On Super Edge-Magic Graphs*

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Abstract

A lemma of Enomoto, Llado, Nakamigawa and Ringel gives an upper bound for the edge number of a super edge-magic graph with $p > 1$ vertices. In this paper we give some results which come out from answering some natural questions suggested by this useful lemma.

1 Introduction

All graphs in this paper are finite and have no loops or multiple edges. We use $G(p, q)$ to denote a graph G with p vertices and q edges. Following [8], we call a one-to-one function f from elements of a graph G to integers as a *labeling* of G. When the domain of f is $V(G)$ ($E(G)$, resp.), f is called a *vertex* (*edge*, resp.) *labeling* of G. When the domain of f is $V(G) \cup E(G)$, f is called a *total labeling* of G. In general we follow the graph-theoretic notation and terminology of [1] unless otherwise specified.

According to Ringel and Llado's definition [6], a graph $G(p, q)$ is called *edge-magic* if there is a bijection f from $V(G) \cup E(G)$ to $\{1, 2, ..., p + q\}$ such that the sum $f(uv) + f(u) + f(v)$ is a constant s for all edges uv of G. We shall follow [8] to call such a function f an *edge-magic total labeling* (to avoid confusion), and call the constant s the *magic sum* of G corresponding to f. A graph $G(p, q)$ is called *super edge-magic* if there is an edge-magic total labeling f satisfying $f(V(G)) = \{1, 2, ..., p\}$.

The idea of edge-magic graphs was introduced and studied by Kotzig and Rosa [4] nearly three decades ago with a different name as graphs with *magic valuation*. It has regained researchers' attention since Ringel and Llado's paper [6] was published in 1996. In Gallian's most recent version

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of his inclusive dynamic survey [3] on graph labelings we can find a variety of results on this topic. It should be noted that the words "edge-magic" and "super" have been used with different meaning in the literature, see for example [5] and [7].

In [2], there is a useful lemma giving an upper bound for the edge number of a super edge-magic graph with $p > 1$ vertices:

Lemma 1 *If $G(p,q)$ is non-trivial and super edge-magic, then $q \leq 2p-3$.*

This lemma naturally suggests the following questions.

(1) Is this upper bound the best possible?

Notice that for any tree T the join $K_1 + T$ satisfies the condition $q = 2p-3$. (Recall the definition: The join of graphs G_1 and G_2, denoted as $G_1 + G_2$, is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2.) Can we find a family of infinitely many trees T such that $K_1 + T$ is super edge-magic for each T in the family?

(2) A more general question is, does there exist a connected super edge-magic graph $G(p,q)$ for any positive integers $p > 1$ and $p-1 \leq q \leq 2p-3$?

(3) What can we say about super edge-magic regular graphs?

(4) What can we say about the join $G_1 + G_2$ for general G_1,G_2?

These questions are discussed in the next section. In addition, a necessary condition for regular edge-magic graphs is also presented, which enables us to present two infinite families of regular graphs that are not edge-magic.

2 Main results

There is an obvious duality in the edge-magic total labelings (see [8]): If f is an edge-magic total labeling of $G(p,q)$, then so is f^*, where $f^*(x) = p+q+1-f(x)$. Obviously, the dual of a super edge-magic total labeling is an edge-magic total labeling where $V(G)$ receives labels $q+1, q+2, \ldots, q+p$ and $E(G)$ receives labels $1, 2, \ldots, q$. In fact one can easily prove:

A graph $G(p,q)$ is super edge-magic if and only if there is an edge-magic total labeling f such that $f(E(G)) = \{1, 2, \ldots, q\}$.

This gives an alternative definition for the super edge-magic graphs.

For convenience, we shall call a set S of integers *consecutive* if S consists of consecutive integers.

Lemma 2 *A graph $G(p,q)$ is super edge-magic if and only if there is a vertex labeling f such that the two sets $f(V(G))$ and $\{f(u)+f(v) : uv \in E(G)\}$ are both consecutive.*

The proof of Lemma 2 is easy and is left to the reader.

Lemma 3 [3] *A complete bipartite graph $K_{m,n}$ is super edge-magic if and only if m=1 or n=1.*

In other words, the stars are the only complete bipartite graphs that are super edge-magic.

Now we are ready to answer the questions raised in section 1.

