Presuperschemes and Colored Directed Graphs*

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Abstract

The scheme associated with a graph is an association scheme iff the graph is strongly regular. Consider the problem of extending such an association scheme to a superscheme in the case of a colored, directed graph. The obstacles can be expressed in terms of t-vertex conditions. If a graph does not satisfy the t-vertex condition, a presuperscheme associated with it cannot be erected beyond the $(t-3)$rd level.

1 Introduction

An association scheme (Q, Γ), comprising a certain kind of partition Γ of the direct square of a finite, nonempty set Q, corresponds to a binary relation algebra. A superscheme (Q, Γ^*) comprises a certain family of partitions of the direct powers Q^{n+2} for each natural number n. Superschemes correspond to Krasner (relation) algebras. J.D.H. Smith [9] showed that an association scheme (Q, Γ) can be extended to a superscheme (Q, Γ^*) iff it is a permutation group scheme, i.e. there exists a transitive, multiplicity-free permutation group G acting on Q such that each partition Γ^n of the superscheme is the set of orbits of G acting componentwise on Q^{n+2}.

A natural question arises: what can prevent an association scheme from being built up any further to a superscheme beyond the partition Γ^t? A scheme can be associated with a graph. This scheme is an association scheme if and only if the graph is strongly regular. A height t presuperscheme consists of the bottom t levels of a superscheme. The current paper

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contains a construction of a presuperscheme associated with a colored, directed graph. If the scheme associated with a colored directed graph is an association scheme, the graph is strongly regular. The main theorem shows that if a presuperscheme associated with a colored, directed graph is of height t, then the graph satisfies the $(t+3)$-vertex condition. We provide an example of a colored graph whose presuperscheme cannot be extended beyond the association scheme level.

2 Basic definitions

Definition 2.1 [1] Let Q be a finite, non-empty set. A *(non-commutative) association scheme* (Q,Γ) on Q is a partition $\Gamma = \{C_0, \ldots, C_s\}$ of the direct square Q^2 such that three conditions are satisfied:

$$(A1) \quad C_0 = \{(x,x) | x \in Q\};$$
$$(A2) \quad \forall C_i \in \Gamma,\ \{(y,x) | (x,y) \in C_i\} \in \Gamma;$$
$$(A3) \quad \forall C_i \in \Gamma,\ \forall C_j \in \Gamma,\ \forall C_k \in \Gamma,\ \exists c(i,j,k) \in \mathbf{N}.\ \forall (x,y) \in C_k,$$
$$|\{z \in Q \mid (x,z) \in C_i,\ (z,y) \in C_j\}| = c(i,j,k).$$

Definition 2.2 [9] Let Q be a finite non-empty set. A *(non-commutative) superscheme* (Q,Γ^*) on Q is a family of partitions $\Gamma^n = \{C_0^n, \ldots, C_{i_s}^n\}$ of the direct power Q^{n+2}, for each natural number n, such that:

$$(S1)\, C_0^0 = \{(x,x) | x \in Q\};$$
$$(S2)\, \forall m,n \in \mathbf{N},\ \forall f : \{1,\ldots,m+2\} \longrightarrow \{1,\ldots,n+2\},\ \forall C_j^n \in \Gamma^n,$$
$$f^*(C_j^n) = \{(x_1,\ldots,x_{m+2}) \mid \exists (y_1,\ldots,y_{n+2}) \in C_j^n.\ \forall 1 \le i \le m+2,\ x_i = y_{f(i)}\}$$
is an element of Γ^m;
$$(S3)\, \forall (m,n) \in \mathbf{N}^2,\ \forall C_i^m \in \Gamma^m,\ \forall C_j^n \in \Gamma^n,\ \forall C_k^{m+n} \in \Gamma^{m+n},$$
$$\exists c(i,j,k;m,n) \in \mathbf{N}.\ \forall (x_0,\ldots,x_m,y_0,\ldots,y_n) \in C_k^{m+n},$$
$$|\{z \in Q \mid (x_0,\ldots x_m,z) \in C_i^m,\ (z,y_0,\ldots,y_n) \in C_j^n\}| = c(i,j,k;m,n).$$

