A Survey of Results on Mandatory Representation Designs*

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Abstract

A mandatory representation design MRD $(K; v)$ is a pairwise balanced design PBD $(K; v)$ in which for each $k \in K$ there is at least one block in the design of size k. The study of the mandatory representation designs is closely related to that of subdesigns in pairwise balanced designs. In this paper, we survey the known results on MRDs and pose some open questions.

1 Introduction

A pairwise balanced design (PBD) is a pair (X, B), where X is a set of *points* and B is a collection of subsets of X called *blocks*, such that each pair of distinct points from X occurs in a unique block. A PBD $(K; v)$ is a PBD on v points in which each block has size an integer in the set K. A *mandatory representation design* MRD $(K; v)$ is a PBD $(K; v)$ in which for each $k \in K$ there is at least one block in the design of size k. Define $B(K) = \{v \in \mathbb{Z}^+ :$ there exists a PBD $(K, v)\}$ and $MB(K) = \{v \in \mathbb{Z}^+ :$ there exists an MRD $(K, v)\}$. Then clearly $MB(K) \subseteq B(K)$, but the reverse is *not* generally true. For example, $7 \in B(\{3,6\})$ (let $X = \mathbb{Z}_7$ and develop the triple 0 1 3 mod 7) but $7 \notin MB(\{3,6\})$ (as soon as a block of size 6 is prescribed, all remaining blocks must have size 2).

A pairwise balanced design PBD $(K; v)$ with $K = \{k\}$ and $v > k$ is called a balanced incomplete block design and is denoted $(v, k, 1)$-BIBD. A $(v, 3, 1)$-BIBD is called a Steiner triple system STS (v); it is well known

*Paper presented by invitation at the Thirteenth Midwestern Conference on Combinatorics, Cryptography and Computing, Normal, Illinois, October 1999.

that there exists an STS (v) if and only if $v \equiv 1$ or 3 mod 6. The following analogues for block sizes 4 and 5 were established by Hanani [15].

Theorem 1.1 *A $(v,4,1)$-BIBD exists if and only if $v \equiv 1$ or 4 mod 12; a $(v,5,1)$-BIBD exists if and only if $v \equiv 1$ or 5 mod 20.*

If (X, B) and (X', B') are PBDs, then (X', B') is a *subdesign*, or *flat*, of (X, B) if $X' \subseteq X$ and $B' \subseteq B$. For example, if we construct a one-factorization of the complete graph K_8 with one-factors $F_j, j \in \mathbb{Z}_7$, and adjoin a new point j to each pair in F_j, we can then construct an STS (7) on the 7 new points to yield an STS (15) with a sub-STS (7). The first major result on the existence of subdesigns is the following well-known theorem, established by Doyen and Wilson [10] in 1973.

Theorem 1.2 *An STS (v) with a sub-STS (w) with $v > w$ exists if and only if $v, w \equiv 1$ or 3 mod 6 and $v \geq 2w + 1$.*

The following analogue for block size 4 was established as a result of the combined efforts of Brouwer and Lenz [3, 4], Wei and Zhu [22], and Rees and Stinson [21].

Theorem 1.3 *A $(v,4,1)$-BIBD containing a sub-$(w,4,1)$-BIBD with $v > w$ exists if and only if $v, w \equiv 1$ or 4 mod 12, and $v \geq 3w + 1$.*

We can rephrase Theorems 1.2 and 1.3 in terms of mandatory representation designs, as follows.

Corollary 1.4 *(i) For each $k \equiv 1$ or 3 mod 6 with $k > 3$, there exists an MRD $(\{3, k\}; v)$ if and only if $v \equiv 1$ or 3 mod 6 and $v \geq 2k + 1$;*

(ii) For each $k \equiv 1$ or 4 mod 12 with $k > 4$, there exists an MRD $(\{4, k\}; v)$ if and only if $v \equiv 1$ or 4 mod 12 and $v \geq 3k + 1$.

