ON $F(j)$-GRAPHS AND THEIR APPLICATIONS*

Narong Punnim
Department of Mathematics
Srinakharinwirot University
Sukhumvit Soi 23, Bangkok 10110, Thailand

Abstract

Erdös and Gallai (1963) showed that any r-regular graph of order n, with $r < n-1$, has chromatic number at most $3n/5$, and this bound is achieved by precisely those graphs with complement equal to a disjoint union of 5-cycles.

We are able to generalize this result by considering the problem of determining a $(j-1)$-regular graph G of minimum order $f(j)$ such that the chromatic number of the complement of G exceeds $f(j)/2$. Such a graph will be called an *$F(j)$-graph*. We produce an $F(j)$-graph for all odd integers $j \geq 3$ and show that $f(j) = 5(j-1)/2$ if $j \equiv 3 \pmod 4$, and $f(j) = 1 + 5(j-1)/2$ if $j \equiv 1 \pmod 4$.

1. Introduction

All graphs considered in this paper are finite, simple and loopless. We use the terminology of Parthasarathy [4]. A *covering* of a graph G is a partition P of $V(G)$ such that for each $V_i \in P$, the induced subgraph $G[V_i]$ in G is a complete graph. The *covering number* of G is denoted by $c(G)$ and is defined by

$$c(G) := \min\{|P| : P \text{ is a covering of } G\}.$$

Evidently $c(G) = \chi(\bar{G})$ for any graph G and its complement $\bar{G}$, where χ denotes chromatic number. Erdös & Gallai [1] used this relationship when proving that any r-regular graph of order n with $r < n-1$ has chromatic number at most $3n/5$, and this bound is achieved by precise those graphs with complement equal to a disjoint union of 5-cycles. This means that the

*Paper presented at the Thirteenth Midwestern Conference on Combinatorics, Cryptography and Computing, Normal, Illinois, October 1999.

bound $3n/5$ is the best for an r-regular graph of order n when $r = n-3$, so $j := n-r = 3$. In this paper, we define an *$F(j)$-graph* to be a $(j-1)$-regular graph of minimum order $f(j)$ with the property that its covering number exceeds $f(j)/2$. We determine $F(j)$-graphs for all odd integers $j \geq 3$. In fact, we shall prove the following theorem:

Theorem 1. *For any odd integer $j \geq 3$, we have $f(j) = \frac{5}{2}(j-1)$ if $j \equiv 3 \pmod 4$ and $f(j) = 1 + \frac{5}{2}(j-1)$ if $j \equiv 1 \pmod 4$.*

As an application, we shall have the following theorem:

Theorem 2. *Any r-regular graph of order n with odd $j := n - r \geq 3$ has chromatic number at most $\frac{f(j)+1}{2f(j)} \cdot n$, and this bound is achieved by precise those graphs with complement equal to a disjoint union of $F(j)$-graphs.*

2. Upper bound

In this section, we shall give an upper bound for $f(j)$ by constructively showing the existence of a $(j-1)$-regular, triangle-free hamilton graph on $g(j)$ vertices for odd $j \geq 3$, where $g(j) = \frac{5}{2}(j-1)$ if $j \equiv 3 \pmod 4$, and $g(j) = 1 + \frac{5}{2}(j-1)$ if $j \equiv 1 \pmod 4$. Before going to the proof, we introduce some graph construction notation. Let A, B be non-empty sets. Let $K(A, B)$ be the complete bipartite graph with bipartitioning sets A and B. Let G and H be any two graphs, with $V(G)$ and $V(H)$ not necessarily disjoint. Then $G + H$ is the graph obtained from G and H such that $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H)$. It is clear that the binary operation "+" is associative.

If $j \equiv 3 \pmod 4$, put $j = 4k+3$. Let V_1, V_2, V_3, V_4, V_5 be pairwise disjoint sets, each of which has cardinality $2k + 1$. We define

$$G(j) := K(V_1, V_2) + K(V_2, V_3) + K(V_3, V_4) + K(V_4, V_5) + K(V_5, V_1).$$

Each $v \in G(j)$, has degree $d(v) = 4k+2 = j-1$, so $G(j)$ is $(j-1)$-regular. Furthermore $G(j)$ is a triangle-free and hamiltonian. Thus $c(G(j)) = \frac{1}{2}(g(j) + 1)$, where $g(j) = \frac{5}{2}(j-1)$.

