On Group-Magic Graphs

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Abstract: For any abelian group A, we call a graph G=(V,E) as A-magic if there exists a labeling l:E(G) → A−{0} such that the induced vertex set labeling l^+: V(G) → A

$$l^+(v)=\Sigma\ \{l(u,v) : (u,v) \text{ in } E(G)\}$$

is a constant map. We denote the set of all A such that G is A-magic by AM(G) and call it as group-magic index set of G..

Key words: Abelian group, group-magic graph, perfect matching, group-magic index sets.

1. Introduction. For any abelian group A, written additively we denote A* = A−{0}. Any mapping l: E(G) → A* is called a labeling. Given a labeling on edge set of G we can induced a vertex set labeling l^+: V(G) → A as follows:

$$l^+(v)=\Sigma\ \{l(u,v) : (u,v) \text{ in } E(G)\}$$

A graph G is known as A-magic if there is a labeling l: E(G) → A* such that for each vertex v, the sum of the labels of the edges incident with v are all equal to the same constant; i.e., $l^+(v)$ = c for some fixed c in A. We will called <G,l> a A-magic graph with sum c. In general, a graph G may admits more than one labeling to become a A-magic graph. We denote the class of all graphs (either simple or multiple graphs) by **Gph**. For each abelian group A we denote the class of all A-magic graphs by $_A$**MGp**. We denote the class of all abelian groups by **Ab**.

When A = Z, the Z-magic graphs were considered in Stanley [22,23]; he pointed out that the theory of magic labelings can be put into the more general context of linear homogeneous diophantine equations.

When the group is Z_k, we shall refer to the Z_k –magic graph as *k-magic*. Graphs which are k-magic had been studied in [2,6,9,10,12,15]. For convenience, we will consider Z-magic as 1-magic. A Z-magic graph <G.l> is caled supermagic if l(E(G))={1,2,...,|E(G)|}. The following wheel is Z_8-magic and Z-magic. (Figure 1)

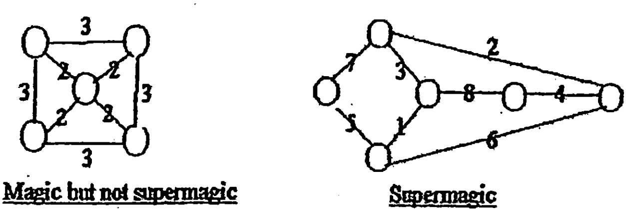


Figure 1.

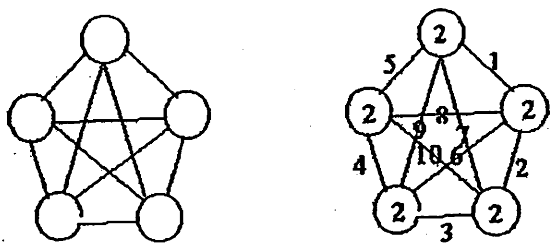
Doob [2] also considered A-magic graphs where A is an abelian group. He determined which wheels are Z-magic.

Given the graph G , the problem of deciding whether G admits a magic labeling is equivalent to the problem of deciding whether a set of linear homogeneous Diophantine equation has a solution [23]. At present, given an abelian group, no general efficient algorithm is known for finding magic labelings for general graphs.

The original concept of A-magic graph is due to J. Sedlacek [19, 20], who defined it to be a graph with real-valued edge labeling such that (i) distinct edges have distinct nonnegative labels, and (ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

It is well-known that a graph G is Z-magic if and only if each edge of G is contained in a 1-factor (a perfect matching) or a {1,2}-factor. Some special classes of Z-magic graphs had been considered in the literature.(see [5],[10],[12],[24],[25],[26]). A study of magic strengths of graphs are carried out in [10, 12].

Stewart called a Z-magic graph $\langle G,l \rangle$ as supermagic if $l(E(G))=\{1,2,\ldots,|E(G)|\}$. He showed that complete graphs K_5 is not supermagic and when $n \equiv 0 \bmod 4$, K_n is not supermagic. A (p,q)-graph G= (V, E) with p vertices and q edges is called **edge-magic** if there is a bijection $f: E \to \{1,2,\ldots,q\}$ such that the induced mapping $f^+: V \to Z_p$, given by $f^+(u) = \Sigma\{f(u,v): (u,v) \text{ in } E\} \pmod p$ is a constant mapping. The concept of edge-magic graphs was introduced by Lee, Seah and Tan in 1992 [13]. Clearly a supermagic graph



not super-magic **edge-magic**

Figure 2

is edge-magic but not conversely (Figure 2).