Proof. Note that when $k \equiv 1$ or 3 mod 6, we can obtain an STS (v) with a sub-STS (k) from an MRD $(\{3, k\}; v)$ by constructing an STS (k) on the points of the block(s) of size k; on the other hand, we can obtain an MRD $(\{3, k\}; v)$ from an STS (v) with a sub-STS (k) by removing the blocks (but not the points) from the sub-STS (k). The proof of (ii) is similar. □

Now suppose that we start with a one-factorization of the complete graph K_6 with one-factors $F_j, j = 1, 2, \ldots, 5$. If we adjoin a new point j to each pair in F_j and define $\{1, 2, 3, 4, 5\}$ to be a block, we obtain a PBD $(\{3, 5^*\}; 11)$, that is, a PBD $(\{3, 5\}; 11)$ with exactly *one* block of size 5. The following was established by Huang, Mendelsohn, and Rosa [17].

Theorem 1.5 *For each $k \equiv 5$ mod 6, there exists a PBD $(\{3, k^*\}; v)$ with $v > k$ if and only if $v \equiv 5$ mod 6 and $v \geq 2k + 1$.*

Rees and Stinson [21] established an analogue to Theorem 1.5 for block size 4.

Theorem 1.6 *For each $k \equiv 7$ or 10 mod 12, there exists a PBD $(\{4, k^*\}; v)$ with $v > k$ if and only if $v \equiv 7$ or 10 mod 12 and $v \geq 3k + 1$.*

Theorems 1.5 and 1.6 can be translated directly into the language of mandatory representation designs:

Corollary 1.7 *(i) For each $k \equiv 5$ mod 6, there exists an MRD $(\{3, k\}; v)$ having exactly* one *block of size k if and only if $v \equiv 5$ mod 6 and $v \geq 2k + 1$.*

(ii) For each $k \equiv 7$ or 10 mod 12, there exists an MRD $(\{4, k\}; v)$ having exactly one *block of size k if and only if $v \equiv 7$ or 10 mod 12 and $v \geq 3k + 1$.*

Mandatory representation designs were formally introduced by Mendelsohn and Rees [18], who considered in detail the case where $3 \in K$. A lot of information can be determined about this case by concentrating on the particular case $K = \{3, k\}$. Now Corollaries 1.4 (i) and 1.7 (i) do not quite cover all possibilities for MRD $(\{3, k\}; v)$ when k is odd. For example, if we start with a transversal design TD $(3, 4)$ (which is equivalent to a Latin square of side 4) and adjoin a common point to the groups of size 4 we get an MRD $(\{3, 5\}; 13)$. That is, there do exist MRD $(\{3, k\}; v)$ with $k \equiv 5$ mod 6 and $v \equiv 1$ or 3 mod 6; this requires, however, that there be at least 3 blocks of size k (the number of blocks of size k must be a multiple of 3) and therefore that $v \geq 3k - 2$. The following was established in [18].

Theorem 1.8 *Let $k \equiv 5$ mod 6. Then for every $v \equiv 1$ or 3 mod 6 with $v \geq 3k - 2$, there exists an MRD $(\{3, k\}; v)$ having exactly 3 blocks of size k.*

"Most" of the designs in Theorem 1.8 are easy to obtain, using the Doyen-Wilson Theorem (Theorem 1.2/Corollary 1.4 (i)), as follows. Take a transversal design TD $(3, k - 1)$ (which is equivalent to a Latin square of side $k - 1$) and adjoin a common point to the groups to obtain an MRD $(\{3, k\}; 3k - 2)$. Since $3k - 2 \equiv 1$ mod 6, we can apply Corollary 1.4 (i) to obtain an MRD $(\{3, 3k - 2\}; v)$ for all $v \equiv 1$ or 3 mod 6 with $v \geq 2(3k - 2) + 1 = 6k - 3$; now construct a copy of the MRD $(\{3, k\}; 3k - 2)$ on the block of size $3k - 2$ in this design to yield an MRD $(\{3, k\}; v)$ for all $v \equiv 1$ or 3 mod 6 with $v \geq 6k - 3$.

Designs in the remaining interval $3k - 2 < v < 6k - 3$ were then constructed by using a combination of direct and simple recursive constructions, most notably Wilson's Fundamental Construction; see Construction 1.10.

Grütttmüller and Rees [14] considered the "analogue" to Theorem 1.8 for block size 4, noting that Corollaries 1.4 (ii) and 1.7 (ii) do not cover the remaining possibility for $k \equiv 1 \bmod 3$, namely that $k \equiv 7$ or $10 \bmod 12$ and $v \equiv 1$ or $4 \bmod 12$. This case requires that there be an even number (therefore, at least two) of blocks of size k and therefore that $v \geq 4k - 3$. The following was established in [14].