Theorem 1 *The join of K_1 with any star is super edge-magic.*

Proof. Let $G = K_1 + T$ where T is a star. It is a trivial case when $|V(T)| = 1$. So we assume $T = K_{1,p-2}$ with $p > 2$.

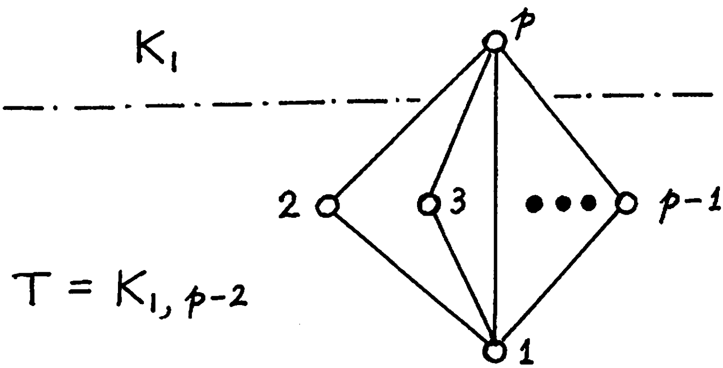

Figure 1. $K_1 + T$

As depicted in Figure 1, we define a vertex labeling f of G such that

$$f(v) = \begin{cases} 1 & \text{if } v \text{ is the center of the star } T \\ p & \text{if } v \text{ is the vertex of the } K_1, \end{cases}$$

and the pendant vertices of T are labeled as $\{2, 3, \ldots, p-1\}$.
Clearly,
$f(V(G)) = \{1, 2, \ldots, p\}$ and
$\{f(u) + f(v) : uv \in E(G)\} = \{3, 4, \ldots, 2p-1\}$.
Therefore, the graph $G = K_1 + T$ is super edge-magic by Lemma 2.

Theorem 1 answers the first question completely. The answer for the second question is given in the following

Theorem 2 *For integers $p > 1$ and $q > 0$, there is a connected super edge-magic graph with p vertices and q edges if and only if $p - 1 \leq q \leq 2p - 3$.*

Proof. The necessity of the inequality is directly from Lemma 1 and the connectedness of the graph.

From Theorem 1, there is a connected super edge-magic graph $H = K_1 + K_{1,p-2}$ with p vertices and $q = 2p-3$ edges. So we only need to show there exits a connected super edge-magic graph $G(p,q)$ when $p-1 \le q < 2p-3$. In fact, $G(p,q)$ can be obtained from the graph H (with the labeling given in the proof of Theorem 1) by deleting the k edges $\{(1,2),(1,3),\ldots,(1,k+1)\}$ where $k = 2p-3-q$. By Lemma 2, $G(p,q)$ is super edge-magic.

Note: From the above proof we see that Theorem 1 can be generalized to the more general form as follows.

Theorem * 1 *The join of K_1 with any non-empty subgraph of a star is super edge-magic.*

Now we come to consider the third question.

Theorem 3 *If a k-regular graph is super edge-magic then $k \le 3$.*

Proof. By contradiction. Let $G(p,q)$ be a k-regular, super edge-magic graph with $k \ge 4$. Then $q = kp/2 \ge 2p$, contradicting Lemma 1.

Kotzig and Rosa [5] showed that qK_2 is edge-magic if and only if q is odd. In fact, for $G = qK_2$ with q odd, let $E(G) = \{e_1, e_2, \ldots, e_q\}$ with $e_i = u_i v_i$ for $i = 1,2,\ldots,q$. Define a total labeling f of G such that

$$f(e_i) = 3q + 1 - i,$$

$$f(u_i) = \begin{cases} \lceil i/2 \rceil & \text{when } i \text{ is odd} \\ (q+1+i)/2 & \text{when } i \text{ is even,} \end{cases}$$

$$f(v_i) = (1+3q)/2 + i - f(u_i),$$

for $i = 1,2,\ldots,q$.

It is easy to show that f is a super edge-magic total labeling. Thus we have proven:

Theorem 4 *A 1-regular graph G is super edge-magic if and only if $G = qK_2$ with q odd.*

That is, each 1-regular, super edge-magic graph G is the union of an odd number of disjoint edges.

