Symmetric designs, sets with two intersection numbers and Krein parameters of incidence graphs*

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Abstract. Let $(\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a symmetric (v, k, λ) block design. The incidence graph G of this design is distance-regular, hence belongs to an association scheme. In this paper, we use the algebraic structure of this association scheme to analyse certain symmetric partitions of the incidence structure.
A set with two intersection numbers is a subset $\mathcal{K} \subseteq \mathcal{P}$ with the property that $|B \cap \mathcal{K}|$ takes on only two values as B ranges over the blocks of the design. In the special case where the design is a projective plane, these objects have received considerable attention. Two intersection theorems are proven regarding sets of this type which have a certain type of dual. Applications to the study of substructures in finite projective spaces of dimensions two and three are discussed.

1 Introduction

Consider a *symmetric block design* $(\mathcal{P}, \mathcal{B}, \mathcal{I})$. Here $\mathcal{P}$ denotes a set of v *points*, $\mathcal{B}$ denotes a set of v *blocks*, and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$ is an incidence relation with the following properties: any point (resp., block) is incident with k blocks (resp., k points); any two points (resp., blocks) are incident with exactly λ common blocks (resp., λ common points). Examples include the point-hyperplane incidence structure of any finite projective geometry and designs arising from Hadamard matrices.

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A *set of type* (s,t) in our design is a subset $\mathcal{K}$ of $\mathcal{P}$ with the property that every block of the design is incident to either s or t points of $\mathcal{K}$. In the special case where $(\mathcal{P},\mathcal{B},\mathcal{I})$ is a projective plane, such sets have been studied extensively. See [9] and the references cited therein.

Example: In an arbitrary symmetric design, consider a pair of distinct points, p and q. The *line* through p and q is the collection of all points r having the property

$$(p\mathcal{I}B \text{ and } q\mathcal{I}B) \Rightarrow r\mathcal{I}B$$

for every block $B \in \mathcal{B}$. Let ℓ_{pq} denote the line through p and q. Then $|\ell_{pq}| \leq (v-\lambda)/(k-\lambda)$. (See [8, p15].) A line meeting this bound with equality is called *maximal.* It is known that ℓ_{pq} is maximal if and only if every block is incident with some point of ℓ_{pq}. Thus a maximal line is a set of type $(1, \frac{v-\lambda}{k-\lambda})$ in this design. □

The following is well-known. If $\mathcal{K}$ is a set of type (s,t), then $\mathcal{P}-\mathcal{K}$ is a set of type $(k-s, k-t)$. Moreover, there are two kinds of blocks, depending on incidence with $\mathcal{K}$, and each of these is a set of type (s',t') in the dual symmetric design $(\mathcal{B},\mathcal{P},\mathcal{I}^T)$ for some integers s' and t', depending on which set is considered.

In this paper, we examine a special case of this scenario using two elementary results from the theory of association schemes. The incidence graph of any symmetric design is distance-regular. Thus its adjacency algebra is a Bose-Mesner algebra. The main observation of this paper is roughly as follows: the fact that a certain Krein parameter of this association scheme is equal to zero has non-trivial combinatorial consequences for the intersection properties of certain subconfigurations in the design. It is hoped that this technique will someday be applied more widely, for instance in the study of block designs or error-correcting codes.

The paper is laid out as follows. Two intersection theorems for configurations in association schemes are proved in Section 2. In Section 3, we review basic information regarding the incidence graph of a symmetric design. Next, we explain in Section 4 how the configurations we study arise from a special class of sets with two intersection numbers. In Section 5, we give examples and demonstrate how the intersection theorems apply in the case of finite projective planes and projective geometries.

2 Vanishing Krein parameters

Let X be a finite set of size $m > 0$ and let $\mathcal{A} = \{A_0, \ldots, A_d\}$ be a set of $d+1$ symmetric 01-matrices with rows and columns indexed by X. We say $(X,\mathcal{A})$ is a *symmetric association scheme* provided

(i) $A_0 = I$;
(ii) $\sum A_i = J$, the all-ones matrix of order m;
(iii) for any choice of i and j ($0 \le i, j \le d$), the matrix product $A_i A_j$ lies in the linear span of $\mathcal{A}$.