Theorem 1.9 *Let $k \equiv 7$ or 10 mod 12. There exists an MRD $(\{4, k\}; v)$ for every $v \equiv 1$ or 4 mod 12 with $v \geq 4k - 3$, with at most 56 expectations.*

In "most" cases, the MRDs constructed in Theorem 1.9 have exactly two (intersecting) blocks of size k; see ahead to Section 3 for details.

We complete this section with some terminology and notation that we will use in the sequel. General references for design theory are [2] and [6]; a general reference for graph theory is [23].

A *group-divisible design* (GDD) is a triple (X, G, B), where X is a set of points, G is a partition of X into *groups*, and B is a collection of subsets of X (blocks) such that any pair of distinct points occur together either in some groups or in exactly one block, but not both. A K-GDD of type $g_1^{t_1} g_2^{t_2} \ldots g_s^{t_s}$ is a GDD in which each block has size from the set K and in which there are t_i groups of size $g_i, i = 1, 2, \ldots, s$. We may also say K-GDD of type S, where S is the multiset containing t_i copies of $g_i, i = 1, 2, \ldots, s$. A *transversal design* TD (k, n) is a $\{k\}$-GDD of type n^k; it is well known that a TD (k, n) is equivalent to a set of $k - 2$ MOLS of order n.

One of the most important recursive constructions used in the proofs of Theorems 1.8 and 1.9 is the following, due to Wilson (see [6]).

Construction 1.10 *Let (X, G, B) be a K-GDD of type S and let $w : X \to \mathbb{Z}^+ \cup \{0\}$ (w is called a* weighting*). If for each block $b \in B$, there exists a $\{k\}$-GDD of type $\{w(x) : x \in b\}$, then there exists a $\{k\}$-GDD of type $\{\sum_{x \in G_i} w(x) : G_i \in G\}$.*

If G is a graph, then we denote by $V(G)$ and $E(G)$ the vertex set of G and the edge set of G, respectively. The *line graph* G^* of G is the graph whose vertex set is $E(G)$, in which two vertices are adjacent if and only if the corresponding edges share a vertex in G. A *clique* in a graph G is a complete subgraph of G. We denote the complement of G by G^c.

2 Mandatory Representation Designs MRD $(K; v)$ With $3 \in K$

As we stated in the introduction, we can obtain a lot of information about MRD $(K; v)$ with $3 \in K$ by paying particular attention to the case where

$K = \{3, k\}$, see [18, Section 1] for details. Now Corollaries 1.4 (i), 1.7 (i) and Theorem 1.8 establish the spectrum for these designs when k is odd. Thus, we henceforth assume that $K = \{3, k\}$ where k is even.

2.1 v is Odd

In this case, each point must be contained in an even number of blocks of size k. Since there is at least one block of size k, there must therefore be at least $k+1$ such blocks. This forms the basis for the following lower bound.

Theorem 2.1 *[18, Theorem 2.5 (iii)] Suppose that there exists an MRD* $(\{3, k\}; v)$ *with v odd. If $k \equiv 0$ or 4 mod 6, then $v \equiv 1$ or 3 mod 6 and $v \geq \frac{1}{2}k(k+1)$; if $k \equiv 2$ mod 6, then either $v \equiv 1$ or 3 mod 6 and $v \geq \frac{1}{2}k(k+1)$, or $v \equiv 5$ mod 6 and $v \geq \frac{1}{2}k(k+2)+1$.*

Now Gronau, Mullin, and Pietsch [11] and Grüttmüller [12] have shown that the conditions in Theorem 2.1 are sufficient for all even k, $4 \leq k \leq 50$, *except* for $(k, v) = (6, 21)$, by using a combination of hill-climbing and recursive constructions. Subsequently, Rees [20] established the following result.

Theorem 2.2 *The necessary conditions of Theorem 2.1 are sufficient whenever $v > \frac{1}{2}k^2 + 6k$.*

With regards the case $(k, v) = (6, 21)$, no such design exists ([see, e.g. [20]):

Theorem 2.3 *There does not exist an MRD* $(\{3, 6\}; 21)$.

