Total Colorings of Some Join Graphs*

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Abstract

In this paper, we investigate the total colorings of the join graph $G_1 + G_2$ where $G_1 \cup G_2$ is a graph with maximum degree at most 2. As a consequence of the main result, we prove that if $G = (2l+1)C_m + (2l+1)C_n$, then G is Type 2 if and only if $m = n$ and n is odd where $(2l+1)C_m$ and $(2l+1)C_n$ represent $(2l+1)$ disjoint copies of C_m and C_n, respectively.

1 Introduction

All graphs in this paper will be finite simple graphs. Given two graphs G_1 and G_2, we define the *join graph* of G_1 and G_2, denoted by G_1+G_2, to be the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $\{uv | uv \in E(G_1) \cup E(G_2)$ or $u \in V(G_1), v \in V(G_2)\}$. We note that $G_1 + G_2$ is a complete bipartite graph if both G_1 and G_2 are sets of independent vertices. Let C_n and P_n be cycle and path of n vertices, respectively.

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An *edge coloring* of a graph G is a map $\phi : E(G) \to C$, where C is the set of colors, such that no two edges with the same color are incident with the same vertex. The chromatic index $\chi'(G)$ of G is the least value of $|C|$ for which G has an edge coloring. In [22] Vizing showed that the edge chromatic number, $\chi'(G)$, of a graph G with maximum degree $\Delta(G)$, is bounded by $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. A fairly long-standing problem has been the attempt to classify which graphs G are class one ($\chi'(G) = \Delta(G)$), and which graphs G are class two ($\chi'(G) = \Delta(G) + 1$). It is known that the problem of determining the edge chromatic number of a graph is NP-hard.

A *total coloring* of a graph G is a coloring of the vertices and edges of G so that no two edges incident with the same vertex receive the same color, no two adjacent vertices receive the same color, and no incident edge and vertex receive the same color. The *total chromatic number* $\chi_T(G)$ of a graph G is the least number of colors needed in a total coloring of G.

There is no known analogue for the total chromatic number of Vizing's theorem about the chromatic index. Instead we have the Total Chromatic Number Conjecture (TCC) of Behzad [1] and Vizing [23] that

$$\Delta(G) + 1 \leq \chi_T(G) \leq \Delta(G) + 2.$$

The lower bound here is very easy to prove. This conjecture, now more than thirty years old, has been verified for graphs G satisfying $\Delta(G) \geq \frac{3}{4}|V(G)|$ by Hilton and Hind [12], and for graphs G satisfying $\Delta(G) \leq 5$ by Kostochka [17]. It has very recently been shown by Molloy and Reed [19] that there is a constant c such that $\chi_T(G) \leq \Delta(G) + c$.

A graph G is called *Type 1* if $\chi_T(G) = \Delta(G) + 1$ and *Type 2* otherwise. To classify Type 1 and Type 2 graphs, Chetwynd and Hilton [5], introduced the following concepts. They defined the *deficiency* of a graph G, denoted by def(G), to be

$$def(G) = \sum_{v \in V(G)} (\Delta(G) - d_G(v)).$$

A vertex coloring of a graph G with $\Delta(G) + 1$ colors is called *conformable* if the number of color classes of parity different from that of $|V(G)|$ is at most def(G). Note that empty color classes are permitted in this definition. A graph G is *conformable* if it has a conformable vertex coloring. It is not very hard to see that G is Type 2 if G is non-conformable. The Conformability Conjecture of Chetwynd and Hilton [5], modified by Hamilton, Hilton and Hind [9] is:

Conjecture 1 *Let G be a graph satisfying $\Delta(G) \geq \frac{1}{2}(|V(G)| + 1)$. Then G is Type 2 if and only if G contains a subgraph H with $\Delta(G) = \Delta(H)$ which* *is either non-conformable, or, when $\Delta(G)$ is even, consists of $K_{\Delta(G)+1}$ with one edge subdivided.*

Conjecture 1 has been verified for several cases when $\Delta(G)$ is big and close to the order of a graph G (see [11], [4], [26], [25], [13], [9], [15] and [14]). It would be very interesting to provide nontrivial evidence for Conjecture 1 when $\Delta(G)$ is close to one half of the order of a graph G.