If $j \equiv 1 \pmod 4$, put $j = 4k+1$. Let V_1, V_2, V_3, V_4, V_5 be pairwise disjoint sets such that $|V_1| = 2k-1$, $|V_2| = |V_5| = 2k$ and $|V_3| = |V_4| = 2k+1$. We define

$$G(j) := K(V_1, V_2) + K(V_2, V_3) + H + K(V_4, V_5) + K(V_5, V_1)$$

where H is a $2k$-regular bipartite graph with V_3 and V_4 as its bipartitioning sets. Each $v \in G(j)$ has degree $d(v) = 4k = j-1$. Once again $G(j)$ is $(j-1)$-regular triangle-free hamilton graph on $g(j) = 10k+1 = 1 + \frac{5}{2}(j-1)$. Thus $c(G(j)) = \frac{1}{2}(g(j) + 1)$.

Hence we have $f(j) \leq g(j)$ for all odd $j \geq 3$.

3. Elementary facts about $F(j)$-graph

In Section 4 we shall show that $f(j) = g(j)$ by assuming that $f(j) < g(j)$ and obtaining a contradiction. But first, we need to develop some elementary facts about $F(j)$-graph which we shall require. From now on, if G is an $F(j)$-graph, then G will have order $f(j)$ and we shall assume that $f(j) < g(j)$. It is clear that if G is a graph of order n with 1-factor then $c(G) \leq \frac{n}{2}$. Thus an $F(j)$-graph has no 1-factor. Wallis [6] showed that if G is a $(j-1)$-regular graph of even order and G has no 1-factor, then $n \geq 3(j-1)+4$ if j is odd and $n \geq 3(j-1)+7$ if j is even. This exceeds the bound $f(j) \leq g(j) \leq 1 + \frac{5}{2}(j-1)$ obtained in Section 2, so if G is an $F(j)$-graph it follows that its order $f(j)$ must be odd. We had $g(j) = 10k+5 = \frac{5}{2}(j-1)$ when $j = 4k+3$, and $g(j) = 10k+1 = 1+\frac{5}{2}(j-1)$ when $j = 4k+1$, so because $f(j)$ is odd, in both cases $f(j) < g(j)$ requires $f(j) < \frac{5}{2}(j-1)$. Hence $f(j) < \frac{5}{2}(j-1)$.

Jackson [3] showed that a 2-connected r-regular graph with at most $3r$ vertices is hamiltonian.

Proposition 1 *Any $F(j)$-graph is hamiltonian.*

Proof The claim holds when $j = 3$ since it is clear that $f(3) = 5$ and the $F(3)$-graph is a 5-cycle. Now assume that $j \geq 5$. It is enough to show that if G is an $F(j)$-graph, then G is 2-connected. Moreover, by the minimality of the order $n := f(j)$ of G, we may assume that G is connected. If G has a cut vertex v and $G \setminus \{v\} = G_1 \cup G_2$, where $|G_1| = n_1$ and $|G_2| = n_2$, then $\delta(G_1) = \delta(G_2) = j-2$. Thus $\Delta(\overline{G_1}) = n_1 - j + 1$ and $\Delta(\overline{G_2}) = n_2 - j + 1$, where $\overline{G_1}$ and $\overline{G_2}$ are the complements of G_1 and G_2 in K_{n_1} and K_{n_2} respectively. By Brooks's Theorem [4, p. 279], we have $\chi(\overline{G_1}) \leq n_1 - j + 1$ and $\chi(\overline{G_2}) \leq n_2 - j + 1$. Thus

$$\begin{aligned} c(G) &\leq 1 + c(G_1) + c(G_2) \\ &= 1 + \chi(\overline{G_1}) + \chi(\overline{G_2}) \\ &\leq 1 + n_1 - j + 1 + n_2 - j + 1 \\ &= n - 2j + 3. \end{aligned}$$

