On c-Bhaskar Rao Designs With Block Size 3 and negative c

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Abstract: It is shown that the necessary conditions are sufficient for the existence of all c-BRD(v, 3, λ) for negative c-values. This completes the study of c-BRDs with block size three as previously the authors and J. Seberry have shown that the necessary conditions are sufficient for $c \geq -1$.

Keywords: Bhaskar Rao Designs, BIBD, c-BRD, balanced orthogonal matrices.

1. Introduction

The incidence matrix of a BIBD(v, b, r, k, λ) becomes a Bhaskar Rao design, a BRD(v, k, λ), when the one's are assigned a plus or minus sign in such a way that the rows are orthogonal under the standard inner product. We consider designs with an assignment of plus/minus signs which yield a constant inner product c, but c is not necessarily zero. Such matrices were introduced by Dey and Midha [6], who referred to them as GBM's or Generalized Balanced Matrices, but, to be more consistent with present terminology, we choose to call them c-BRD's. Seberry [13] was the first to prove the necessary conditions were sufficient for the existence of 0-BRD(v, 3, λ). In Hurd and Sarvate [8] it was shown that the necessary conditions were sufficient for the existence of 1-BRD(v, 3, λ), and in [9] extended to all $c \geq -1$.

Originally 0-BRD's were introduced in [1] and [2]. Such matrices and their generalizations have been studied by numerous authors, e.g., see [3], [4], [5], [10], [11], [12], and [14] and the references therein.

As usual, we do not distinguish between the incidence matrix of a BIBD and the BIBD. The incidence matrix of a BIBD(v, k, λ) with no minus signs is a λ-BRD. Recall that all BIBD's satisfy (1) vr = bk and (2) $\lambda(v - 1) =$

r(k - 1). We assume throughout the paper that $v \geq 3$.

In **[8]** it was shown that:

Lemma 1: *(A) For every c-BRD(3, 3, λ), $c \equiv \lambda \pmod 4$; this extends the well-known condition that $c \equiv \lambda \pmod 2$ for every c-BRD.*

(B) For every 1-BRD(v, 3, 3), $v \equiv 1 \pmod 4$.

(C) For k odd, every c-BRD(v, k, λ) satisfies $b(k - 1) + cv(v - 1) \equiv 0 \pmod 8$.

These were used in **[9]** to get

Lemma 2: *(A) For every c-BRD, $vr + cv(v - 1) \geq 0$.*

(B) For $k = 3$, $c = -1$, $2b \equiv v(v - 1) \pmod 8$.

When we applied these ideas to the case $c = -1$, seeking a condition analogous to Lemma 1(B), we found a striking contrast to the $c = 1$ case **[8]**. Suppose $\lambda = 3$. As $k = 3$, and as $\lambda(v - 1) = r(k - 1)$, we have $r = 3(v - 1)/2$. But from $vr = bk$, we see $b = vr/k = v(v - 1)/2$. From Lemma 1(C), and as $c = -1$, we have, for some t, $8t = 2b - v(v - 1) = 0$. But this is no restriction at all. Thus, for all odd v, a (-1)-BRD(v, 3, 3) should exist!

Lemma 3: [9] *A (-1)-BRD(v, 3, 3) exists for all odd v.*

Proof: From **[11, p.49]**, let $(Q, \circ)$ be an idempotent commutative quasigroup of order v, for v odd. Then

$$\{\{a, b, a \circ b\} \mid a < b \in Q\}$$

forms a BIBD(v, 3, 3), (or actually a 3-BRD(v, 3, 3)).We sign $a \circ b$ with -1 in the incidence matrix. This forms a (-1)-BRD(v, 3, 3) since all pairs (a, b) with $a < b$ occur only once as the first two entries in a block.

■

We will need the following from **[8]**.

Lemma 4: [8] *Let PB(v, K, λ) be any pairwise balanced design. If there exists a c-BRD(k$'$, k, μ) for each k$'$ in K, then there exists a cλ-BRD(v, k, $\lambda\mu$)..*

■

2. New necessary conditions

In view of Lemma 1, we expect a (-3)-BRD(v, 3, 5) to exist as $c \equiv \lambda \pmod 4$. However, Lemma 2(A) gives $vr \geq 3v(v - 1)$ or, $r \geq 3(v - 1)$, which is not satisfied since $r = 5$ in the case $v = 3$. So, a (-3)-BRD(3, 3, 5) does not exist after all. This example shows not only that the condition is useful but also that consideration of negative c-values leads to surprising new results and is thus an important topic in itself.