It is also known [3] that the connected 2-regular super edge-magic graphs are odd cycles. Considering the connected 3-regular super edge-magic graphs, we have the following result.

Theorem 5 *There is a connected 3-regular super edge-magic graph G with p vertices if and only if $p \equiv 2(mod\ 4)$.*

Proof.

$\Rightarrow$. Assume there is a connected 3-regular super edge-magic graph $G(p,q)$. Then p must be even so that either $p \equiv 0(mod\ 4)$ or $p \equiv 2(mod\ 4)$. If $p \equiv 0(mod\ 4)$, we have $p = 4h$ for some positive integer h. Then $q = 3p/2 = 6h$. Let f be a super edge-magic total labeling of G with the magic sum s. Then

$$\begin{aligned} qs &= 3 \sum_{v \in V(G)} f(v) + \sum_{e \in E(G)} f(e) \\ &= 2 \sum_{v \in V(G)} f(v) + (1 + 2 + \ldots + (p+q)) \\ &= p(1+p) + (p+q)(1+p+q)/2, \end{aligned}$$

implying that $s = 11h + 3/2$. It is impossible. So $p \equiv 2(mod\ 4)$.

$\Leftarrow$. Let $p \equiv 2(mod\ 4)$, i.e., $p = 4k+2$. For convenience we let $n = p/2 = 2k+1$. We construct a connected 3-regular graph G with p vertices from two cycles C_n and C'_n as follows. Let

$V(C_n) = \{a_0, a_1, \ldots, a_{n-1}\}, E(C_n) = \{a_i a_{i+1} : i = 0, 1, , \ldots, n-1\}$,

$V(C'_n) = \{a'_0, a'_1, \ldots, a'_{n-1}\}, E(C'_n) = \{a'_i a'_{i+1} : i = 0, 1, , \ldots, n-1\}$.

(Note that in the proof the addition or substraction in indexes is always taken modulo n.) Now we define $V(G) = V(C_n) \cup V(C'_n), and$ $E(G) = E(C_n \cup E(C'_n) \cup \{a_i a'_{n-ik} : i = 0, 1, , \ldots, n-1\}$. Clearly, G is a connected graph with p vertices. It is easy to show that G is 3-regular as follows. For $0 \leq i \neq j \leq n-1, n$ does not divide $(j-i)k$, since $gcd(k,n) = 1$ and n does not divide $(j-i)$. Then $(n-ik)$ is not congruent to $(n-jk)$ modulo n. Therefore G is 3-regular.

Now we give a vertex labeling f of G as follows (see Figure 2 for the case $p = 10$.) Let

$$f(a_i) = \begin{cases} (i+2)/2 & \text{if } i \text{ is even} \\ k + (i+3)/2 & \text{if } i \text{ is odd,} \end{cases}$$

$$f(a'_i) = \begin{cases} 2n - i/2 & \text{if } i \text{ is even} \\ 2n - k - (i+1)/2 & \text{if } i \text{ is odd,} \end{cases}$$

for $i = 0, 1, 2, \ldots, n-1$.

Then it is not difficult to verify that
$f(V(G)) = \{1, 2, \ldots, 2n\}$, and
$\{f(u) + f(v) : uv \in E(G)\} = A \cup B \cup C = \{k+2, k+3, \ldots, 4n-k\}$,
where
$A = \{f(a_i) + f(a_{i+1}) : i = 1, 2, \ldots, n-1\} = \{k+2, k+3, \ldots, k+n+1\}$,
$B = \{f(a'_i) + f(a'_{i+1}) : i = 1, 2, \ldots, n-1\} = \{3n-k+1, 3n-k+2, \ldots, 4n-k\}$,
$C = \{f(a_i) + f(a'_{n-ik}) : i = 1, 2, \ldots, n-1\} = \{n+k+2, n+k+3, \ldots, 3n-k\}$.
Therefore, G is super edge-magic by Lemma 2.

