On classification of 2-(8,3) and 2-(9,3) trades[†]

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Abstract

In this paper, the standard basis for trades is used to develop an algorithm to classify all simple 2-(8,3) trades. The existence of a total number of 15,011 trades reveals the rich structure of trades in spite of a small number of points. Some results on simple 2-(9,3) trades are also obtained.

1. Introduction

For given v, k, and t, let $X = \{1, 2, \ldots, v\}$ and let $P_k(X)$ denote the set of all k-subsets of X. The elements of X and $P_k(X)$ are called points and blocks, respectively.

A t-(v,k) *trade* $T = \{T_1, T_2\}$, consists of two disjoint collections of blocks T_1 and T_2 such that for every $A \in P_t(X)$, the number of blocks containing A is the same in both T_1 and T_2.

The *foundation* of a trade is the set of all elements covered by T_1 and T_2 and is denoted by $found(T)$. In a t-(v,k) trade, we take v to be the foundation size. The number of blocks in $T_1(T_2)$ is called the *volume* of the trade T and is denoted by $vol(T)$.

A t-(v,k) trade T is called *Steiner*, if each element $A \in P_t(X)$ occurs at most once in $T_1(T_2)$. T is called *simple*, if there are no repeated blocks in $T_1(T_2)$. Here, we are concerned only with simple trades.

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Two trades $T = \{T_1, T_2\}$ and $T' = \{T_1', T_2'\}$ are called isomorphic, if there exists a bijection σ : found$(T) \to$ found(T') such that $\sigma(T) = \{\sigma(T_1), \sigma(T_2)\} = \{T_1', T_2'\} = T'$.

For each point $x \in$ found(T), we consider the set of all blocks containing it and omit x from them. The result is a $(t-1)$-$(v-1, k-1)$ trade and we call it the *derived trade* with respect to x.

In [4], Khosrovshahi *et al.*, using some analytic arguments, classified 2-$(v, 3)$ trades for $v = 6$ and 7 and showed that there exist 3 trades with foundation size 6 and 12 trades with foundation size 7. In this paper, we give a complete classification of 2-$(8, 3)$ trades and provide some results on 2-$(9, 3)$ trades. Our method is computational and utilizes the *standard basis* for trades introduced in [5]. Our computational results confirm those of [4] and produce a total number of 15,011 simple 2-$(8, 3)$ trades which reveals the rich structure of trades even with small foundation size.

2. The standard basis for trades

Let $1 \leq t < k < v$, and let X be a v-set. Let $P_{t,k}^v = [p_{A,B}]$ be the $\binom{v}{t} \times \binom{v}{k}$ *inclusion matrix* where, for A a t-subset of X and B a block,

$$p_{A,B} = \begin{cases} 1 & \text{if } A \subseteq B, \\ 0 & \text{otherwise.} \end{cases}$$

It is known that the rank of $P_{t,k}^v$ is $\binom{v}{t}$, for $t < k < v - t$ and hence its kernel $N_{t,k}^v$ is a $\mathbb{Z}$-module of dimension $\binom{v}{k} - \binom{v}{t}$. The trade $T = \{T_1, T_2\}$ corresponds to the $\binom{v}{k}$-integral vector F which is a solution of the equation $P_{t,k}^v F = 0$. That is, the set of all t-(v, k) trades is the kernel of $P_{t,k}^v$.

Various authors have introduced different bases for $N_{t,k}^v$. For a brief description of these bases the reader is referred to [5]. There, Khosrovshahi and Maysoori introduced a new basis which they called the *standard basis*. We utilize this basis to calssify t-(v, k) trades. The $\binom{v}{k} - \binom{v}{t}$ trades of the standard basis constitute the columns of a matrix $M_{t,k}^v$ which has the following block structure:

$$M_{t,k}^v = \left[\frac{I}{\overline{M}_{t,k}^v} \right], \tag{1}$$

where I is the identity matrix of order $\binom{v}{k} - \binom{v}{t}$. The rows coresponding to I are indexed by the so-called ***starting blocks*** and the remaining rows by the ***non-starting blocks***. By (1), the following observation is obvious.

Lemma. Let T be a trade. Then $T \neq 0$ if and only if T contains at least one starting block.

The starting blocks corresponding to the triple (v, k, t) on the point set $\{1, \ldots, v\}$ have the following property. If we choose from among these starting blocks, the ones containing 1 and omit 1 from them, the resulting blocks are the starting blocks for the triple $(v-1, k-1, t-1)$ on the point set $\{2, \ldots, v\}$. Hence, we have the following block structure for $M^v_{t,k}$:

$$\begin{bmatrix} I & 0 \\ 0 & I \\ K & L \\ Q & R \end{bmatrix} \tag{2}$$

The indices corresponding to the first and the third rows of this block structure are the starting and non-starting blocks for the triple $(v-1, k-1, t-1)$, respectively.