Supermagic graphs had been considered in [5, 24,25]. For other works on edge-magic graphs the readers are reffered to [13, 14,21]

2. k-magic Graphs. When the group is Z_k, we shall refer to the Z_k –magic graph as *k-magic* . There exits graph G which is not k-magic for any k.

Definition 1. A graph G is called nonmagic if it is not A-magic for any group A in Ab.

Lemma 1. P_3 is nonmagic.

Proof. For, if one labels the two edges by x and y then we have

$$x+y=x=y$$

and x and y must be 0.

We recall that a star St(k) is a tree with k+1 vertices with k pairs of leaf vertices. A tree T is called a spider if there exists a vertex r such that $deg(v) \leq 2$, for all v in V(T) $-\{r\}$.

Corollary 2. All spiders except stars are non-magic.

In fact, we can see that any graph G is an induced subgraph of a non-magic graph $G^{\#}$. We can extend G to $G^{\#}$ by glue the end vertex of P, to any vertex u of G the resulting graph $(G,u) \circ P_3 = G\#$ is nonmagic. Thus we have

Theorem 3. Any graph G is an induced subgraph of a non-magic graph $G^{\#}$.

We have the following characterization of 2-magic graphs.

Theorem 4. A graph is 2-magic if and only if all the degrees of its nodes have the same parity; i.e., either all odd or all even.

Proof. It is obvious since we may label all edges by 1.

Corollary 5. All Eulerian graphs are 2-magic.

Theorem 6. A tree is 2-magic if and only if all its vertices have odd degree.

Proof. This is a special case of Theorem 4. Since the branches have degree 1, so all its nodes must have odd degrees.

For any $k \geq 2$, to classify k-magic graphs, we have the following useful sufficient condition:

Theorem 7. For any integer $k \geq 2$, if G is a graph with degree set $\{d_1, d_2, \ldots, d_m\}$ such that $d_i \equiv d_j \pmod{k}$ for all i, j, then G is k-magic.

Proof. We merely label each edge by 1.

Corollary 8. A tree is k-magic if and only each vertex has degree congruent to 1 modulo k.

Corollary 9. K-magic trees exist for all $k \geq 2$.

In general, it is difficult for trees to be k-magic. Since any graph with P_3 standing out is nonmagic, for any tree T, $T \circ P_3$ is a nonmagic tree

Corollary 10. For any k , l ≥2, there exist trees that are k-magic and not l –magic.

3. **All pseudo-omino graphs are k-magic for k>2.** In 1953 S.W. Golomb introduced the concept of polyominoes. They are shapes made by connecting certain numbers of equal-sized squares, each joined together with at least one other square along an edge and each square can travel either horizontally or vertically in one move successively to another square [4]. If the connected set squares has the property that each square besides travel horizontally or vetically , it can move diagonally to reach any other square in finite number of moves then it is called a pseudo-omino. A pseudo- omino with n squares is called pseudo-n-omino. We consider a class of connected planar graphs which are subgraphs of a grid-graph $P_m x P_n$ for some m≥1, n≥1 and named them as pseudo-omino graphs if they are planar representation of some pseudo-ominoes.

Examples of psudo-n-omino graphs:

(1) Grid-graphs $P_m x P_n$

(2) Young tableau graphs, $Y(n_1, n_2, \dots ,n_k)$

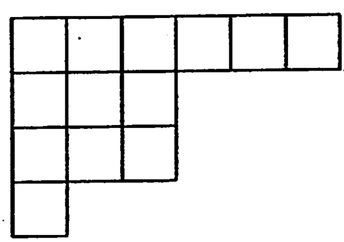


Figure 3. Y(6,3,3,1)

(3) Some pseudo-n-omino graphs.(Figure 4.)