Later on we will need modified versions of (S2) and (S3). Define

$$\mathbf{N}_t = \{n \in \mathbf{N} | n \le t\} \text{ and } \mathbf{N}_t^2 = \{(m,n) \in \mathbf{N}^2 | m+n \le t\}.$$

Then consider the following conditions:

$$(S2_t)\, \forall m,n \in \mathbf{N}_t,\ \forall f : \{1,\ldots,m+2\} \longrightarrow \{1,\ldots,n+2\},\ \forall C_j^n \in \Gamma^n,$$
$$f^*(C_j^n) = \{(x_1,\ldots,x_{m+2}) \mid \exists (y_1,\ldots,y_{n+2}) \in C_j^n.\ \forall 1 \le i \le m+2,\ x_i = y_{f(i)}\}$$
is an element of Γ^m;
$$(S3_t)\, \forall (m,n) \in \mathbf{N}_t^2,\ \forall C_i^m \in \Gamma^m,\ \forall C_j^n \in \Gamma^n,\ \forall C_k^{m+n} \in \Gamma^{m+n},$$

$$\exists c(i,j,k;m,n) \in \mathbf{N}.\, \forall (x_0,\ldots,x_m,y_0,\ldots,y_n) \in C_k^{m+n},$$
$$|\{z \in Q \mid (x_0,\ldots x_m,z) \in C_i^m,\ (z,y_0,\ldots,y_n) \in C_j^n\}| = c(i,j,k;m,n).$$

A *height t (non-commutative) presuperscheme* (Q,Γ_t^*) on Q is a family of partitions $\Gamma^n = \{C_1^n,\ldots,C_{i_s}^n\}$ of the direct power Q^{n+2}, for each number $n \in \mathbf{N}_t$, such that the conditions $(S1)$, $(S2_t)$ and $(S3_t)$ are satisfied.

Remark 2.3 Additional commutativity axioms were used in [9] and [10]:

$$(A4) \quad \forall 1 \le i,j,k \le s,\ c(i,j,k) = c(j,i,k),$$
$$(S4) \quad \forall 1 \le i,j,k \le s_0,\ c(i,j,k;0,0) = c(j,i,k;0,0).$$

However, they are not used in proofs of theorems in [9], therefore results from this paper can be translated to the non-commutative case.

Definition 2.4 [6] Let $\mathbf{G}$ be a *colored, directed graph* with vertex set V and mapping $E : V^2 \longrightarrow \{0,1,\ldots,d\}$ which assigns to each ordered pair (x,y) of vertices the value $E(x,y)$, called the *color* of the arc from x to y.

Let $W \subseteq V$. Then W induces a *colored, directed subgraph* $\mathbf{G}(W)$ of $\mathbf{G}$ with vertex set W and mapping $E_W = E|_{W^2} : W^2 \longrightarrow \{0,1,\ldots,d\}$.

Definition 2.5 [6] Let $W_1, W_2 \subseteq V$ and x and y be two (not necessarily distinct) vertices of V which belong both to W_1 and W_2. We say that subgraphs $\mathbf{G}(W_1)$ and $\mathbf{G}(W_2)$ are *of the same type with respect to the pair* (x,y) if there is an isomorphism from the subgraph $\mathbf{G}(W_1)$ to $\mathbf{G}(W_2)$ which maps x to x and y to y.

Definition 2.6 [6] We say that a colored graph $\mathbf{G}$ satisfies the *t-vertex condition on the arcs of color l,* $0 \le l \le d$, if for every k, $2 \le k \le t$, the number of k-vertex subgraphs of each fixed type, with respect to an ordered pair of vertices (x,y) joined by an arc of color l, is the same for all arcs of color l.

We call a colored graph $\mathbf{G}$, which satisfies the t-vertex condition on the arcs of all its colors, a graph with the *t-vertex condition* (or a *t-regular graph* or a *graph with depth t*).

