1	2	2	1					
7	1	2	4	2	1				
8	1	3	6	6	3	1			
9	1	3	9	10	9	3	1		
10	1	4	12	19	19	12	4	1	
11	1	4	16	28	38	28	16	4	1

It is interesting to note the various number patterns in this triangle. To obtain n($n(\bigcup_{j=1}^{m-2} C(m,j))$ we compute $\sum_{j=1}^{m-2} n(C(m,j))$.

Previously the term caterpillar has been defined in abstract graph theory. A caterpillar is an abstract graph which is a tree with at least three vertices such that the tree T' formed by deleting all the endpoints is a path. The number of caterpillars with $n+4$ vertices is equal to $N(n) = 2^n + 2^{\lfloor \frac{n}{2} \rfloor}$ (see [4]). However, this counting scheme does not work for our purposes. For instance, $V = \{v_0v_1, v_1v_2, v_1v_3\}$ and $V' = \{v_0v_1, v_1v_1', v_1'v_2, v_1v_3\}$ are distinct caterpillars. However, the geometric realization of each of these caterpillars is a simple 3-odd.

For other related results see [3].

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