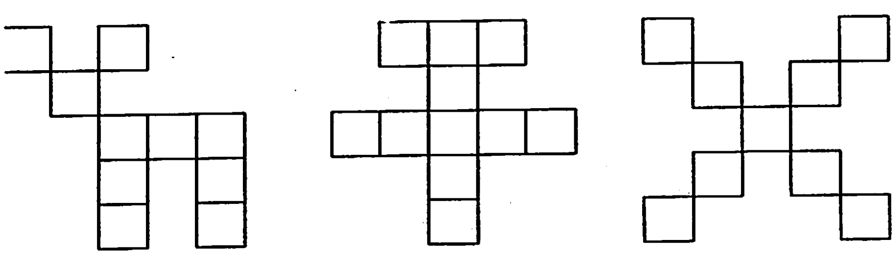


Figure 4.

Theorem 11 For any $n > 1$ and $k > 2$. All pseudo–n-omino graphs are k-magic.

Proof. For $k > 2$ and a given pseudo–n-omino graph G, we define the labeling scheme as follows:

First we label all the horizontal edges which are boundary of G by 1, and for each interior horizontal edges we label them by 2.

Then we label all the vertical edges which are boundary of G by k-1, and for each interior vertical edges we label them by k-2.

We see that the vertex set of G can be decompose into sets of three types:

(A) Those of degree 2 —— a vertex has a vertical boundary edge and a horizontal boundary edge, its vertex label is $1+(k-1) \equiv 0 \pmod{k}$.

(B) Those of degree 3 —— a vertex is either has two vertical boundary edges and a horizontal interior edge or has two horizontal boundary edges and a vertical interior edge. For the previous case the vertex has the label $(k-1) + (k-1) + 2 \equiv 0 \pmod{k}$. and for the later case it has $1 + 1 + k-2 \equiv 0 \pmod{k}$.

(C) Those of degree 4 —— a vertex is an interior point of G. It has two vertical interior edges and two horizontal interior edges. The vertex has the label $2 + 2 + k-2+k-2)\equiv 0 \pmod{k}$.

Thus G is k-magic.

We illustrate the above result by the following example.

Example 1. A pseudo-omino graph with Z_3-magic labeling. (Figure 5.)

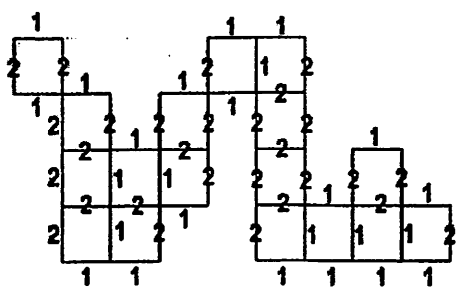


Figure 5.

Note: For $k=2$, the condition in Theorem 4 is also a sufficient condition for a graph G to be 2-magic. However, when $k>2$ we see from Theorem 11 that there exists k-magic graphs which are not satisfy the above condition

4. General Embedding Theorem.

Given any abelian group A, we want to show that any graph G in **Gph** can be embedded in a A-magic graph G^* as an induced subgraph.

Theorem 12. For any abelian group A, and any G in Gph with a labeling $f: E(G) \rightarrow A^*$, we can embedded G into a A-magic graph (G^*, f^*) such that $f^*|_G = f$.

Proof. Suppose $V(G)=\{x_1,\ldots,x_n\}$. Let $G x K_2$ where $V(K_2)=\{u,v\}$ is the complete graph. For the two subgraphs $G \times \{u\}$ and $G \times \{v\}$ we let them inherit the same labels of G. Let c be any element in A. If for any x_i in V(G), $f^+(x_i)=d \neq 0$. We define

$f^*(((x_i,u),(x_i,v)))=c-d$

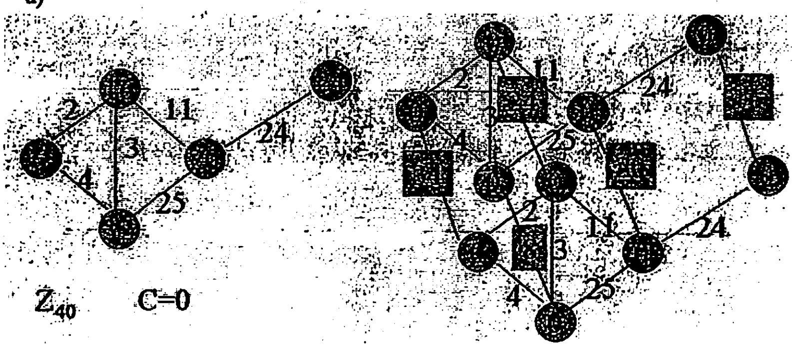
Let $G^*=G \times K_2$. We see that (G^*,f^*) is a A-magic graph. (see **Figure 6 (a)**)