Note that if a graph $\mathbf{G}$ satisfies the t-vertex condition, then for every $k < t$, $\mathbf{G}$ satisfies the k-vertex condition.

Definition 2.7 A colored, directed graph $\mathbf{G} = (V,E)$ is *regular* if three conditions are satisfied:

$$(R1) \quad \forall u,v \in V,\ E(u,v) = 0 \Leftrightarrow u = v,$$
$$(R2) \quad \forall 0 \le i \le d,\ \exists! 0 \le j \le d.\, \forall u,v \in V,\ E(u,v) = i \Leftrightarrow E(v,u) = j,$$
$$(R3) \quad \forall 0 \le i \le d, \exists c_i \in \mathbf{N}.\, \forall u \in V,\ |\{v \in V \mid E(u,v) = i\}| = c_i.$$

Clearly, in a regular graph $\mathbf{G}$, the number of arcs of a given color coming into a vertex is the same for all vertices.

Example 2.8 [4, 6, 8] If a colored, directed graph $\mathbf{G}$ is regular, then it satisfies the 2-vertex condition.
Conversely, if a colored, directed graph satisfies the 2-vertex condition and the condition $(R1)$ of Definition 2.7, then the graph is regular. □

Remark 2.9 A different description of a colored, directed graph with the 2-vertex condition can be found in the literature:

"Γ is a graph with the 2-vertex condition if and only if all nonisolated vertices in each of its one colored subgraphs have the same valence."

However the following example shows that this description is incorrect.

Example 2.10 Consider the colored, directed graph on four vertices, with 2-colored arcs: $\mathbf{G} = (V, E)$, where $V = \{a, b, c, d\}$ and $E(a, a) = E(b, b) = E(c, c) = E(d, d) = E(a, b) = E(b, c) = E(c, d) = E(d, a) = 0$ and $E(b, a) = E(a, d) = E(d, c) = E(c, b) = E(a, c) = E(c, a) = E(b, d) = E(d, b) = 1$. The graph $\mathbf{G}$ does not satisfy the 2-vertex condition on the arcs of color 1. Indeed, consider the type T of the 2-vertex subgraph $\mathbf{K} = (K, E_K)$, where $K = \{x, y\}$ and $E_K(x, x) = E_K(y, y) = E_K(y, x) = 0$ and $E_K(x, y) = 1$. If we choose the pair (x, y) to be (b, a) then the number of 2-vertex subgraphs of the type T with respect to the pair (x, y) joined by an arc of color 1 is 1. If we choose the pair (x, y) to be (b, d) then the number of 2-vertex subgraphs of the type T with respect to the pair (x, y) joined by an arc of color 1 is 0.

3 The 3-vertex condition

Remark 3.1 [3, 5] An undirected graph $\mathbf{G}$ is strongly regular (or an SRG) with the parameters (n, k, λ, μ) if $\mathbf{G}$ is regular, and each pair of adjacent vertices has λ common neighbors and each pair of nonadjacent vertices has μ common neighbors. The parameters n and k respectively denote the number of vertices of $\mathbf{G}$ and the valence of each vertex.

Proposition 3.2 [2, 4, 6, 8, 10] An undirected graph $\mathbf{G}$ is strongly regular iff it satisfies the 3-vertex condition. □

Definition 3.3 We say that a colored, directed graph $\mathbf{G} = (V, E)$ is *strongly regular (or an SRG)* if it satisfies the 3-vertex condition.

Definition 3.4 [5, 10] Let $\mathbf{G} = (V, E)$ be a colored, directed graph. The *scheme associated with* $\mathbf{G}$ is (V, Γ), where $\Gamma = \{C_0, C_1, \ldots, C_d\}$, where $C_l = E^{-1}(l)$ for $0 \le l \le d$.

Proposition 3.5 [6] If the scheme associated with a colored, directed graph is an association scheme, then the graph is strongly regular.
Conversely, if a colored, directed graph is strongly regular and satisfies the condition $(R1)$ of Definition 2.7, then the scheme associated with the graph is an association scheme.