A good characterization of all Type 1 graphs is unlikely as Sanchez-Arroyo [21] showed that the problem of determining the total chromatic number of a graph is NP-hard. Not only that, there are few results about the total chromatic numbers of even very nice graphs, for example, the complete multipartite graphs. In [2], the authors classified which complete bipartite graphs are Type 1. A natural question is to ask which graph G, which is obtained from a complete bipartite graph by adding in some edges in the two parts, is Type 1. Such a graph G can be represented as a join of two graphs.

In this paper, we investigate the total chromatic number of graphs of the form $G_1 + G_2$, where G_1 and G_2 are graphs with maximum degree at most 2.

The following results will be used in this paper.

Theorem 1.1 (Behzad, Chartrand and Cooper [2]) *Let $K_{m,n}$ be the complete bipartite graph. Then $K_{m,n}$ is Type 1 if $m \neq n$ and Type 2 otherwise.*

Theorem 1.2 (König's Theorem) *If G is bipartite graph with maximum degree $\Delta(G)$, then $\chi'(G) = \Delta(G)$.*

Let $\bar{G}$ be the complement of G, $e(G) = |E(G)|$ and $j(G)$ be the edge independence number of G.

Theorem 1.3 (Hamilton, Hilton and Hind [9]) *Let G be a graph of even order $2n$. If $e(\bar{G}) + j(\bar{G}) \leq n(2n - \Delta(G)) - 1$, then $\chi_T(G) > \Delta(G) + 1$.*

Theorem 1.4 [18] *Let $G = G_1 + G_2$, where the maximum degree of G_i is at most one for $i = 1, 2$. Then G is Type 1 if and only if G is not isomorphic to $K_{n,n}$ or K_4.*

Theorem 1.5 [18] *Let $G = G_1 + G_2$. Then G is Type 1 if one of the following holds.*

(1) G is not isomorphic to K_4 or $K_{n,n}$ and both G_1 and G_2 are unions of paths with $|\Delta(G_1) - \Delta(G_2)| \leq 1$.

(2) Both G_1 and G_2 are unions of even cycles with $|V(G_1)| = |V(G_2)|$.

The following lemma is about the list colorings of cycles, which plays an important role in the proof of our main results. Surprisingly, we could not find it anywhere although we believe that it exists.

Lemma 1.6 *Let G be a cycle. If each edge e of G is assigned a list $l(e)$ of at least two colors and the total number of colors available to G is at least 3, then there is a proper edge coloring c of G such that $c(e) \in l(e)$ for each edge e of G.*

Proof. If an edge e has 3 colors in its list $l(e)$, then we list color the path $G - \{e\}$ and then assign e a color from $l(e)$ which is not assigned to its neighbors. Thus, we assume each list has two colors. Because at least three colors are available, there are consecutive edges e_1, e_2 and e_3 in G such that e_2 has a color A that is not in $l(e_1)$. We can list color the path $G - \{e_2\}$ without using color A on e_3, then assign e_2 color A to achieve the desired coloring.

This completes the proof. ∎

2 Type 1 graphs

Lemma 2.1 *Let $G = G_1 + G_2$ where $\Delta(G_1) = \Delta(G_2) = 2$. Then G is Type 1 if $|V(G_1)| \neq |V(G_2)|$.*

Proof. Let $|V(G_1)| < |V(G_2)| = n$. Then we have that $\Delta(G) + 1 = n + 3$. Let $B = G - (E(G_1) \cup E(G_2))$ and M a maximum matching of B ($|M| = |V(G_1)|$). There is at least one vertex in G_2, say u, which is not M-saturated.