Now assume some of $f^+(x_i)=0$> If the numbers of such i are even, we will connect all this vertices by a cycle. Then we label the edges by c, -c,c,-c,... successively For the edges ($(x_i,u),(x_i,v)$) we label it by c. We see that the graph $G^*=G \times K_2$ U the cycle is A-magic. (see Figure 6 (b)).

If the numbers of such i are odd, we will pick a vertex x_m with $f^+(x_m)=d \neq 0$ with all the other vertices xi with $f^+(x_i)=0$ and form an even cycle. Then we label the edges of this cycle by c, -c,c,-c,... successively. For the edges ($(x_i,u),(x_i,v)$) where $f^+(x_i)=0$ we label it by c. We see that the graph $G^*=(G \times K_2)$ U the cycle is A-magic.

Example 2.

a)



b)

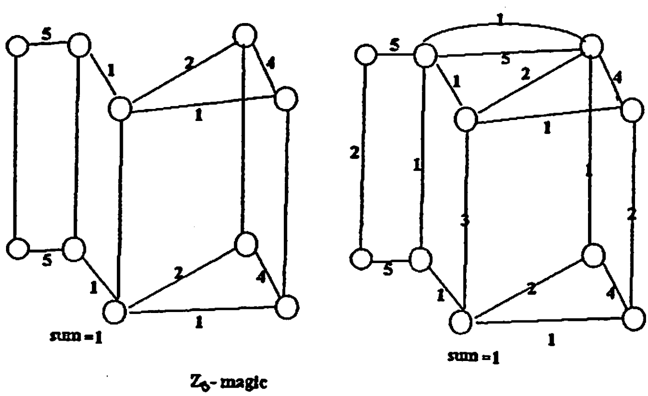


Figure 6. *(a)* G and its G*. The graph G* is Z_{40}-magic. (b) H and its H*.

Corollary 13. For any graph G with $|V(G)| > 1$, the graph $G \times K_2$ is A-magic for all group A.

We recall that a category of graphs Ω is said to be universal in Gph if for any graph G in Gph there exists an object C(G) in Ω such that the association $G \to C(G)$ is a faithful functor.

What we have shown in the above result is that the category $_A$**MGp** of all A-magic graphs is a universal category for any abelian group A.

5. Special Construction.

Theorem 14. The graph $P_3 + K_1$ is Klein 4 group-magic and 2k-magic for any k> 1 but not Z-magic and 2k+1-magic for all k>1.

Proof. The Klein 4 group-magic labeling is shown by Figure 7.

⊕	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,1)	(1,0)	(1,1)
(0,1)	(0,1)	(0,0)	(1,1)	(1,0)
(1,0)	(1,0)	(1,1)	(0,0)	(0,1)
(1,1)	(1,1)	(1,0)	(0,1)	(0,0)

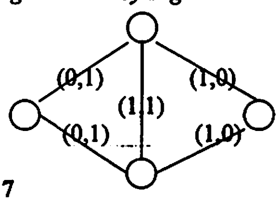


Figure 7

Note that the graph is not 2-magic. The Z_{2k}-magic labeling (k>1) is shown by

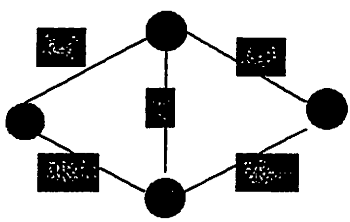


Figure 8

Figure 8.

To see that $P_3 + K_1$ is not 2k+1-magic for all k>1, assume it has a labeling which is 2k+1-magic for some k (see Figure 9)

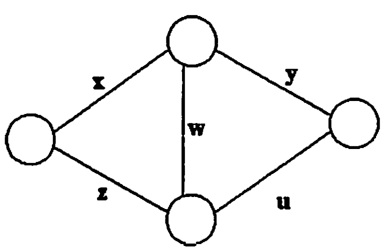


Figure 9.