Proof. Let (V, Γ), with $\Gamma = \{C_0, C_1, \ldots, C_d\}$, be the scheme associated with the colored, directed graph $\mathbf{G} = (V, E)$. Since (V, Γ) is an association scheme, the condition $(A1)$ of Definition 2.1 implies the condition $(R1)$ of Definition 2.7. The condition $(A2)$ of Definition 2.1 implies the condition $(R2)$ of Definition 2.7. Finally the condition $(A3)$ of Definition 2.1 with $k = 0$ implies the condition $(R3)$ of Definition 2.7. Therefore the graph is regular, and hence satisfies the 2-vertex condition. The condition $(A3)$ with $1 \leq k \leq d$, together with $(R2)$, implies that the number of 3-vertex subgraphs of each fixed type with respect to an ordered pair of vertices (x, y), joined by an arc of color k, is the same for all arcs of color k. Therefore the graph satisfies the 3-vertex condition.

Suppose the graph $\mathbf{G} = (V, E)$ is strongly regular. Then it satisfies the 3-vertex condition, and therefore the 2-vertex condition. It also satisfies the condition $(R1)$ of Definition 2.7. Let (V, Γ), with $\Gamma = \{C_0, C_1, \ldots, C_d\}$, be the scheme associated with the colored, directed graph $\mathbf{G} = (V, E)$. Since the graph $\mathbf{G}$ satisfies the condition $(R1)$ of Definition 2.7 and the 2-vertex condition, it is regular (see Example 2.8). Therefore the conditions $(R1)$, $(R2)$ and $(R3)$ of Definition 2.7 are satisfied. But the condition $(R1)$ means that the scheme (V, Γ) satisfies the condition $(A1)$ of Definition 2.1. The condition $(R2)$ means that the scheme (V, Γ) satisfies the condition $(A2)$ of Definition 2.1. The condition $(R3)$ means that the scheme (V, Γ) satisfies the condition $(A3)$, with $k = 0$, of Definition 2.1. And finally, since the graph $\mathbf{G}$ satisfies the 3-vertex condition, for every color k, $1 \leq k \leq d$, the number of 3-vertex subgraphs of each fixed type, with respect to an ordered pair of vertices (x, y) joined by an arc of color k, is the same for all arcs of color k. But this means that the condition $(A3)$ (with $1 \leq k \leq d$) of Definition 2.1 is satisfied, and therefore the scheme (V, Γ) is an association scheme. □

4 The main theorem

The question arises: can Proposition 3.5 be generalized? The answer is given by a connection between height t presuperschemes and $(t+3)$-regular graphs.

Let $\mathbf{G} = (V, E)$ be a colored, directed graph.

Definition 4.1 Let $\mathbf{G} = (V, E)$ be a graph with totally ordered vertex set $V = \{v_1 < v_2 < \ldots < v_n\}$. Then $\mathbf{G}$ with such ordered vertices is called a *locally ordered graph.*
Let $\mathbf{G}$ be a locally ordered graph. The *skeleton* $S(\mathbf{G})$ *of the locally ordered graph* $\mathbf{G}$ is the incidence matrix of the graph $\mathbf{G}$ with (i, j)-th entry equal to $E(v_i, v_j)$.

Next, we will generalize the scheme associated with a graph $\mathbf{G}$ to the concept of a *presuperscheme associated with the graph* $\mathbf{G}$. The generalization involves an inductive construction of $\mathbf{G}$-*compatible partitions* Γ^t of V^{t+2}. This procedure is illustrated by a flow-diagram (Figure 1). For $t \geq 0$, a $\mathbf{G}$-compatible partition Γ^t of V^{t+2} satisfies some of the properties of Definition 2.2. However, it should be noted that a $\mathbf{G}$-compatible partition at level $t > 0$ is not necessarily unique and may not be a partition of a presuperscheme; hence, the need for the left loop in the flow-diagram.

Definition 4.2 For $t = 0$, the $\mathbf{G}$-compatible partition Γ^0 of V is the scheme associated with the graph, $\Gamma^0 = \{C_0, C_1, \ldots, C_d\}$.