By Lemma, we have $L = 0$, therefore $K = \overline{M}^{v-1}_{t-1,k-1}$. Clearly $R = \overline{M}^{v-1}_{t,k}$. Hence by permuting the rows of $M^v_{t,k}$, one obtains :

$$M^v_{t,k} = \begin{bmatrix} M^{v-1}_{t-1,k-1} & 0 \\ N & M^{v-1}_{t,k} \end{bmatrix}$$

An example of the standard basis for $v = 8, k = 3$ and $t = 2$ in the form (2) is given in Table 1.

A direct way to produce all simple t-(v, k) trades is to compute linear combinations of the columns of $M^v_{t,k}$ with coefficients 0, 1, and -1 and then to decide whether the result is a simple t-(v, k) trade or not.

Suppose all $(t-1)$-$(v', k-1)$ trades have been classified for $v' \leq v-1$ so that we have one representative for each isomorphism class. Let T be a t-(v, k) trade and T' its derived trade with respect to the point 1. T' is clearly isomorphic to one of the representative $(t-1)$-$(v', k-1)$ trades such as T''. So there exists a permutation π such that $T'' = \pi T'$. Therefore, πT (an isomorphic copy of T) will be the extension of T''. Hence, to classify t-(v, k) trades, up to isomorphism, it suffices to extend only representatives of the isomorphism classes of $(t-1)$-$(v', k-1)$ trades. The recursive structure of $M_{t,k}^{v}$ helps us determine t-(v, k) trades by extending $(t-1)$-$(v', k-1)$ trades for $v' \leq v-1$. Let T' be a representative $(t-1)$-$(v', k-1)$ trade. Then the coefficients of the first $\binom{v-1}{k-1} - \binom{v-1}{t-1}$ columns of $M_{t,k}^{v}$ are specified by the starting blocks of T'. To extend T', it suffices to set the coefficients of the remaining columns and then to decide whether the resulting trade is a simple t-(v, k) trade or not. Clearly, these extensions may be isomorphic and we must apply permutations to extract one representative of each class of isomorphism.

Using this approach, we were able to classify 2-(8, 3) trades and to obtain some results on 2-(9, 3) trades which are presented in the next sections. This method is also applied to count the number of ways of halving the complete design 2-(10, 3, 8) [1].

3. Classification of 2-(8, 3) trades

Using the approach described in Section 2, we are able to classify, up to isomorphism, all simple 2-(8, 3) trades. First, 1-$(v, 2)$ trades are categorized for $v \leq 7$. There exist 47 of these trades. In Table 2, the size of autmorphism groups and the number of extensions of these 47 trades are given. In the next step, by using the standard basis, the possible simple extensions of 1-$(v, 2)$ trades are computed. The total number of 2-(8, 3) trades in this step is equal to 301,625. While the total number of distinct simple solutions obtained from Table 2 by summing over all products of the size of extensions automorphism group of each of 1-$(v, 2)$ trades and the number of its is 564,946,923. Then the size of automorphism group of each trade is determined. After that, we are able to seperate the trades into disjoint classes with respect to volume and size of automorphism group. In each class, there are yet isomorphic trades. Hence, in the last step which is somewhat time-consuming we choose, by applying permutations, only one representative of each isomorphism class to obtain all non-isomorphic trades.

Table 1. $M^{8}_{2,3} = \begin{bmatrix} I & 0 \\ 0 & I \\ \overline{M}^{7}_{1,2} & 0 \\ N & \overline{M}^{7}_{2,3} \end{bmatrix}$

123	1													
124		1												
125			1											
126				1										
127					1									
134						1								
135							1							
136								1						
137									1					
145										1				
146											1			
147												1		
156													1	
157														1
234														
235														
236														
237														
245														
246														
247														
256														
257														
345														
346														
347														
356														
357														
128	-1	-1	-1	-1	-1									
138	-1					-1	-1	-1	-1					
148		-1				-1				-1	-1	-1		
158			-1				-1			-1			-1	-1
167	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
168	1	1	1		1	1	1		1	1		1		1
178	1	1	1	1		1	1	1		1	1		1	
238	-1													
248		-1												
258			-1											
267	-1	-1	-1	-1	-1									
268	1	1	1		1									
278	1	1	1	1										
348						-1								
358							-1							
367	-1					-1	-1	-1	-1					
368	1					1	1		1					
378	1					1	1	1						
456	-1	-1	-1	-1		-1	-1	-1		-1	-1		-1	
457	-1	-1	-1		-1	-1	-1		-1	-1		-1		-1
458	2	2	2	1	1	2	2	1	1	1	1	1	1	1
467	2	1	2	1	1	1	2	1	1	1			1	1
468	-1		-1		-1		-1		-1					-1
478	-1		-1	-1			-1	-1					-1	
567	2	2	1	1	1	2	1	1	1	1	1	1		
568	-1	-1			-1	-1			-1			-1		
578	-1	-1		-1		-1		-1			-1			
678	-1	-1	-1			-1	-1			-1				