The vector space spanned by $A_0, \ldots, A_d$ is called the *Bose-Mesner algebra* of the association scheme $(X, \mathcal{A})$. This is a $(d+1)$-dimensional commutative algebra of symmetric $m \times m$ matrices which is also closed under Schur (entrywise) multiplication and contains both I and the all-ones matrix J. This algebra has a unique basis $\{E_0, \ldots, E_d\}$ of primitive idempotents ($E_i E_j = \delta_{i,j} E_i$). By convention, we take $E_0 = \frac{1}{m} J$. If $\circ$ denotes the Schur product of two matrices, then the *Krein parameters* of the scheme are defined as the structure constants given by

$$E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^{d} q_{ij}^k E_k. \tag{1}$$

We use the notation and results of Chapter 2 in [3]. The following lemma is an easy generalization of Equation (9) on p61 of [3] (cf. Equation (2.28) in [6]).

Lemma 1. *Let $(X, \mathcal{A})$ be a d-class symmetric association scheme and $u \in \mathbb{R}^m$. Then, for $0 \le i, j \le d$,*

$$||E_i \Delta_u E_j||^2 = \frac{1}{|X|} \sum_{k=0}^{d} q_{ij}^k \, u^T E_k \, u$$

where Δ_u denotes the diagonal matrix with (x,x)-entry equal to u_x and $||M|| = \sqrt{\mathrm{trace}(M^* M)}$ denotes the ℓ_2-norm of a matrix M.

Proof. We have

$$\begin{aligned} ||E_i \Delta_u E_j||^2 &= \mathrm{trace}\,\{E_j \Delta_u E_i \Delta_u E_j\} \\ &= \sum_{y \in X} \sum_{z \in X} u_y u_z (E_i)_{yz} \sum_{x \in X} (E_j)_{xy} (E_j)_{zx} \\ &= \sum_{y \in X} \sum_{z \in X} u_y u_z (E_i)_{yz} (E_j)_{yz} \\ &= \frac{1}{|X|} \sum_{k=0}^{d} q_{ij}^k \sum_{y} \sum_{z} u_y (E_k)_{yz} u_z \\ &= \frac{1}{|X|} \sum_{k=0}^{d} q_{ij}^k u^T E_k u. \quad \Box \end{aligned}$$

A well-known property of Krein parameters is the following: $q^{\ell}_{ij} = 0$ if and only if $q^{i}_{\ell j} = 0$ (see [3, Lemma 2.3.1(iv)]). The following theorem is originally due to Cameron, Goethals, and Seidel [5, Theorem 5.1]. (See also Proposition II.8.3(i) in [1].) We supply an elementary proof based on the above lemma. Here, V_j denotes the j^{th} eigenspace, i.e., the column space of the idempotent E_j.

Theorem 1. *If $u \in V_\ell$ and $v \in V_j$ and $q^{\ell}_{ij} = 0$, then $u \circ v$ is orthogonal to V_i where $u \circ v$ denotes the entrywise product of vectors u and v.*

Proof. We have $v = E_j v$ and $u \circ v = \Delta_u v$. So it suffices to show that $E_i \Delta_u E_j = 0$. Now $u \in V_\ell$, so

$$E_k u = \delta_{\ell,k} u \ .$$

Since $q^{\ell}_{ij} = 0$, Lemma 1 gives the result. □

For a subset $C \subseteq X$ with characteristic vector x, define the *dual degree set* of C as

$$s(C) = \{j : E_j x \neq 0\}.$$

This term is due to Godsil. For a fixed d-class association scheme $(X, \mathcal{A})$ and $F, G \subseteq \{0, \ldots, d\}$, define

$$F * G = \{k : q^{k}_{ij} > 0 \text{ for some } \ i \in F, j \in G\}$$

(cf. Zieschang [11]). Here are our two intersection theorems.