Proof. An MRD $(\{3, 6\}; 21)$ would have exactly 7 blocks of size 6, with each pair of blocks intersecting in a point. These 7 blocks of size 6 therefore can be regarded as an edge-partition of K_7^* (i.e. the line graph of K_7) into cliques with 6 vertices, and the triples of the MRD would form an edge-partition of $(K_7^*)^c$ into triangles (K_3s). But no such edge-partition of $(K_7^*)^c$ exists, by [1]. □

In view of the foregoing, we are prompted to pose the following.

Conjecture 2.4 *The conditions of Theorem 2.1 are sufficient for the existence of an MRD* $(\{3, k\}; v)$, *with the single exceptional case of* $(k, v) = (6, 21)$.

This conjecture promises to be difficult to prove, particularly for values of v close to the lower bounds in Theorem 2.1. It will involve analyzing the complements of line graphs of complete graphs with a view to decomposing the edge sets of these graphs into triangles and one-factors. In the extreme case, one will be forced to consider the following very difficult problem, posed by Alspach and Heinrich in 1990 [1]:

Problem 2.5 *Does there exist an N such that if four transversals of a partial Latin square of order $n \geq N$ are prescribed, then the square can always be completed?*

For a full discussion of the foregoing, we refer the reader to [20].

2.2 v is Even

This is the most difficult case of all, as each point must be contained in an odd number (hence, at least one) of blocks of size k. The following necessary conditions for existence were established in [18] and [12]. Let $p(m) = \min\{n > 0 : K_n \text{ contains } m \text{ edge-disjoint } K_3\text{s}\}$.

Theorem 2.6 *Suppose that there exists an MRD $(\{3,k\}; v)$ with v even. Then*

(i) if $k \equiv 0$ or 4 mod 6, then $v \equiv 0$ or 4 mod 6 and $v \geq kp(t) - 2t$ when $v \equiv -2t$ mod $k, 0 \leq t < k/2$;

(ii) if $k \equiv 2$ mod 6, then $v \geq kp(t + \frac{1}{2}dk) - 2(t + \frac{1}{2}dk)$ when $v \equiv -2t$ mod $k, 0 \leq t < k/2$, where $d \equiv -\lceil v/k \rceil$ mod 3 if $v \equiv 0$ or 4 mod 6, while $d \equiv -\lceil v/k \rceil + 1$ mod 3 if $v \equiv 2$ mod $6, 0 \leq d < 3$;

(iii) $v \neq 4k - 2$.

Note that when $t = 1$, we get a lower bound of $v \geq 3k - 2$ (which can always be achieved by adjoining a common point to the groups in a TD $(3, k-1)$). On the other hand, since $p(m) \sim \sqrt{6m}$, the expression $kp(t) - 2t$ is of order $k^{\frac{3}{2}}$ when $t = k/2 - 1$. This means that for each value of k we can expect many 'gaps' in the spectrum of MRD $(\{3,k\}; v)$ for v between (roughly) $3k$ and $k^{\frac{3}{2}}$. Gronau, Mullin, and Pietsch [11] and Grüttmüller [12] have established that the necessary conditions given by Theorem 2.6 are sufficient for all even $k, 4 \leq k \leq 50$. Now, in general, because each point must lie in an odd number of blocks of size k, we cannot make effective use of the Doyen-Wilson Theorem, as was the case in the proof of Theorem 1.8, for example. Instead, we can make use of the following classes of 3-GDDs, constructed in [8].

Theorem 2.7 *(i) If $g \equiv 0$ mod 6, then there exists a 3-GDD of type $g^t m^1$ for all $t \geq 3$ and all $m \equiv 0$ mod 2 with $0 \leq m \leq g(t-1)$;*

(ii) if $g \equiv 2$ mod 6, then there exists a 3-GDD of type $g^t m^1$ for all $t \geq 3$ and all $0 \leq m \leq g(t-1)$, where $m \equiv 0$ mod 2 if $t \equiv 0$ mod $3, m \equiv 0$ mod 6 if $t \equiv 1$ mod 3, and $m \equiv 2$ mod 6 if $t \equiv 2$ mod 3;

(iii) *if* $g \equiv 4$ *mod* 6, *then there exists a 3-GDD of type* $g^t m^1$ *for all* $t \geq 3$ *and all* $0 \leq m \leq g(t-1)$, *where* $m \equiv 0$ *mod* 2 *if* $t \equiv 0$ *mod* $3, m \equiv 0$ *mod* 6 *if* $t \equiv 1$ *mod* 3, *and* $m \equiv 4$ *mod* 6 *if* $t \equiv 2$ *mod* 3.