To show that G is a Type 1 graph, we need to form a total coloring of G using $n + 3$ colors. Such a total coloring can be found by the following steps:

1. We total color G_1 with colors $1, 2, 3$ and 4 so that the vertices in G_1 are colored by colors $1, 2$ and 3. Color the vertex u in G_2 by the color 4.

2. Color the edges of M with colors $1, 2, 3$ and 4 (there is a unique way to do this if G_1 consists of only cycles).

3. We edge color $E(G_2)$ with colors 1, 2, 3, and 4. This can be done without adjacent edges (including those from M) having the same color because G_2 is the union of cycles and paths and at least two colors from $\{1, 2, 3, 4\}$ are available on each edge of G_2 . Because the total number of colors available for G is at least 3, by Lemma 1.6, we can edge color G_2 using only available colors.

4. Let $G^1 = B - E(M)$. Form a new graph G^* by adding in a new vertex v^* and joining v^* to $V(G_2)$ except the vertex u. Then $d_{G^*}(v^*) = n-1$, $d_{G^*}(x) = n - 1$ for any $x \in V(G_1)$ and $d_{G^*}(y) \leq n - 1$ for any $y \in V(G_2)$. Therefore, G^* is a bipartite graph of $\Delta(G^*) = n - 1 = \Delta(G) - 3$.

   By Theorem 1.2, we can color the edges of G^* with $n - 1$ colors $5, \ldots, n + 3$. We note that this also gives an edge coloring of B.

5. Color the vertex $v \in V(G_2) - \{u\}$, by the color on vv^*.

It is easy to see that we have obtained a proper total coloring of G using $\Delta(G) + 1$ colors. This completes the proof. ∎

Lemma 2.2 *Let $G = G_1 + G_2$ where G_1 is an independent set and $\Delta(G_2) = 2$. Then G is Type 1 if $|V(G_1)| \geq |V(G_2)| - 1$.*

Proof. If $|V(G_1)| > |V(G_2)|$, the lemma follows from Lemma 2.1 by simply adding in some cycles in $V(G_1)$. Let $B = G - E(G_2)$ and M be a maximum matching of B. Let $|V(G_2)| = n$.

Case 1. $|V(G_1)| = |V(G_2)|$.

In this case, $\Delta(G) + 1 = n + 3$. A total coloring of G using $n + 3$ colors can be obtained as follows.

1. Let $H = G_2 \cup M$. We color the edges of H and the vertices in G_1 by colors 1, 2 and 3.

2. Let $G^1 = B \backslash E(M)$. Then $\Delta(G^1) = n - 1$. Form a new graph G^* by adding in a new vertex v^* and joining v^* to each vertex in $V(G_2)$. Then G^* is a bipartite graph with $\Delta(G^*) = n$.

   There is a proper edge coloring of G^* using n colors $4, \ldots, n + 3$. We note that this also gives an edge coloring of B.

3. Color the vertex $v \in V(G_2)$, then v is colored by the color on vv^*.

Case 2. $|V(G_1)| = |V(G_2)| - 1$.

In this case, $\Delta(G) + 1 = n + 2$. There is a vertex $u \in V(G_2)$ which is not M-saturated. A total coloring of G using $n + 2$ colors can be obtained as follows.

1. Color u by the color 3. Color the vertices in G_1 by the color 1. Without loss of generality, assume $d_{G_2}(u) = 2$ and let x and y be the two neighbors in G_2. Color the edges in M which are incident to x and y by color 3. Now the number of available colors from $\{1, 2, 3\}$ for each of the edge of G_2 is at least 2, therefore by Lemma 1.6, we can color the edges in G_2 using colors in $\{1, 2, 3\}$.

2. Let $G^1 = B\backslash E(M)$. Form a new graph G^* by adding in a new vertex v^* and joining v^* to each vertex in $V(G_2)-\{u\}$. Then G^* is a bipartite graph with $\Delta(G^*) = n-1$.

   There is a proper edge coloring of G^* using $n-1$ colors $4, \ldots, n+2$. We note that this also gives an edge coloring of B.