Theorem 2. *For any two subsets, C and D, of X, we have*

$$s(C \cap D) \subseteq s(C) * s(D).$$

Proof. Denote the characteristic vectors of C and D by x and y, respectively. Express each as a sum of its projections onto the various eigenspaces:

$$x = \sum_{j \in s(C)} x_j, \qquad y = \sum_{j \in s(D)} y_j$$

where $x_j = E_j x$ and $y_j = E_j y$. The characteristic vector of $C \cap D$ is

$$x \circ y = \sum_{i \in s(C)} \sum_{j \in s(D)} x_i \circ y_j.$$

Theorem 1 guarantees that all terms appearing in the double sum on the right-hand side lie in

$$\bigoplus_{k \in s(C) * s(D)} V_k. \ \square$$

Theorem 3 (Roos [10, Corollary 3.3]). *Let C and D be subsets of X. If $s(C)$ and $s(D)$ contain no common element other than zero, then $|C \cap D| = |C| \cdot |D|/|X|$.*

Proof. Recall that $E_0 = |X|^{-1}J$ where J is the all-ones matrix. Thus, for any set $S \subseteq X$ with characteristic vector z, $E_0 z = (|S|/|X|)\mathbf{1}$ where $\mathbf{1}$ denotes the vector of all ones.

Suppose C has characteristic vector x and D has characteristic vector y. Then $C \cap D$ has characteristic vector $x \circ y$. As in the proof of the previous theorem, we express each of x and y as a sum of its projections into the various eigenspaces:

$$x = \sum_{j \in s(C)} x_j, \qquad y = \sum_{j \in s(D)} y_j.$$

Now, with $\hat{s}(C) = s(C) - \{0\}$ and $\hat{s}(D) = s(D) - \{0\}$,

$$x \circ y = (x_0 \circ y_0) + \sum_{j \in \hat{s}(D)} x_0 \circ y_j + \sum_{j \in \hat{s}(C)} x_j \circ y_0 + \sum_{i \in \hat{s}(C)} \sum_{j \in \hat{s}(D)} x_i \circ y_j.$$

Now $q_{ij}^0 = 0$ unless $i = j$, so $(x_i \circ y_j) \perp V_0$ unless $i = j = 0$ (Theorem 1). Hence, $E_0(x \circ y) = x_0 \circ y_0$ and the result follows from the observation above. □

3 The incidence graph of a symmetric design

Let $(\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a symmetric design with incidence graph G. That is, G has vertex set $X = \mathcal{P} \cup \mathcal{B}$ and edge set $\mathcal{I} \cup \mathcal{I}^T$. This is a bipartite distance-regular graph of diameter three with intersection array

$$\{k, k-1, 1; 1, \lambda, k\} .$$

Let A denote the adjacency matrix of G. The eigenvalues of A are $\theta_0 = k$, $\theta_1 = \sqrt{n}$, $\theta_2 = -\sqrt{n}$ and $\theta_3 = -k$ where we define $n = k - \lambda$. We order the eigenspaces V_0, V_1, V_2, V_3 of A according to decreasing eigenvalue: $Av = \theta_j v$ for $v \in V_j$.

Since G is distance-regular, the algebra generated by A is a Bose-Mesner algebra. It has two standard bases:

$$\{A_0 = I, A_1 = A, A_2, A_3\} \quad \text{and} \quad \{E_0, E_1, E_2, E_3\} ,$$

where, as usual, A_i is the i^{th} distance matrix of G. The distance-two graph is a union of two cliques of size v and the distance-three graph is the incidence graph of the complementary symmetric design. The matrix E_j represents orthogonal projection onto V_j. Clearly $E_0 = \frac{1}{2v}J$. We note that $E_3 = \frac{1}{2v}J \otimes \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is also a rank one matrix. It is thus easy to verify that the following Krein parameters vanish: $q_{ij}^3 = 0$ unless $i + j = 3$.