Theorem 5. *For any c-BRD(v, k, λ),* $\dfrac{\lambda}{k-1} \geq |c|$.

Proof. By Lemma 2(A), $vr + cv(v - 1) \geq 0$. Substituting $r = \lambda(v - 1)/(k - 1)$ we get the result.

■

When $k = 3$, the condition gives $|c| \leq \lambda/2$. It turns out that this potentially restrictive condition is rather weak when $k = 3$ which is exactly the case of interest here. There is a new inequality which we now develop which is sharper but which only applies when $k = 3$.

Theorem 6. *For any c-BRD(v, 3, λ),* $\lambda \geq 3|c|$.

Proof: Now suppose X denotes any c-BRD(v, 3, λ) with c negative. Any two rows overlap in λ non-zero positions. Let x be the number of these λ positions where the two rows have the same entry - either two ones or two minus ones. Let y be the number of these λ positions where the two rows have different entries. Thus there are at least y minus ones between the two rows. Clearly $x + y = \lambda$, and $y - x = |c|$. From these equations we get $y = (\lambda + |c|)/2$.

The entries of any column in the incidence matrix of a BRD can be multiplied by minus one without changing any inner products. We may thus assume that there is exactly zero or one, minus ones in a column. Since there are at least y minus ones per pair of rows and there are $v(v - 1)/2$ pairs of rows, there must be at least $yv(v - 1)/4$ minus ones in the complete incidence matrix. The extra factor of two in the denominator accounts for the fact that each minus sign occurs in two different row pairs. So we have $yv(v - 1) \leq 4b$ where b is the number of blocks in the incidence matrix. As $b = \lambda v(v - 1)/6$, we get $y \leq 2\lambda/3$. But $y = (\lambda + |c|)/2$. Thus, $|c| \leq \lambda/3$.

■

We point out that when $|c| = \lambda/3$, each column must use the minus sign allotted to it - no column will consist entirely of plus ones.

Table 1 below is used throughout the remaining parts of the paper. The first half is taken from [11] and the second half from [13].

Necessary and Sufficient Conditions for λ-fold Triple Systems and 0-BRD's

	λ	v
A.	0 mod 6	all $v \neq 2$
B.	1, 5 mod 6	$v \equiv 1, 3 \pmod 6$
C.	2, 4 mod 6	$v \equiv 0, 1 \pmod 3$
D.	3 mod 6	all odd v

E. 0-BRD(v, 3, 2) exist if and only if $v(v-1) \equiv 0 \pmod{12}$.

F. 0-BRD(v, 3, 4) exist if and only if $v(v-1) \equiv 0 \pmod 3$.

G. 0-BRD(v, 3, 6) exist if and only if $v(v-1) \equiv 0 \pmod 4$.

H. 0-BRD(v, 3, 2t) exist if and only if $2tv(v-1) \equiv 0 \pmod{24}$.

Table 1

3. The case $\lambda = 5$.

A *Hadamard matrix* of order n, denoted H(n), is a square matrix of plus ones and minus ones such that every pair of rows has an inner product of zero. We define a *c-H(n) of order n and index c* to be a Hadamard matrix of $\pm$ 1's such that every pair of rows has inner product c.

Theorem 7. *For c-H(n) to exist, $n \equiv c \pmod 4$.*

Proof. Any three rows of c-H(n) form a c-BRD(3, 3, n). By Lemma 1(A), the result is immediate. ■

It is well-known that, if H(n) is a Hadamard matrix, what we call a 0-H(n), then $n \equiv 0 \pmod 4$. This now follows as an immediate corollary from Theorem 7. We can extend the theorem more generally as follows.

Corollary. *For any c-BRD(v, v, λ), $\lambda \equiv c \pmod 4$.*

As an application of the reults so far, we wish to conclude this section with a closer examination of the conditions concerning $\lambda = 5$. Lemma 1(A) shows that (-1)-BRD(3, 3, 5) does not exist since $-1 \not\equiv 5 \pmod 4$. In the previous section we saw more generally that, although $-3 \equiv 5 \pmod 4$, a (-3)-BRD(3, 3, 5) did not exist either - in some sense because |c| and λ are too close to each other. We conclude that the only c-BRD(3, 3, 5) occurs when c = 1.