For $t > 0$ a $\mathbf{G}$-compatible partition Γ^t of V is defined in the following way.

The diagonal classes of Γ^t are of the form:

$$f^*(C_j^{t-1}) = \{(x_1, \ldots, x_{t+2}) | \exists (y_1, \ldots, y_{t+1}) \in C_j^{t-1}. \forall 1 \leq i \leq t+2, x_i = y_{f(i)}\}$$

for some function $f : \{1, \ldots, t+2\} \longrightarrow \{1, \ldots, t+1\}$ and some $C_j^{t-1} \in \Gamma^{t-1}$.

A non-diagonal class $C_k^t \in \Gamma^t$ consists of lists of $t + 2$ vertices of $\mathbf{G}$ which belong to locally ordered $(t+2)$-vertex subgraphs of $\mathbf{G}$ with the same skeleton. Thus there is a correspondence between non-diagonal classes C_k^t and isomorphism types of skeletons.

Definition 4.3 A set of $\mathbf{G}$-compatible partitions which satisfies the conditions of a presuperscheme from Definition 2.2 is called a *presuperscheme associated with the graph* $\mathbf{G}$.

Theorem 4.4 If a presuperscheme associated with a graph $\mathbf{G}$ has t levels, then the graph $\mathbf{G}$ satisfies the $(t + 3)$-vertex condition.

5 Proof of the main theorem

Case $t = 0$. If the $\mathbf{G}$-compatible partition Γ^0 is an association scheme (V, Γ^0), then by Proposition 3.5 the graph $\mathbf{G}$ satisfies the 3-vertex condition.
Case $t > 0$. Suppose there exists a height t presuperscheme $\{\Gamma^0, \ldots, \Gamma^{t-1}, \Gamma^t\}$ associated with the graph $\mathbf{G}$. Then the partitions $\{\Gamma^0, \ldots, \Gamma^{t-1}\}$ form