123														
124														
125														
126														
127														
134														
135														
136														
137														
145														
146														
147														
156														
157														
234	1													
235		1												
236			1											
237				1										
245					1									
246						1								
247							1							
256								1						
257									1					
345										1				
346											1			
347												1		
356													1	
357														1
128														
138														
148														
158														
167														
168														
178														
238	-1	-1	-1	-1										
248	-1				-1	-1	-1							
258		-1			-1			-1	-1					
267	-1	-1	-1	-1	-1	-1	-1	-1	-1					
268	1	1		1	1		1		1					
278	1	1	1		1	1		1						
348	-1									-1	-1	-1		
358		-1								-1			-1	-1
367	-1	-1	-1	-1						-1	-1	-1	-1	-1
368	1	1		1						1		1		1
378	1	1	1							1	1		1	
456	-1	-1	-1		-1	-1		-1		-1	-1		-1	
457	-1	-1		-1	-1		-1		-1	-1		-1		-1
458	2	2	1	1	1	1	1	1	1	1	1	1	1	1
467	1	2	1	1	1			1	1	1			1	1
468		-1		-1					-1					-1
478		-1	-1					-1					-1	
567	2	1	1	1	1	1	1			1	1	1		
568	-1			-1			-1					-1		
578	-1		-1			-1					-1			
678	-1	-1			-1					-1				

The computational results are presented in Tables 3 and 4 which give a detailed account of the number of 2-$(8,3)$ trades. The number of trades with at least 24 automorphisms are given in Table 3 and the rest in Table 4.

Table 2.

No. of automorphisms and extensions of 1-$(v,2)$ trades for $v \leq 7$.

trade no.	#Aut	#extensions	trade no.	#Aut	#extensions
1	48	18039	25	2	5812
2	16	14139	26	1	6016
3	12	14144	27	2	6436
4	4	10546	28	2	6028
5	16	9937	29	1	4706
6	4	10834	30	1	4238
7	2	10526	31	1	4672
8	4	7846	32	2	4682
9	2	8055	33	2	3754
10	4	7848	34	2	4682
11	2	7858	35	4	4200
12	1	8095	36	16	3550
13	2	8340	37	14	4455
14	4	8356	38	2	4498
15	40	8833	39	2	4073
16	1	6032	40	4	4324
17	6	5400	41	2	3323
18	2	5650	42	2	3682
19	2	5288	43	2	3280
20	4	5482	44	2	2877
21	12	5103	45	8	2302
22	2	6262	46	2	2266
23	2	6274	47	6	2598
24	8	6284			

There are only 10 trades whose size of automorphism groups exceed 24. These trades are given in Table I of the Appendix. These tables show that there exists a trade of all volumes ranging from 6 to 24, except for the volume 23. From the total of 15,011 trades, 13,190 are rigid.

Table 3.

The number of 2-(8, 3) trades with at most 24 automorphisms.

		#Aut										
volume	total	1	2	3	4	6	7	8	12	14	16	24
6	1											1
7	1		1									
8	13		6		5			1			1	
9	21	14	6			1						
10	114	49	43		11	2		4	2		3	
11	212	184	28									
12	669	487	140		26	5		5			4	2
13	1057	1003	51	2		1						
14	2108	1798	263		37		1	7			2	
15	2368	2298	67	2		1						
16	3137	2683	386	2	38	10		10	3		5	
17	2164	2112	52									
18	1815	1498	281	2	24	5	6		3			
19	750	717	28	5								
20	406	252	134		14			6				
21	85	77	7							1		
22	71	22	33		5	6		3	2			
24	7		1			2		4				
total:	15,001											

Table 4. The number of 2-(8, 3) trades with more than 24 automorphisms.

volume	total	#Aut 32	40	48	64	128
8	3	1			1	1
16	4	1		2	1	
20	1		1			
24	2	1		1		
total:	10					

4. Some results on 2-(9, 3) trades

In this section, we consider 2-(9, 3) trades. Our approach is basically the one described in the previous sections. Here, the dimension of $N^9_{2,3}$ is 48 which suggests the existence of a huge number of trades. Therefore, the difficulties involved in testing isomorphism among solutions led us to focus on the following two cases (although a complete classification is not yet out of reach) :

(i) classification of Steiner trades,

(ii) classification of all trades with volume at most 9.