More general conclusions are possible for $\lambda = 5$. If a 0-BRD(v, 3, 2) exists and if a (-1)-BRD(v, 3, 3) exists, then by *juxtaposition* (i.e., combining the incidence matrices by placing them side-by-side) we get a (-1)-BRD(v, 3, 5). The first condition is satisfied (Table 1) when $v(v - 1) \equiv 0 \pmod{12}$, and the second is satisfied for all odd v (Lemma 3). But these overlap only when $v \equiv 1, 9 \pmod{12}$. For any BIBD(v, 3, 5), v must be odd [since from $\lambda(v-1) = r(k-1)$ we get $5(v-1) = 2r$]. Also, $v(v-1) \equiv 0$ mod 6 [since $vr = bk$ and $\lambda(v-1) = r(k-1)$ give $5v(v - 1) = 6b$]. So, $v \not\equiv 5, 11 \pmod{12}$. What happens if $v \equiv 3,7 \pmod{12}$? To answer this, we first need another necessary condition from [9] which is extensively applied in the next section.

Theorem 8. [9] *Suppose $v(v - 1) \not\equiv 0 \pmod{12}$. Then for any c-BRD(v, 3, λ), $c \equiv \lambda \pmod 4$. Further, (A) For any (-2s-1)-BRD(v, 3, $6t \pm 1$), $s \not\equiv t \pmod 2$. (B) Suppose $\lambda = 6t + 3$. If $c = -2s - 1 < 0$, then $t \equiv s \pmod 2$. (C) If $c = 2s$ and $\lambda = 2t$, then $s \equiv t \pmod 2$.*

Theorem 9. *Suppose $v \geq 3$. (-1)-BRD(v, 3, 5) exist if and only if $v \equiv 1, 9 \pmod{12}$.*

Proof. When $v(v - 1) \equiv 0 \pmod{12}$, the construction above suffices. If $v(v - 1) \not\equiv 0 \pmod{12}$, then, since $-1 \not\equiv 5 \pmod 4$, the BRD does not exist by Theorem 8.

■

4. Main results: the case c ≤ -2.

We use the next theorem to construct c-BRD's for many negative values of c.

Theorem 10. *For all v ≥ 3, (-2)-BRD(v, 3, 6) exist.*

Proof: Tables 2 and 3 below show examples of a (-2)-BRD(3, 3, 6) and (-2)-BRD(4, 3, 6), respectively. Two copies of (-1)-BRD(5, 3, 3) give (-2)-BRD(5, 3, 6). Table 4 shows an example of (-2)-BRD(6, 3, 6), or for a different example, apply Lemma 4.

(-2)-BRD(3,3,6)					
-1	-1	1	1	1	1
1	1	-1	-1	1	1
1	1	1	1	-1	-1

Table 2

(-2)-BRD(4,3,6)											
-1	-1	1	1	-1	1	1	1	1	0	0	0
1	1	-1	-1	-1	1	0	0	0	1	1	1
1	1	1	0	0	0	1	-1	-1	-1	-1	-1
0	0	0	-1	1	-1	-1	1	1	1	-1	-1

Table 3

Table 5 at the end gives an example of a (-2)-BRD(8, 3, 6). Theorem 10 now follows from Lemma 4 and Hanani's Lemma 5.3 [7, p.289] which states for every integer v ≥ 3, a pairwise balanced design PB(v, K_3, 1) exists with block sizes from K_3 = {3, 4, 5, 6, 8}.

■

Theorem 11. The necessary conditions are sufficient for the existence of all c-BRD with c ≤ -2.

Proof: The necessary conditions are c ≡ λ (mod 2), and the retrictions given in Lemmas 1 and 2, Theorem 5, Theorem 6, the Corollary to Theorem 7, Theorem

8, and Table 1. We will construct examples for all cases allowed by these restrictions. There are two general cases, each with subcases.
Case 1. $v(v - 1) \equiv 0 \pmod{12}$. A juxtaposition construction for each possible c and λ is given by Table 6 just below. Since $v(v - 1) \equiv 0 \pmod{12}$, each of the BRD's in Table 6 necessarily exist, by Table 1. In Tables 6, 7, and 8, *s means use s copies.
Case 2. $v(v - 1) \not\equiv 0 \pmod{12}$. Here Theorem 8 applies. First suppose c is even. Say c = -2s. If $\lambda = 6t$ we can juxtapose s-copies of (-2)-BRD(v,3,6), and {(t-s)/2}-copies of 0-BRD(v,3,12). If $\lambda = 6t + 4$, compose s-copies of a (-2)-BRD(v,3,6), {(t-s)/2}-copies of a 0-BRD(v,3,12), and a 0-BRD(v,3,4). When $\lambda = 6t + 2$, we must consider t even and t odd separately, and we do this in Table 7. For c odd, we list the cases in Table 8.