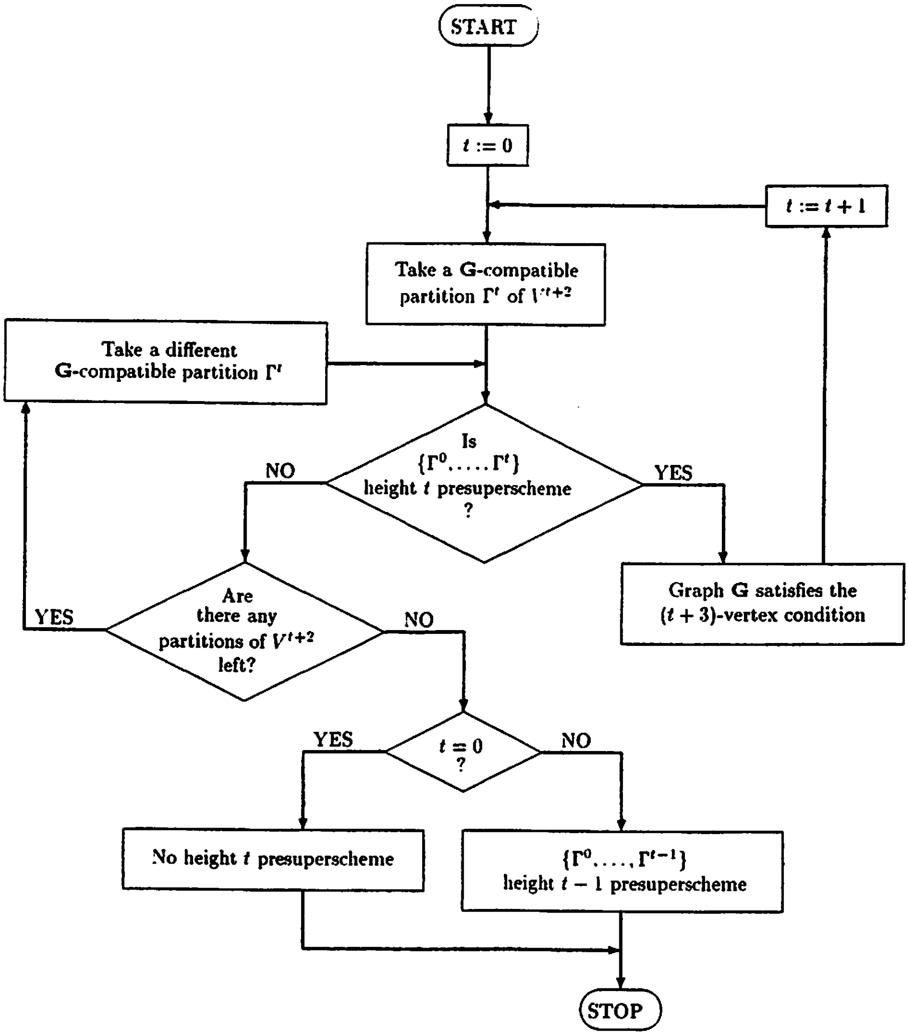

Figure 1: Flow-diagram of the construction of a presuperscheme associated with the graph **G**

a height $t-1$ presuperscheme, and by induction, the graph $\mathbf{G}$ satisfies the $(t+2)$-vertex condition.

To prove the $(t+3)$-vertex condition, it suffices to show that the number of $(t+3)$-vertex subgraphs of $\mathbf{G}$ of the same type with respect to an ordered pair of vertices (x, y) joined by an arc of color l, $0 \leq l \leq d$, does not depend on the choice of the arc of color l.

Let $\mathbf{F}$ be a $(t+3)$-vertex subgraph of $\mathbf{G}$ of some type T, containing $\{x, y\}$. Fix an ordering of the vertices of $\mathbf{F}$. WLOG, the ordered vertices of $\mathbf{F}$ are: $\{x < z < y < a_1 < \ldots < a_t\}$.

Let $\mathbf{I}$ be the locally ordered 2-vertex subgraph of $\mathbf{F}$ with vertices $\{x < z\}$.

Let $\mathbf{J}$ be the locally ordered $(t+2)$-vertex subgraph of $\mathbf{F}$ with vertices $\{z < y < a_1 < \ldots < a_t\}$.

Let $\mathbf{K}$ be the locally ordered $(t+2)$-vertex subgraph of $\mathbf{F}$ with vertices $\{x < y < a_1 < \ldots < a_t\}$.

Let $S(\mathbf{I})$, $S(\mathbf{J})$, $S(\mathbf{K})$ be the skeletons of the graphs $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ respectively.

Let C_i^0 be the class of the partition Γ^0 corresponding to the skeleton $S(\mathbf{I})$.

Let $C_{j_1}^t, \ldots, C_{j_u}^t$ be all the classes of the partition Γ^t corresponding to the skeleton $S(\mathbf{J})$.

Let $C_{k_1}^t, \ldots, C_{k_v}^t$ be all the classes of the partition Γ^t corresponding to the skeleton $S(\mathbf{K})$.

Then the number of $(t+3)$-vertex subgraphs of type T with respect to (x, y) joined by an arc of color l is equal to:

$$(5.1) \quad \frac{\sum_{q=1}^{v} \sum_{p=1}^{u} |C_{k_q}^t| c(i, j_p, k_q; 0, t)}{n c_l |\mathrm{Stab}_{x,y}(\mathrm{Aut}(\mathbf{F}))|} \qquad \text{for } E(x, y) = l,\ 1 \leq l \leq d,$$

where $\mathrm{Stab}_{x,y}(\mathrm{Aut}(\mathbf{F})) \subseteq \mathrm{Aut}(\mathbf{F})$ is the stabilizer of x and y in the automorphism group of the graph $\mathbf{F}$, and c_l denotes the number of arcs of color l going out from a vertex.

Note that in each case, the numbers do not depend on the choice of (x, y), but only on the type T (on the skeleton $S(\mathbf{F})$).