3. Color the vertex $v \in V(G_2) - \{u\}$ by the color on vv^*.

It is easy to see that in both cases we have obtained a proper total coloring of G using $\Delta(G)+1$ colors.

This completes the proof. ∎

3 Type 2 graphs

Lemma 3.1 *Let $G = G_1 + G_2$, where G_1 is an independent set and G_2 is a union of cycles and both $|V(G_1)|$ and $|V(G_2)|$ are odd. Then G is Type 2 if $|V(G_2)| = |V(G_1)| + 2$.*

Proof. Let $|V(G_1)| = m$ and $|V(G_2)| = n$. Then $\Delta(G) = m+2$ and $n = m+2$.

$$\begin{aligned} & e(\overline{G}) + j(\overline{G}) - [\frac{|V(G)|}{2}(|V(G)| - \Delta(G)) - 1] = \\ = \; & \frac{m(m-1)}{2} + \frac{n(n-1)}{2} - n + \frac{m-1}{2} + \frac{n-1}{2} \\ & -(\frac{m+n}{2}(m+n-(m+2)) - 1) \\ = \; & \frac{m^2+n^2-2n-2}{2} - \frac{m^2+nm-2}{2} \\ = \; & 0. \end{aligned}$$

Therefore, by Theorem 1.3, G is Type 2. ∎

Lemma 3.2 *Let $G = G_1 + G_2$, where $|V(G_1)| = |V(G_2)| = n$ and G_1 and G_2 consist of cycles only. Then G is Type 2 if n is odd.*

Proof. We note that $\Delta(G) = n+2$.

$$\begin{aligned} e(\bar{G}) + j(\bar{G}) &= (\frac{2n(2n-1)}{2} - n^2 - 2n) + 2(\frac{n-1}{2}) \\ &= n(2n - \Delta(G)) - 1. \end{aligned}$$

Therefore, $\chi_T(G) \geq \Delta(G)+2$ by Theorem 1.3 and hence G is Type 2. This completes the proof. ∎

We can actually to show that the total chromatic number of these Type 2 graphs is $\Delta(G)+2$.

Lemma 3.3 *Let* $G = G_1 + G_2$, *where* $\Delta(G_1) \leq 2$ *and* $\Delta(G_2) \leq 2$. *Then* $\chi_T(G) \leq \Delta(G)+2$.

Proof. Let $B = G - (E(G_1) \cup E(G_2))$ and M be a maximum matching of B. By Theorems 1.4, 1.5 and Lemmas 2.1, 2.2, we need consider only the following cases.

Case 1. $|V(G_1)| = |V(G_2)| = n$ and $\Delta(G_1) = \Delta(G_2) = 2$.

In this case, $\Delta(G)+2 = n+4$. We total color G using $n+4$ colors as following.

1. We total color G_1 with colors $1, 2, 3$ and 4. The edges of M receive colors $1, 2, 3$ and 4. Then we edge color G_2 using colors $1, 2, 3$, and 4 (this is possible because of Lemma 1.6).

2. Let $G^1 = B \backslash E(M)$ and let G^* be the graph obtained from G^1 by adding in a new vertex v^* and linking all possible edges from v^* to $V(G_2)$ in G^1. It is easy to see that $\Delta(G^*) = n$ and G^* is a bipartite graph. Therefore, there is a proper edge coloring of G^* using n colors $5, 6, \ldots, n+4$. We note that this also gives colors of the edges in G^1.

3. We color the vertices in $V(G_2)$ as follows. For any $v \in V(G_2)$, color v with the color vv^*.

It is easy to see that the above coloring is a total coloring using $n+4$ colors.

Case 2. $|V(G_1)| < |V(G_2)| = n$, $\Delta(G_1) \leq 1$ and $\Delta(G_2) = 2$.

We note that $\Delta(G) = n + \Delta(G_1)$. We need total color G using $n+2+\Delta(G_1)$ colors. Let u be a vertex of $V(G_2)$ which is not M-saturated.