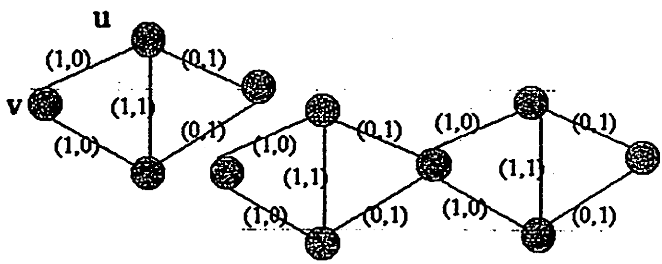
Since x+z = x+w+y and y+u= z+w+u , then we have z = w+y and y=z+w. From which we derive z=w+y=w+z+w=2w +z . Hence we have 2w=0. But there exits no such element in Z_{2k+1} with order 2. This is a contradiction.

If (G_1,L_1) and (G_2,L_2) ∈ $_A$MGp with sum 0. We can construct a new A-magic graph as follows: fixed a u in $V(G_1)$ and v in $V(G_2)$, we union G1 and G2 by identifying u and v, the resulting graph which is denoted by (G_1,u) # (G_2,v) is called the one-point union of (G_1,u) and (G_2,v). It is obvious that

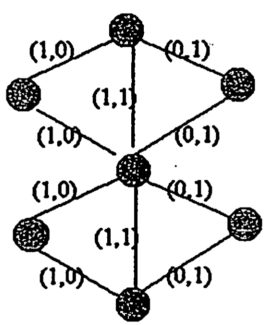
Theorem 15. If (G_1,L_1) and (G_2,L_2) ∈ $_A$MGp with sum 0 and for any u in V (G_1) and v in $V(G_2)$ the one-point union (G_1,u) # (G_2,v) is A-magic.

The above construction provides infinitely many A-magic graphs.

<u>Example 3.</u> Consider the graph $P_3 + K_1$ we can construct the following Klein-4 group-magic graphs.



(G,v) # (G,v)



(G,u) # (G,u)

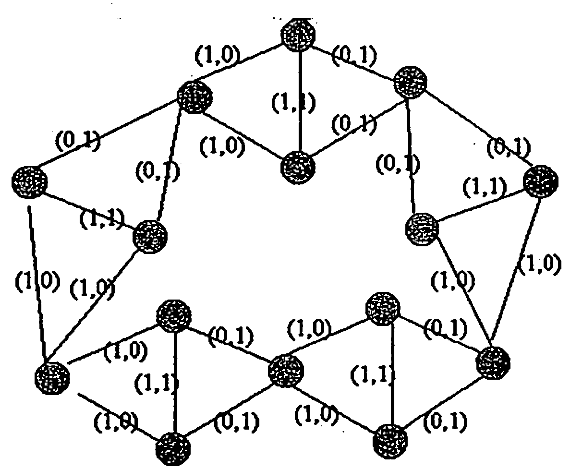


Figure 10

6. Some conjectures and unsolved problems.

We propose here some conjectures and unsolved problems for further research.

Conjecture. For any abelian group A, almost all graphs are non-A-magic.

If we denote the set of all A-magic graphs of order n by ${}_A MGph(n)$, the above conjecture means that

$$\lim_{n \to \infty} \frac{|{}_A MGph(n)|}{|Gph(n)|} = 0$$

Problem 1. For any $A \neq Z_2$, give the characterization of A-magic graphs.

Theorem 12 shows that it is impossible to have a characterization of A-magic graphs just as the Kuratowski type for planar graphs.

Given a graph G, we denote the set of all k> 0 such that G is k-magic by IM(G). We call this set as **integer-magic spectrum** of G. Likewise, we denote AM(G)={ A ε **Ab**: G is A-magic} the **group-magic spectrum** of G.

Problem 2. Determine AM(G), in particular IM(G) for non-regualr graph G.

AM sets and IM sets of some special graphs had been considered in [15,16].

Problem 3. Decide what subsets of N can be IM(G) for some graph G.

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