4 Sets with two intersection numbers

Let G denote the incidence graph of design $(\mathcal{P}, \mathcal{B}, \mathcal{I})$. An *equitable partition* of G is a partition π of the vertex set X having the property that, for any two cells C and C' of π there exists a constant $r_{C,C'}$ such that every vertex in C is adjacent to exactly $r_{C,C'}$ vertices in C'. A simple but useful lemma is the following:

Lemma 2 ([7, Lemma 5.2.2]). *If π is an equitable partition of G and B is the matrix with rows and columns indexed by the cells of π having (C, C')-entry equal to $r_{C,C'}$, then every eigenvalue of B is an eigenvalue of the graph G.* □

The matrix B appearing in the lemma is called the *quotient matrix* of the partition. For more information on equitable partitions, see Chapter 5 in [7].

We are interested in equitable partitions of the incidence graph G having exactly four cells, two of which partition $\mathcal{P}$. (In fact, this latter condition is superfluous.) Such a partition will be denoted $\pi = \{P_1, P_2, B_1, B_2\}$. See Figure 1. The parameters for this partition will be denoted r_{ij} and k_{ij}: each point in P_i is incident to r_{ij} blocks from B_j and each block in B_j is incident to k_{ji} points in P_i. As a straightforward extension of the identities listed in [9], we have the following equations, with $p_i = |P_i|$ and $b_i = |B_i|$:

$$r_{11} + r_{12} = k, \tag{2}$$

$$r_{21} + r_{22} = k, \tag{3}$$

$$r_{21}k_{11} + r_{22}k_{21} = \lambda p_1, \tag{4}$$

$$r_{11}(k_{11} - 1) + r_{12}(k_{21} - 1) = \lambda(p_1 - 1). \tag{5}$$

Any solution $\{r_{ij} : 1 \le i, j \le 2\}$ to this system also satisfies

$$r_{12}k_{22} + r_{11}k_{12} = \lambda p_2, \tag{6}$$

$$r_{21}(k_{12} - 1) + r_{22}(k_{22} - 1) = \lambda(p_2 - 1) \tag{7}$$

given that $p_2 = v - p_1$, $k_{12} = k - k_{11}$ and $k_{22} = k - k_{21}$. Here are expressions for all remaining parameters in terms of k, n, k_{11}, k_{21}, and p_1:

$$r_{11} = \frac{k_{21}k - \lambda p_1 - n}{k_{21} - k_{11}}, \tag{8}$$

$$r_{12} = \frac{\lambda p_1 + n - k_{11}k}{k_{21} - k_{11}}, \tag{9}$$

$$r_{21} = \frac{k_{21}k - \lambda p_1}{k_{21} - k_{11}}, \tag{10}$$

$$r_{22} = \frac{\lambda p_1 - k_{11}k}{k_{21} - k_{11}}, \tag{11}$$

$$b_1 = p_1 r_{11}/k_{11}, \tag{12}$$

$$b_2 = p_1 r_{12}/k_{21}. \tag{13}$$

Upon simplification, we find $r_{12} - r_{22} = n/(k_{21} - k_{11})$ with the fundamental consequence that $k_{21} - k_{11}$ is a divisor of the order n of the design.

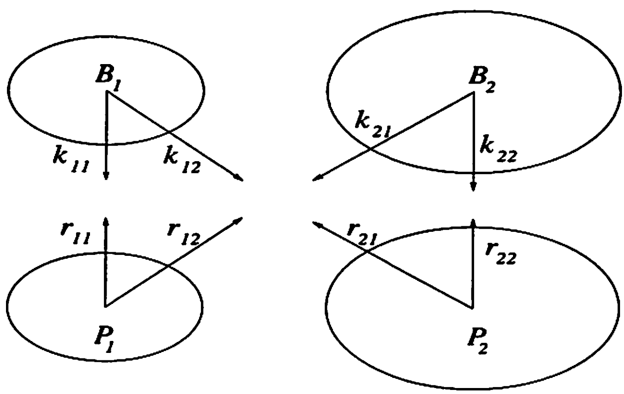

Fig. 1. A typical partition arising from a set of type (k_{11}, k_{21}).