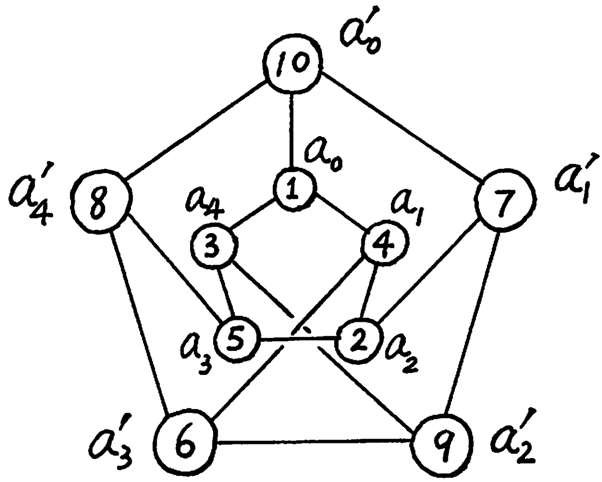

Figure 2. The case p=10 (Petersen graph).

Note: The graph depicted in Figure 2 is the Petersen graph. So the Petersen graph is super edge-magic.

Regarding the fourth question, we have the following result.

Theorem 6 *The join of nontrivial graphs G_1 and G_2 is super edge-magic if and only if*
(i) at least one of G_1 and G_2 has exactly two vertices, and
(ii) $G_1 \cup G_2$ has exactly one edge.

Proof.
$\Rightarrow$. Assume $G_1 + G_2$ is super edge-magic with $|V(G_1)| = m \geq 2$ and $|V(G_2)| = n \geq 2$. Let $|E(G_1 \cup G_2)| = k$. Then $|V(G_1 + G_2)| = m + n$ and $|E(G_1 + G_2)| = mn + k$. By Lemma 1, we have $mn + k \leq 2(m+n) - 3$. It

follows that

$0 \le (m-2)(n-2) \le 1-k$ .... (*)

So, $k \le 1$.

Note that if $k = 0$, $G_1 + G_2$ is a complete bipartite graph so that it can not be super edge-magic by Lemma 3. Hence, we must have $k = 1$, i.e., $|E(G_1 \cup G_2)| = 1$. Then it follows from (*) that $(m-2)(n-2) = 0$. Therefore, we must have $m = 2$ or $n = 2$.

$\Leftarrow$. Let $G = G_1 + G_2$ satisfy the condition that $2 = |V(G_1| \le |V(G_2|$ and $|E(G_1 \cup G_2)| = 1$. Then there are two cases as depicted in Fig. 3, depending on the unique edge of $G_1 \cup G_2$ belonging to G_1 or G_2. It is easy to verify that the vertex labeling given for each case in Figure 3 satisfies the condition of Lemma 2. Therefore, G is super edge-magic by Lemma 2.

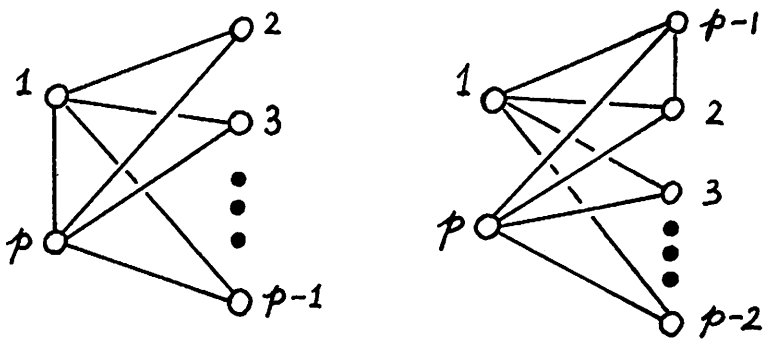

Figure 3. Two cases of $G_1 + G_2$

It is known (see [5] or [6]) that all caterpillars are super edge-magic. Enomoto et al. [3] conjecture that all trees are super edge-magic. Our next result gives support to this conjecture.

Theorem 7 *For any $n \ge 7$, there is a non-caterpillar tree with n vertices that is super edge-magic.*

Proof. For $n = 7$, there is a unique non-caterpillar tree , which is depicted in Figure 4. It is easy to verify that the vertex labeling given in Figure 4 satisfies the condition of Lemma 2 so that the tree is super edge-magic. For $n > 7$, let $n = 7 + k$ with $k \ge 1$. The graph depicted in Figure 5 is a non-caterpillar tree with n vertices. The given vertex labeling also shows that the graph is super edge-magic by Lemma 2.

Finally we give a necessary condition for a regular graph to be edge-magic.