In constructing these tables, care must be given to the necessary conditions in Table 1. For example, if $\lambda = 12y + 1$ and a BIBD(v, 3, 12y + 1) exists, then by Table 1(B), $v \equiv 1, 3 \pmod 6$. But in this case, a 0-BRD(v, 3, 4) exists by Table 1(F). Therefore, in Table 8 for example, we may use a 0-BRD(v, 3, 4) in the constructions in the row for $\lambda = 12y + 1$. We may not use a 0-BRD(v, 3, 4) in the rows for $\lambda = 12y + 3$ and $\lambda = 12y + 9$, however, since in these cases v may be any odd number (Table 1(D)), and $v(v - 1)$ need not be a multiple of 3. By Table 1(H), a 0-BRD(v, 3, 12) necessarily exists and may be used anywhere in the Tables.

■

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(-2)-BRD(8, 3, 6)														
v\b	b1	b2	b3	b4	b5	b6	b7	b8	b9	b10	b11	b12	b13	b14
v1	1	1	1	1	1	1	1	0	1	1	1	1	1	1
v2	1	1	-1	-1	-1	-1	0	1	0	0	0	0	0	0
v3	-1	-1	-1	1	1	0	-1	1	0	0	0	0	0	0
v4	0	0	0	0	0	0	0	0	1	1	-1	-1	-1	-1
v5	0	0	0	0	0	0	0	0	-1	-1	-1	1	1	0
v6	0	0	0	0	0	0	0	0	0	0	0	0	0	0
v7	0	0	0	0	0	0	0	0	0	0	0	0	0	0
v8	0	0	0	0	0	1	1	-1	0	0	0	0	0	-1
v\b	b15	b16	b17	b18	b19	b20	b21	b22	b23	b24	b25	b26	b27	b28
v1	1	0	1	1	1	1	1	1	1	0	0	0	0	0
v2	0	0	0	0	0	0	0	0	0	0	1	1	-1	-1
v3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
v4	0	1	0	0	0	0	0	0	0	0	-1	0	0	0
v5	-1	1	0	0	0	0	0	0	0	0	0	-1	0	0
v6	0	0	1	1	-1	-1	-1	-1	0	1	0	0	-1	0
v7	0	0	-1	-1	-1	1	1	0	-1	1	0		0	-1
v8	-1	-1	0	0	0	0	0	-1	-1	-1	1	1	1	1
v\b	b29	b30	b31	b32	b33	b34	b35	b36	b37	b38	b39	b40	b41	b42
v1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
v2	0	0	0	0	0	0	0	0	1	1	1	1	1	1
v3	1	1	-1	-1	0	0	0	0	0	0	0	0	0	0
v4	-1	0	0	0	1	-1	0	0	1	1	-1	-1	-1	0
v5	0	-1	0	0	0	0	-1	1	0	0	0	0	0	1
v6	0	0	-1	0	-1	0	1	0	-1	-1	-1	-1	1	0
v7	0	0	0	-1	0	1	0	-1	0	0	0	0	0	-1
v8	1	1	1	1	1	1	1	1	0	0	0	0	0	0
v\b	b43	b44	b45	b46	b47	b48	b49	b50	b51	b52	b53	b54	b55	b56
v1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
v2	1	1	1	1	0	0	0	0	0	0	0	0	0	0
v3	0	0	0	0	1	1	1	1	1	1	1	1	1	1
v4	0	0	0	0	1	1	-1	-1	-1	0	0	0	0	0
v5	1	-1	-1	-1	0	0	0	0	0	1	1	-1	-1	-1
v6	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	1
v7	-1	-1	-1	1	-1	-1	-1	-1	1	0	0	0	0	0
v8	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 5

	λ= 6t	λ= 6t + 2	λ= 6t + 4
c = -2s	(-2)-BRD(v,3,6)*s 0-BRD(v,3,6)*(t-s)	(-2)-BRD(v,3,6)*s 0-BRD(v,3,6)*(t-s) 0-BRD(v,3,2)	(-2)-BRD(v,3,6)*s 0-BRD(v,3,6)*(t-s) 0-BRD(v,3,4)

	λ= 6t + 1	λ= 6t + 3	λ= 6t + 5
c = -2s-1	(-2)-BRD(v,3,6)*s (-1)-BRD(v,3,3) 0-BRD(v,3,6)*(t-s-1) 0-BRD(v,3,4)	(-2)-BRD(v,3,6)*s (-1)-BRD(v,3,3) 0-BRD(v,3,6)*(t-s)	(-2)-BRD(v,3,6)*s (-1)-BRD(v,3,3) 0-BRD(v,3,6)*(t-s) 0-BRD(v,3,2)