Next, observe that b_1 and b_2 satisfy

$$b_1 + b_2 = v \tag{14}$$

$$k_{11}b_1 + k_{21}b_2 = kp_1 \tag{15}$$

$$k_{11}(k_{11} - 1)b_1 + k_{21}(k_{21} - 1)b_2 = \lambda p_1(p_1 - 1). \tag{16}$$

If we multiply equation (14) by $k_{11}k_{21}$, multiply equation (15) by $1 - k_{11} - k_{21}$ and add these two results to equation (16), we find

$$\lambda p_1^2 - [\lambda + k(k_{11} + k_{21} - 1)]p_1 + k_{11}k_{21}v = 0. \tag{17}$$

Thus, given k_{11} and k_{21}, there are at most two choices for $|P_1|$ and these must be integers. The above numerology shows how to determine all remaining parameters from these.

The quotient matrix B of this equitable partition π is

$$B = \begin{pmatrix} 0 & 0 & r_{11} & r_{12} \\ 0 & 0 & r_{21} & r_{22} \\ k_{11} & k_{12} & 0 & 0 \\ k_{21} & k_{22} & 0 & 0 \end{pmatrix}.$$

It is easy to check that any solution to the above equations gives eigenvalues $\pm k, \pm\sqrt{n}$, precisely the eigenvalues of G.

Lemma 3. *The partition $\sigma = \{P_1 \cup B_1, P_2 \cup B_2\}$ of X is an equitable partition of G if and only if $k_{12} = r_{12}$ and either: (i) $k_{11} \neq 0$; or, (ii) $k_{11} = 0$ and $k_{21} = \sqrt{n}$.*

Proof. Clearly, if σ is equitable, then $k_{12} = r_{12}$. If this holds and $k_{11} = 0$, then the quotient matrix of this partition is

$$\begin{pmatrix} 0 & k \\ k_{21} & k - k_{21} \end{pmatrix}.$$

The trace of this matrix is $k - k_{21}$, hence its eigenvalues are k and $-k_{21}$. Since $-k_{21}$ must be an eigenvalue of G by Lemma 2, we have $k_{21} = \sqrt{n}$. Conversely, if $k_{11} \neq 0$, then $k_{11} = r_{11}$ forces $b_1 = p_1$ by simple double counting. Now it is easy to see that $k_{21} = r_{21}$ and σ is equitable. On the other hand, if $k_{11} = 0$ and $k_{21} = s$ with $s = \sqrt{n}$, then $p_1 = s(k-s)/(k-s^2)$ from (17) and

$$b_1 = v - b_2 = \frac{k^2 - s^2}{k - s^2} - \frac{kp_1}{s} = p_1.$$

Again, this is enough to guarantee that σ is equitable. □

In this paper, we want to focus on partitions which satisfy the conditions of Lemma 3. (Of course, with re-labelling, this also includes the case where $k_{22} = r_{11}$.) In this case, the quotient matrix of this coarser partition is

$$B' = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}.$$

Note that the same matrix is found by replacing r_{ij} by k_{ij} and our original 4×4 quotient matrix can be expressed as $B' \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The eigenvalues of B' are k and $r_{11} - r_{21}$. Since the eigenvalues of G are $\pm k$ and $\pm\sqrt{n}$, we must have that n is a square and $r_{11} - r_{21} = \pm\sqrt{n}$. So we henceforth assume that $n = s^2$ for some positive integer s. (By the Bruck-Ryser-Chowla Theorem, this is always the case when v is even.) This gives $\lambda = k - s^2$ and $v = (k^2 - s^2)/(k - s^2)$.