1. If $\Delta(G_1) = 1$, we follow the steps 1, 2 and 3 in the proof of Lemma 2.1 to color the edges in $M \cup G_1 \cup G_2$ and the vertices in $V(G_1) \cup \{u\}$. This takes 4 colors $1, 2, 3$ and 4.

   If $\Delta(G_1) = 0$, we follow the proof of Case 2, step 1 in Lemma 2.2 to color the edges in $M \cup G_1 \cup G_2$ and the vertices in $V(G_1) \cup \{u\}$. This takes 3 colors $1, 2$ and 3.

2. Let $G^1 = B \backslash E(M)$. Form a new graph G^* by adding in a new vertex v^* and joining v^* to $V(G_2)$ except one vertex u. Then $\Delta(G^*) = n-1$.

   Color the edges of G^* with $n-1$ colors $4+\Delta(G_1), \ldots, n+2+\Delta(G_1)$. We note that this also gives an edge coloring of B.

3. Color the vertex $v \in V(G_2) - \{u\}$ by the color on vv^*.

We have obtained a proper total coloring of G using $\Delta(G) + 2$ colors. ∎

4 Conclusions

To summarize, we have proved the following results.

Theorem 4.1 *Let $G = G_1 + G_2$, where $G_1 \cup G_2$ has maximum degree at most two. Then $\chi_T(G) \leq \Delta(G) + 2$ (Lemma 3.3).*

Furthermore, we have

(A) G is Type 1 if G_1 and G_2 satisfy one of the following conditions.

1. *Both G_1 and G_2 are unions of paths with $|\Delta(G_1) - \Delta(G_2)| \leq 1$ and G is not isomorphic to K_4 and $K_{n,n}$. (Theorem 1.5).*
2. *Both G_1 and G_2 are unions of even cycles (Theorem 1.5).*
3. *$|V(G_1)| \neq |V(G_2)|$, $\Delta(G_1) = \Delta(G_2) = 2$ (Lemma 2.1).*
4. *G_1 is an independent set, $\Delta(G_2) = 2$ and $|V(G_1)| \geq |V(G_2)| - 1$ (Lemma 2.2).*

(B) G is Type 2 if one of the following holds.

1. *$|V(G_1)| = |V(G_2)| = n$ is odd and both G_1 and G_2 are unions of cycles (Lemma 3.2).*
2. *G_1 is an independent set and G_2 is a union of cycles and both $|V(G_1)|$ and $|V(G_2)|$ are odd and $|V(G_2)| = |V(G_1)| + 2$ (Lemma 3.1).*

Corollary 4.2 *Let $G = (2l+1)C_m + (2l+1)C_n$. Then G is Type 2 if and only if $m = n$ and n is odd.*

Proof. This is an easy consequence of (2) and (3) of Part (A) and (1) of Part (B) in Theorem 4.1. ∎

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Corrigendum
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1. In page 77, line 9, it is written
   'http://home.coqui.net/dejter/gihidden.zip',
   but it should say
   'http://home.coqui.net/dejterij/gihidden.zip'.

2. The last paragraph in the same page, just above the Acknowledgements, starts with: 'Of the remaining adjacency tables of Phelps-Soloveva codes, we present below the simplest four'. These tables, missing in the published version of the paper, are the following ones:

Adjacency list for bb_7 :