Table 6

λ= 6t+2 = 12y + 2, t even			
	c = -12x - 2 (-2)-BRD(v,3,6)*(6x+1) 0-BRD(v,3,12)*(y-3x-1) 0-BRD(v,3,4)*2	c = - 12x - 6 (-2)-BRD(v,3,6)*(6x+3) 0-BRD(v,3,12)*(y-3x-2) 0-BRD(v,3,4)*2	c = -12x - 10 (-2)-BRD(v,3,6)*(6x+5) 0-BRD(v,3,12)*(y-3x-3) 0-BRD(v,3,4)*2

λ= 6t + 2 = 12y + 8, t odd			
	c = -12x (-2)-BRD(v,3,6)*(6x) 0-BRD(v,3,12)*(y-3x) 0-BRD(v,3,4)*2	c = -12x - 4 (-2)-BRD(v,3,6)*(6x+2) 0-BRD(v,3,12)*(y-3x-1) 0-BRD(v,3,4)*2	c = -12x -8 (-2)-BRD(v,3,6)*(6x+4) 0-BRD(v,3,12)*(y-3x-2) 0-BRD(v,3,4)*2

Table 7

λ	c = -12x - 3	c = -12x - 7	c = -12x - 11
12y + 1	(-2)-BRD(v,3,6)*(6x+1) (-1)-BRD(v,3,3) 0-BRD(v,3,12)*(y-3x-1) 0-BRD(v,3,4)	(-2)-BRD(v,3,6)*(6x+3) (-1)-BRD(v,3,3) 0-BRD(v,3,12)*(y-3x-2) 0-BRD(v,3,4)	(-2)-BRD(v,3,6)*(6x+5) (-1)-BRD(v,3,3) 0-BRD(v,3,12)*(y-3x-3) 0-BRD(v,3,4)
12 y + 5	(-2)-BRD(v,3,6)*(6x+1) (-1)-BRD(v,3,3) 0-BRD(v,3,12)*(y-3x-1) 0-BRD(v,3,4)*2	(-2)-BRD(v,3,6)*(6x+3) (-1)-BRD(v,3,3) 0-BRD(v,3,12)*(y-3x-2) 0-BRD(v,3,4)*2	(-2)-BRD(v,3,6)*(6x+5) (-1)-BRD(v,3,3) 0-BRD(v,3,12)*(y-3x-3) 0-BRD(v,3,4)*2
12y + 9	(-2)-BRD(v,3,6)*(6x+1) (-1)-BRD(v,3,3) 0-BRD(v,3,12)*(y-3x)	(-2)-BRD(v,3,6)*(6x+3) (-1)-BRD(v,3,3) 0-BRD(v,3,12)*(y-3x-1)	(-2)-BRD(v,3,6)*(6x+5) (-1)-BRD(v,3,3) 0-BRD(v,3,12)*(y-3x-2)

λ	c = -12x - 1	c = -12x -5	c = -12x - 9
12y + 3	(-2)-BRD(v,3,6)*6x (-1)-BRD(v,3,3) 0-BRD(v,3,12)*(y-3x)	(-2)-BRD(v,3,6)*(6x+2) (-1)-BRD(v,3,3) 0-BRD(v,3,12)*(y-3x-1)	(-2)-BRD(v,3,6)*(6x+4) (-1)-BRD(v,3,3) 0-BRD(v,3,12)*(y-3x-2)
12y + 7	(-2)-BRD(v,3,6)*6x (-1)-BRD(v,3,3) 0-BRD(v,3,12)*(y-3x) 0-BRD(v,3,4)	(-2)-BRD(v,3,6)*(6x+2) (-1)-BRD(v,3,3) 0-BRD(v,3,12)*(y-3x-1) 0-BRD(v,3,4)	(-2)-BRD(v,3,6)*(6x+4) (-1)-BRD(v,3,3) 0-BRD(v,3,12)*(y-3x-2) 0-BRD(v,3,4)
12y + 11	(-2)-BRD(v,3,6)*6x (-1)-BRD(v,3,3) 0-BRD(v,3,12)*(y-3x) 0-BRD(v,3,4)*2	(-2)-BRD(v,3,6)*(6x+2) (-1)-BRD(v,3,3) 0-BRD(v,3,12)*(y-3x-1) 0-BRD(v,3,4)*2	(-2)-BRD(v,3,6)*(6x+4) (-1)-BRD(v,3,3) 0-BRD(v,3,12)*(y-3x-2) 0-BRD(v,3,4)*2

Table 8