Thus, for example, from Theorem 2.7 (i), we get the following.

Corollary 2.8 *Let* $k \equiv 0$ *mod* 6. *If there exists an MRD* $(\{3, k\}; v_0)$, *then there exists an MRD* $(\{3, k\}; v)$ *for all* $v \equiv v_0$ *mod* k *with* $v \geq 2v_0 + k$.

The strategy for $k \equiv 0$ mod 6 then is to construct a representative MRD $(\{3, k\}; v_t)$ in each fibre $v_t \equiv -2t$ mod $k, 0 \leq t < k/2$ and $t \equiv 0$ or 1 mod 3 (when $k \equiv 0$ mod 6 we must have $v \equiv 0$ or 4 mod 6, see Theorem 2.6 (i)). By using this and similar strategies for $k \equiv 2$ or 4 mod 6, Grüttmüller and Rees have obtained the following result, see [13].

Theorem 2.9 *There exists an MRD* $(\{3, k\}; v)$ *whenever*

(i) $k \equiv 0$ *mod* 6 *and* $v \equiv 0$ *or* 4 *mod* 6 *with* $v \geq 2k^2 - 7k + 8$;

(ii) $k \equiv 4$ *mod* 6 *and* $v \equiv 0$ *or* 4 *mod* 6 *with* $v \geq 2k^2 - 3k + 4$;

(iii) $k \equiv 2$ *mod* 6 *and* v *even with* $v \geq 6k^2 - 11k + 8$.

Now the bound in Theorem 2.9 (i) is obtained as follows. Let $t \equiv 0$ or 1 mod $3, 0 \leq t < k/2$, and let $s = 2t + 1$. Then there is an STS (s). Apply 'weight' $k - 1$ to this STS (that is, replace each point x in the STS by $k - 1$ new points $x_1, x_2, \ldots, x_{k-1}$ and each triple $\{x, y, z\}$ in the STS by a TD $(3, k - 1)$ with groups $\{x_1, x_2, \ldots, x_{k-1}\}, \{y_1, y_2, \ldots, y_{k-1}\}$ and $\{z_1, z_2, \ldots, z_{k-1}\}$) to obtain a 3-GDD of type $(k - 1)^s$. Adjoin a common point to the groups in this GDD to obtain an MRD $(\{3, k\}; s(k - 1) + 1)$. Note that since $s = 2t+1$ we have $v_t = s(k-1)+1 \equiv -2t$ mod k. Moreover, the maximum value for s occurs when $t = k/2 - 2$, i.e. $s = k - 3$; hence, the maximum value for v_t is $(k - 3)(k - 1) + 1 = k^2 - 4k + 4$. Applying Corollary 2.8 then yields the bound of $2k^2 - 7k + 8$. This is, of course, significantly greater than the lower bound in Theorem 2.6 (i), which was noted to have order $k^{\frac{3}{2}}$. In order to come close to this latter bound, we will need an affirmative solution to the following problem, for $k \equiv 0$ mod 6:

Problem 2.10 *Is it possible to construct a family of representative MRD* $(\{3, k\}; v_t)$*s where* $v_t \equiv -2t$ *mod* $k, 0 \leq t < k/2$ *and* $t \equiv 0$ *or* 1 *mod* 3, *so that all* v_t*s have order at most* $k^{\frac{3}{2}}$*?*

Similar problems come to consideration in the cases $k \equiv 2$ or 4 mod 6.

3 Mandatory Representation Designs MRD $(K; v)$ With $4 \in K$

As with the case $3 \in K$, we can obtain a lot of information about MRD $(K; v)$ with $4 \in K$ by concentrating on the case $K = \{4, k\}$. Now Corollaries 1.4 (ii) and 1.7 (ii) and Theorem 1.9 cover the various possibilities when $k \equiv 1 \bmod 3$. In the cases where $k \equiv 0$ or $2 \bmod 3$, Grüttmüller has developed a complex set of necessary conditions for the existence of MRD $(\{4, k\}; v)$s (see [13] or [14, Theorem 1.5]). Mullin et al. [19] have confirmed that these conditions are sufficient for $5 \leq k \leq 9$, with at most 25 exceptions.