Theorem 8 *Let $G(p,q)$ be k-regular ($k \ge 1$) and edge-magic. Let $d = gcd(k-1, q)$. Then $(p+q)(1+p+q) \equiv 0 (mod\ 2d)$.*

Proof. Let f be an edge-magic total labeling of G with the magic sum s. Then

$$\begin{aligned} qs &= k(\sum_{v\in V(G)} f(v)) + \sum_{e\in E(G)} f(e) \\ &= (k-1)\sum_{v\in V(G)} f(v) + (1+2+\ldots+(p+q)). \end{aligned}$$

It follows that

$$(p+q)(1+p+q) = 2(qs - (k-1)\sum_{v\in V(G)} f(v)).$$

Therefore $(p+q)(1+p+q) \equiv 0(mod\ 2d)$.

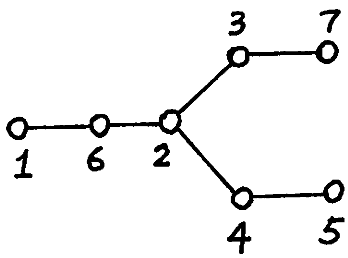

Figure 4. n=7.

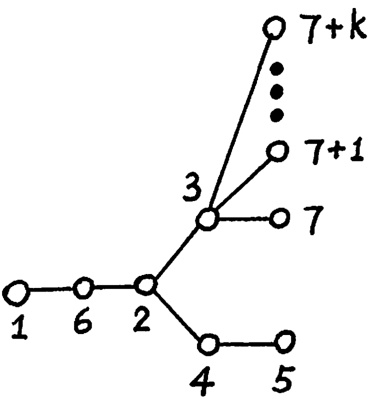

Figure 5. n=7+k.

Now we give some corollaries.

Corollary 1 *The join $nK_2 + nK_2$ is not edge-magic for any $n \geq 1$.*

Proof. Let $G(p,q) = nK_2 + nK_2$. Clearly $p = 4n$, $k = 2n+1$, and $q = kp/2 = 2n(2n+1)$. Then $d = gcd(k-1, q) = 2n$. It is easyly verified that $(p+q)(1+p+q) = 2n(2n+3)(1+2n(2n+3))$ so that $2d$ cannot divide $(p+q)(1+p+q)$.

Therefore, G is not not edge-magic by Theorem 8.

Note that $nK_2 + nK_2$ is K_4 when $n = 1$. So, we see that K_4 is not edge-magic.

Corollary 2 *There are infinitely many 3-regular hamiltonian plane graphs which are not edge-magic.*

Proof. For any odd cycle, replace each vertex by the same graph $K_2 + 2K_1$ to obtain a 3-regular graph as depicted in Figure 6.

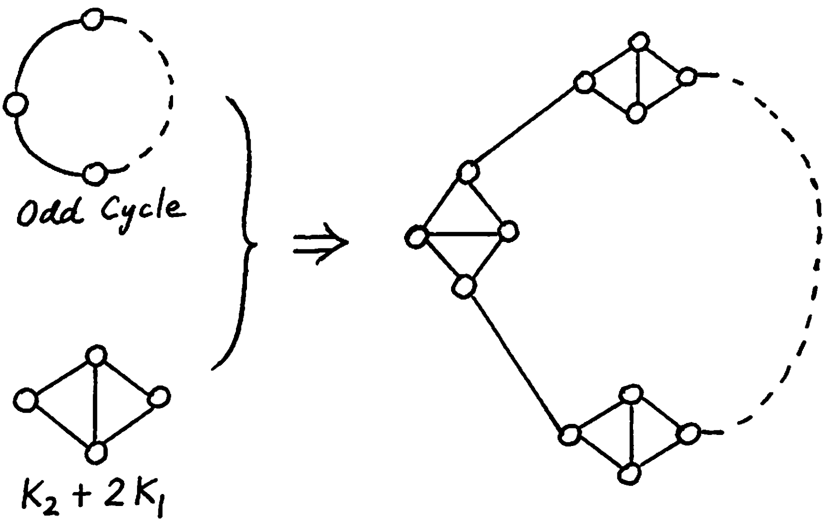

Figure 6

It is easy to see each graph obtained in this way is not edge-magic by Theorem 8.

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