It is not difficult to show that in these trades at least one of the derived trades is Steiner. Therefore, it is sufficient to extend only Steiner 1-$(v, 2)$ trades.

There are only 4 Steiner 1-$(v, 2)$ trades for $v \leq 8$ with volumes 2, 3, and 4. The results of extending these 4 trades yield 17 Steiner 2-(9, 3) trades. The number of these trades in each volume, together with size of automorphism groups are given in Table 5. Khosrovshahi and Maimani in [3] provided, through analytic methods, Steiner 2-(9, 3) trades with volume at most 10. Trades of the remaining volumes, namely 11 and 12, are given in Table II of the Appendix.

Classification of all 2-$(9,3)$ trades with volume at most 9, shows that there exist (apart from Steiner ones) 6 non-Steiner trades of volume 8 and 48 non-Steiner trades of volume 9. We also provide the six non-Steiner trades with volume 8 in Table III of the Appendix.

Table 5.
The number of Steiner 2-$(9,3)$ trades.

		#Aut							
volume	total	2	3	4	6	8	12	18	108
7	1						1		
8	3			2		1			
9	7	2		1	1		1	1	1
10	3		1	1	1				
11	1			1					
12	2				1			1	
total:	17								

5. Appendix

Table I.

2-(8,3) trades with more than 24 automophisms.

volume	8	8	8	16	16	16	16	20	24	24
#Aut	32	64	128	32	48	48	64	40	32	48
	123	123	123	123 467	123 358	123 378	123 357	123 347	123 258	123 267
	157	145	124	124 468	124 456	124 457	124 468	124 348	124 268	124 268
	245	246	156	136 578	125 457	136 478	138 578	136 356	125 278	125 278
	356	247	256	145 678	138 678	145 568	146 678	145 357	137 345	137 346
	148	356	347	156	147	157	156	156 358	138 346	138 347
T_1	268	357	348	236	156	234	158	234 456	146 348	146 356
	378	238	578	245	267	256	235	236 467	156 357	148 358
	467	458	678	256	268	167	247	245 468	167 368	156 378
				347	278	258	256	257	178 457	157 457
				357	347	268	267	258	235 478	236 458
				348	356	348	348	268	247 568	258 567
				358	348	367	347	267	246 567	247 468
	124	124	125	125 458	126 467	125 458	125 467	125 345	126 248	126 248
	135	135	126	126 478	127 478	126 567	126 478	126 346	127 256	127 256
	256	236	134	134 567	128 567	134 578	134 567	134 367	128 257	128 257
	178	237	234	135 568	134 568	137 678	135 568	135 368	134 347	134 348
	238	248	378	146	135	147	148	146 457	135 356	135 357
T_2	367	358	478	234	145	156	168	235 458	136 358	136 367
	457	456	567	235	247	236	234	237 567	147 378	145 368
	468	457	568	246	256	245	237	238 568	157 456	147 456
				367	238	347	246	246	168 467	158 467
				368	345	238	257	247	234 468	237 478
				378	368	248	358	248	238 478	238 568
				457	378	368	378	256	245 679	246 578

Table II.

Steiner 2-(9,3) trades with volumes 11 and 12.

volume	11	12	12
#Aut	4	6	18
T_1	123 456 158 149 248 269 257 359 347 789 368	123 256 149 345 158 369 167 378 167 468 247 478	123 279 149 346 158 359 167 378 248 457 256 689
T_2	124 189 135 279 236 349 258 378 457 569 468	124 278 135 347 168 389 179 458 236 469 259 567	124 278 135 347 168 389 179 458 236 469 259 567

Table III.

Non-Steiner 2-(9,3) trades with volume 8.

#Aut	8	8	2	2	4	4
T_1	123 145 247 268 356 357 239 589	123 145 248 249 267 345 568 579	123 145 248 249 347 356 589 679	123 145 246 248 239 347 356 789	123 145 239 247 268 346 357 489	123 145 248 249 347 356 467 789
T_2	124 135 236 237 289 359 457 568	124 135 234 268 279 458 459 567	124 135 234 289 367 458 479 569	124 135 234 236 289 379 456 478	124 135 236 237 289 349 457 468	124 135 234 289 367 456 478 479

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