With regard to Theorem 1.9, we remarked that in "most" of the cases, the MRDs constructed have exactly two (intersecting) blocks of size k. Many of these designs are easily obtained using Theorem 1.3/Corollary 1.4 (ii), as follows. First we construct an MRD $(\{4, k\}; 4k - 3)$ having exactly two (intersecting) blocks of size k; these exist for all $k \equiv 7$ or $10 \bmod 12$ except possibly for $k = 19$ (see [14, Lemma 3.1]). Since $4k - 3 \equiv 1 \bmod 12$, we can apply Corollary 1.4 (ii) to obtain an MRD $(\{4, 4k - 3\}; v)$ for all $v \equiv 1$ or $4 \bmod 12$ with $v \geq 3(4k - 3) + 1 = 12k - 8$; now construct a copy of the MRD $(\{4, k\}; 4k - 3)$ on the block of size $4k - 3$ in this design to yield an MRD $(\{4, k\}; v)$ having exactly two (intersecting) blocks of size k for all $v \equiv 1$ or $4 \bmod 12$ with $v \geq 12k - 8$, except possibly for $k = 19$. Solutions to Theorem 1.9 with exactly two blocks of size k are of particular interest, as they form particular solutions to the following general problem.

Problem 3.1 *For which integers $v \geq u \geq w \geq z$ does there exist a $\diamond$-IPBD $(4; v; u, w; z)$, that is, an incomplete PBD with block size 4 on v points having* two *holes, one with u points and one with w points, which intersect in a* third *hole with z points?*

Colbourn et al. [9] have considered the analogue to Problem 3.1 for block size three, and while considerable progress was made, much remains to be done. We expect therefore that Problem 3.1 will prove to be very difficult to solve.

4 Conclusion

Very little is known about the spectrum for MRD $(\{k_0, k\}; v)$s where $k_0 \geq 5$. For $k_0 = 5$ and $k \equiv 1 \bmod 4$, for instance, the spectrum for PBD $(\{5, k^*\}; v)$s have been established (or nearly established) for only a few small values of k, see [6, III.1.4].

In another direction, we mention the following related results (see [16] and [5], and [7], respectively).

Theorem 4.1 *Let* $N_{\geq 3} = \{n \in \mathbb{Z} : n \geq 3\}$. *Then a PBD* $(N_{\geq 3} \cup \{k^*\}; v)$ *with* $v > k$ *exists if and only if* $v \geq 2k+1$, *except when*

(i) $v = 2k+1$ *and* $k \equiv 0 \bmod 2$;

(ii) $v = 2k+2$ *and* $k \not\equiv 4 \bmod 6, k > 1$;

(iii) $v = 2k+3$ *and* $k \equiv 0 \bmod 2, k > 6$;

(iv) $(v,k) \in \{(7,2),(8,2),(9,2),(10,2),(11,4),(12,2),(13,2),(17,6)\}$.

Theorem 4.2 *Let* N_{odd} *denote the set of all odd positive integers. Then a PBD* $(N_{odd} \cup \{k^*\}; v)$ *with* $v > k$ *exists if* v *and* k *are odd and* $v \geq 2k+1$.

Remark 4.3 *In each of the foregoing results, the notations imply that the corresponding PBDs have* at least *one block of size* k, *except when* $k = 2$ *in Theorem* 4.1, *in which case the corresponding PBD has* exactly *one block of size* $k = 2$.

At the time of writing then, much work remains to be done on the spectra for MRD $(\{3,k\};v)$s (particularly when k and v are both even) and MRD $(\{4,k\};v)$s (particularly when $k \equiv 0$ or 2 mod 3).

It would also be of interest to investigate the analogue to Theorem 4.1 with $N_{\geq 4} = \{n \in \mathbb{Z} : n \geq 4\}$. That is, for which v and k with $v \geq 3k+1$ does there exist a PBD $(N_{\geq 4} \cup \{k^*\}; v)$?

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