The number of $(t+3)$-vertex subgraphs of type T with respect to (x, x) is equal to:

$$(5.2) \qquad \frac{\sum_{q=1}^{v} \sum_{p=1}^{u} |C_{k_q}^t| c(i, j_p, k_q; 0, t)}{n |\mathrm{Stab}_{x}(\mathrm{Aut}(\mathbf{F}))|} \qquad \text{for } l = 0.$$

So the graph $\mathbf{G}$ satisfies the $(t+3)$-vertex condition. □

Example 5.3 Consider the 3-colored graph $\mathbf{G}$ on 16 vertices $\{v_1, \ldots, v_{16}\}$ given by the following adjacency matrix [7]:

$$\begin{bmatrix} 0 & 3 & 3 & 3 & 3 & 3 & 3 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 0 & 3 & 3 & 2 & 1 & 1 & 3 & 1 & 1 & 3 & 3 & 2 & 2 & 1 & 1 \\ 3 & 3 & 0 & 2 & 1 & 3 & 1 & 1 & 3 & 1 & 3 & 1 & 3 & 1 & 2 & 2 \\ 3 & 3 & 2 & 0 & 1 & 1 & 3 & 1 & 1 & 3 & 1 & 3 & 1 & 3 & 2 & 2 \\ 3 & 2 & 1 & 1 & 0 & 3 & 3 & 3 & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 3 & 1 & 3 & 1 & 3 & 0 & 2 & 1 & 1 & 3 & 2 & 2 & 3 & 1 & 3 & 1 \\ 3 & 1 & 1 & 3 & 3 & 2 & 0 & 1 & 3 & 1 & 2 & 2 & 1 & 3 & 1 & 3 \\ 2 & 3 & 1 & 1 & 3 & 1 & 1 & 0 & 2 & 2 & 3 & 3 & 1 & 1 & 3 & 3 \\ 2 & 1 & 3 & 1 & 1 & 1 & 3 & 2 & 0 & 2 & 3 & 1 & 3 & 3 & 1 & 3 \\ 2 & 1 & 1 & 3 & 1 & 3 & 1 & 2 & 2 & 0 & 1 & 3 & 3 & 3 & 3 & 1 \\ 1 & 3 & 3 & 1 & 1 & 2 & 2 & 3 & 3 & 1 & 0 & 2 & 1 & 3 & 3 & 1 \\ 1 & 3 & 1 & 3 & 1 & 2 & 2 & 3 & 1 & 3 & 2 & 0 & 3 & 1 & 1 & 3 \\ 1 & 2 & 3 & 1 & 2 & 3 & 1 & 1 & 3 & 3 & 1 & 3 & 0 & 2 & 1 & 3 \\ 1 & 2 & 1 & 3 & 2 & 1 & 3 & 1 & 3 & 3 & 3 & 1 & 2 & 0 & 3 & 1 \\ 1 & 1 & 2 & 2 & 3 & 3 & 1 & 3 & 1 & 3 & 3 & 1 & 1 & 3 & 0 & 2 \\ 1 & 1 & 2 & 2 & 3 & 1 & 3 & 3 & 3 & 1 & 1 & 3 & 3 & 1 & 2 & 0 \end{bmatrix}.$$

By direct check, the graph $\mathbf{G}$ satisfies the 2-vertex and the 3-vertex conditions therefore the scheme associated with G is an association scheme. However $\mathbf{G}$ does not satisfy the the 4-vertex condition. Consider the 4-vertex subgraph $\mathbf{F}$ with vertices $\{x, a, b, c\}$ and the adjacency matrix

$$S = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix}.$$

When checking the 4-vertex condition with respect to arcs of color 0 (with respect to a vertex x), we find out that if $x = v_1$, the number of such graphs is 0, while if $x = v_3$, the number of such graphs is 2. Their vertex sets are $\{v_3, v_7, v_8, v_{10}\}$ and $\{v_3, v_{12}, v_5, v_{14}\}$. Thus the presuperscheme associated with $\mathbf{G}$ has only the bottom level $\Gamma = \{C_0, C_1, C_2, C_3\}$.

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