$a^8_8 a_{a^3b^4} b_{a^8} c_{a^4b^8} d_{a^4} e_{b^4}$ $a^8_8 a_{a^4b^3} b_{a^4} c_{a^8b^8} d_{a^4} e_{b^4}$ $a^8_8 a_{c^3} b_{a^8} c_{a^8b^8} d_{a^2} e_{a^2b^2} f_{a^2}$
$b^8_5 a_{a^8b^4c^8} b_{a^3} c_{a^4} e_{a^4} f_{a^4}$ $c^8_4 a_{a^4b^8c^8} b_{a^4} c_{a^3} e_{a^4} f_{a^4}$ $c^8_4 a_{a^8b^8c^8} c_{b^3} d_{a^2} e_{a^2b^2} f_{a^2}$
$d^7_2 a_{a^8b^8c^4} c_{b^4} d_{a^3} e_{a^8}$ $e^7_3 a_{c^4} b_{a^8} c_{a^8b^4} d_{a^8} e_{a^3}$ $e^7_3 a_{a^8b^8c^4} c_{b^4} e_{b^3} f_{a^8}$
$f^7_{16} a_{c^4} b_{a^8} c_{a^8b^4} e_{b^8} f_{a^3}$

Adjacency list for bd_5 :

$a^9_6 a_{a^7b^4} b_{a^{12}b^4e^4} c_{a^4}$ $a^7_6 a_{a^{16}b^3} b_{b^2c^2d^2f^8} c_{a^2}$ $b^9_4 a_{a^{12}} b_{a^7c^4d^4f^4} c_{a^4}$
$b^7_4 a_{a^{16}b^2} b_{b^3c^8e^2f^2} c_{b^2}$ $b^7_4 a_{b^2} b_{a^{16}b^8c^3e^2f^2} c_{b^2}$ $b^7_4 a_{b^2} b_{a^{16}d^3e^2f^2} c_{a^8b^2}$
$b^7_4 a_{a^{16}} b_{b^2c^2d^2e^3} c_{a^2b^8}$ $b^7_4 a_{b^8} b_{a^{16}b^2c^2d^2f^3} c_{a^2}$ $c^7_2 a_{a^{16}b^2} b_{d^8e^2f^2} c_{a^3b^2}$
$c^7_2 b_{a^{16}b^2c^2d^2e^8} c_{a^2b^3}$

Adjacency list for bf_1 :

$a^{9_7}_{21} a_{a^5b^{12}d^2e^{11}} b_{ab} c_{a^3}$ $a^{9_7}_{21} a_{a^{12}b^5c^2e} b_{a^3b^{11}} c_a$ $a^7_{21} a_{b^6c^3d^{10}e^9} b_{ab^3} c_{a^3}$
$a^7_{21} a_{a^6c^{10}d^3e^3} b_{a^3b^9} c_a$ $a^{9_7}_{21} a_{a^{11}bc^3de^5} b_{a^4b^{10}}$ $b^7_{22} a_{a^3b^9cd^3e^{12}} b_{a^3} c_{a^4}$
$b^{9_7}_{22} a_{ab^{11}cd^3e^{10}} b_{b^5} c_{a^4}$ $c^7_{61} a_{a^9b^3c^3d} b_{a^4b^{12}} c_{a^3}$

Adjacency list for dd_3 :

$a^9_6 a_{a^7} b_{a^{12}} c_{c^4d^4} d_{a^4} f_{a^4}$ $b^9_4 a_{a^{12}} b_{a^7} c_{a^4b^4e^4} e_{a^4}$ $c^7_5 b_{a^{16}} c_{a^3d^2e^2} d_{a^8} e_{a^2} f_{a^2}$
$c^7_5 b_{a^{16}} c_{b^3c^8d^2e^2} e_{a^2} f_{a^2}$ $c^7_5 a_{a^{16}} c_{b^8c^3d^2e^2} e_{a^2} f_{a^2}$ $c^7_5 a_{a^{16}} c_{a^2b^2c^2d^3} d_{a^2} e_{a^8}$
$c^7_5 b_{a^{16}} c_{a^2b^2c^2e^3} d_{a^2} f_{a^8}$ $d^7_2 a_{a^{16}} c_{a^8d^2e^2} d_{a^3} e_{a^2} f_{a^2}$ $e^7_7 b_{a^{16}} c_{a^2b^2c^2d^8} d_{a^2} e_{a^3}$
$f^7_3 a_{a^{16}} c_{a^2b^2c^2e^8} d_{a^2} f_{a^3}$