Corollary 1. *If the conditions of Lemma 3 are satisfied, then n is a perfect square.* □

The configurations of interest to us fall into two classes, according as $r_{11}+r_{22}=k+s$ or $r_{11}+r_{22}=k-s$. Before looking at some examples, we make a few observations about the parameters of these configurations.

In the event that $r_{11}+r_{22}=k+s$, we abbreviate r_{11} to r and we can write,

$$B' = \begin{pmatrix} r & k-r \\ r-s & k+s-r \end{pmatrix}.$$

Since $r_{21} \geq 0$, we know $r \geq s+1$. Our equitable partition has cells $C := P_1 \cup B_1$ and $X - C$. Counting the edges between C and $X - C$ in two ways gives

$$|C|(k-r) = (2v - |C|)(r-s)$$

which yields an expression for $|C|$:

$$|P_1 \cup B_1| = \frac{2(k+s)(r-s)}{\lambda}.$$

Similarly, for the case $r_{11}+r_{22}=k-s$, we have

$$B' = \begin{pmatrix} r & k-r \\ r+s & k-s-r \end{pmatrix}$$

so $r \leq k-s$. Furthermore, in this case, we find

$$|C| = \frac{2(k-s)(r+s)}{\lambda}$$

where, again, $C := P_1 \cup B_1$.

5 Intersecting V_1-sets and V_2-sets

The one-dimensional eigenspace V_3 of G is spanned by the vector $x_{\mathcal{P}} - x_{\mathcal{B}}$ where $x_{\mathcal{P}}$ denotes the characteristic vector of $\mathcal{P}$ and $x_{\mathcal{B}}$ denotes the characteristic vector of $\mathcal{B}$. Let us say that a subset $C \subseteq X$ is *square* if C contains equally many points and blocks.

Observation: A subset $C \subseteq X$ is square if and only if its characteristic vector is orthogonal to V_3.

A subset $C \subseteq V$ with characteristic vector x_C is a *V_1-set* if $x_C \in V_0 \oplus V_1$ and a *V_2-set* if $x_C \in V_0 \oplus V_2$. Observe that each V_1-set and each V_2-set is square. With the notation of the previous section, also observe that the two cells of the partition σ are both V_1-sets if $r_{11}+r_{22}=k+s$ and are both V_2-sets if $r_{11}+r_{22}=k-s$. Conversely, if C is a V_1-set or a V_2-set, then $\{C, X-C\}$ is an equitable partition as in Lemma 3. This follows from a result of Delsarte [6] (or see [3, Theorem 11.1.1(iv)]).

Corollary 2. *If C and C' are both V_1-sets (or both V_2-sets), then $C \cap C'$ is square.*

Proof. Suppose C and C' are both V_1-sets. We have $q_{11}^3 = 0$. So by Theorem 2 the characteristic vector of $C \cap C'$ is orthgonal to V_3. Thus $C \cap C'$ is square. Since $q_{22}^3 = 0$, the same argument applies when both C and C' are V_2-sets. □

Corollary 3. *If C is a V_1-set and C' is a V_2-set, then*

$$|C \cap C'| = \frac{|C| \cdot |C'|}{2v}.$$

Proof. Apply Theorem 3. □

Examples:

1. Consider a projective geometry of dimension d and order $n = s^2$. (In the case $d = 2$, this may or may not be Desarguesian.) A *Baer subgeometry* consists of a subset C of $(s^{d+1}-1)/(s-1)$ points and $(s^{d+1}-1)/(s-1)$ blocks such that the induced incidence structure is a projective geometry of the same dimension and order s. It is easy to check that the set C is a V_1-set with quotient matrix
$$\begin{pmatrix} s+1 & s^2-s \\ 1 & s^2 \end{pmatrix}.$$
2. Consider a projective geometry $PG(d,q)$ with d odd. In the affine space $\mathbb{F}_q^{2(d+1)}$, let T be a $(d+1)$-dimensional subspace. Consider the set C consisting of all one-dimensional spaces contained in T and all $(2d+1)$-dimensional spaces containing T. This is a V_1-set in the incidence graph of $PG(d,q)$ with $k_{11} = p_1$. Further examples of V_1-sets can be obtained by choosing a collection $T_1, \ldots, T_i$ of mutually skew $(d+1)$-dimensional spaces and taking the union of the corresponding sets C. In $PG(3,q)$, hyperbolic quadrics arise as a special case: here C consists of $(q+1)^2$ points and $(q+1)^2$ planes.
3. In $PG(3,q)$, an *ovoid* is a set $\mathcal{O}$ of q^2+1 points such that any plane is either tangent to $\mathcal{O}$ or meets it in $q+1$ points. It is easy to see that there is a unique tangent plane at any point of $\mathcal{O}$. So the set C consisting of all points in $\mathcal{O}$ and all planes tangent to $\mathcal{O}$ forms a V_2-set in the incidence graph of $PG(3,q)$.
4. In a projective plane of order $n = s^2$, a *unital* is a set $\mathcal{U}$ of s^3+1 points having the property that every line meets $\mathcal{U}$ in either one or $s+1$ points. The lines of the former type are called *tangents* to $\mathcal{U}$ and it transpires that there are precisely s^3+1 such tangents, forming a unital in the dual plane. If C is the set consisting of all points in $\mathcal{U}$ and all tangents to $\mathcal{U}$, then C is a V_2-set in the incidence graph of the plane.

5. In a projective plane of order $n = s^2$, a *maximal s-arc* is a set $\mathcal{A}$ of $s(s^2 - s + 1)$ points having the property that every line either meets $\mathcal{A}$ in s points or is passant to $\mathcal{A}$. It turns out that there are precisely $s(s^2 - s + 1)$ passants to $\mathcal{A}$, forming a maximal s-arc in the dual plane. If C is the set consisting of all points in $\mathcal{A}$ and all passants to $\mathcal{A}$, then C is a V_2-set in the incidence graph of the plane.
6. In the symmetric (16, 6, 2)-design defined on $\mathbb{Z}_4^2$, the points and blocks of a symmetric (4, 3, 2) subdesign give us a V_1-set with quotient matrix
$$\begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix}.$$
In this same design, we can find a set of type (0, 2) which gives rise to a V_2-set of size eight. This design also admits a V_1-set with quotient matrix $2(I + J)$.
7. In $PG(2, 4)$, it is possible to find two disjoint hyperovals and six secants of each which form two disjoint hyperovals in the dual plane. These 24 elements give us a V_2-set in the incidence graph of $PG(2, 4)$.

We now give five applications of Corollaries 2 and 3 to some of the objects described in the above examples.

Corollary 4 (Bruen [4]). *In a projective geometry of square order, any two Baer subgeometries have equally many points and hyperplanes in common.* □

It is interesting to remark that this result has been obtained with very little information about the structure of $PG(d, s^2)$. In particular, when $d = 2$, the proof applies equally to the Desarguesian and non-Desarguesian cases. In this case, the result is originally due to Bose, Freeman, and Glynn [2]. Note, however, that both [4] and [2] obtain structural information about the intersection beyond what is stated here.

Corollary 5. *In a plane of square order $q = s^2$, any set C consisting of a unital and its tangents contains exactly $2s + 2$ objects in common with any Baer subplane.* □

Corollary 6. *In a plane of square order $q = s^2$, any set C consisting of a maximal s-arc and its passants contains exactly $2s$ objects in common with any Baer subplane.* □

Corollary 7. *In a plane of square order $q = s^2$, if each of C and C' consists of either a unital plus its tangents or a maximal s-arc plus its passants, then $C \cap C'$ contains equally many points and lines.* □

Corollary 8. *Let $\mathcal{O}$ and $\mathcal{O}'$ be two ovoids in $PG(3, q)$ having t points in common. Then $\mathcal{O}$ and $\mathcal{O}'$ have exactly t tangent planes in common as well.* □

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