Since G is an $F(j)$-graph, $\frac{n}{2} < c(G) \leq n - 2j + 3$. Therefore $n > 4j - 6 > f(j)$, which is a contradiction. Hence G is 2-conected, as required. ∎

Results of Hanson, Wang and MacGillivray [2] show for odd $n < \frac{5}{2}(j-1)$ and odd $j \geq 5$ that a $(j-1)$-regular graph of order n must contain a triangle. In proving our result in the next section, a triangle in a hamilton cycle of an $F(j)$-graph will play a crucial role in order to get a contradiction, so, we shall name such a triangle according to the way it fits in the hamilton cycle. Let H be a hamilton cycle and ABC be a triangle as shown in Figure 1. Since $p+q+r+3=f(j)$ is the order of the odd cycle H, it follows that $p+q+r$ must be even. We call ABC an *even triangle* with respect to H if p, q, r are all even; otherwise ABC is an *odd triangle.*

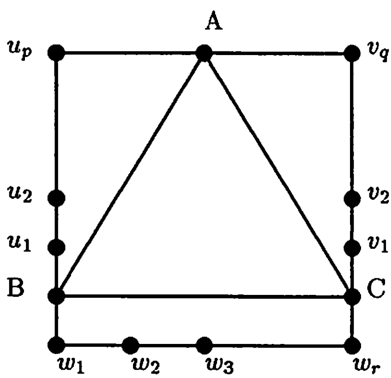

Figure 1

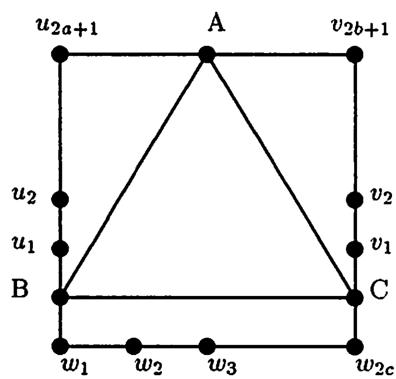

Figure 2

We omit the details of routine argument which proves the following proposition:

Proposition 2 *Let G be an $F(j)$-graph. Then neither of the following is possible:*

1) G has a hamilton cycle H and a triangle ABC which is even with respect to H;

2) G has a hamilton cycle H and two vertices u, v which are adjacent in G and at distance 2 on H.

Hence, without loss of generality we may assume that $p=2a+1, q=2b+1$ and $r=2c$, where a, b, c are positive integers (see Figure 2). Let $U=\{u_1, u_2, \ldots, u_{2a+1}\}$, $V=\{v_1, v_2, \ldots, v_{2b+1}\}$ and $W=\{w_1, w_2, \ldots, w_{2c}\}$. Further, we let $X=\{u_1, u_3, \ldots, u_{2a+1}, v_1, v_3, \ldots, v_{2b+1}\}$ and let $Y=\{u_2, u_4, \ldots, u_{2a}, v_2, v_4, \ldots, v_{2b}\}$.

We also omit the details of routine argument which proves the following proposition:

Proposition 3 *Let G be an $F(j)$-graph. Then neither of the following is possible:*

1) X contains vertices u_i and v_j which are adjacent in G;

2) X contains any vertex which is adjacent to both B and C in G.

Finally, let G be an $F(j)$-graph and H a hamilton cycle of G labeled so that $V(H) = \mathbb{Z}_n$ and $ij \in E(H)$ if and only if $|i-j| = 1$. Since G cannot have an even triangle, each $i \in \mathbb{Z}_n$, can be adjacent to at most one vertex in $\{i+2k-1, i+2k\}$. Since $n = f(j) < g(j)$, each $i \in \mathbb{Z}_n$ must be adjacent to at least one vertex on any path of length at least $\frac{1}{2}(j-1)$ along the hamilton cycle.

4. Proof that $f(j) = g(j)$

By using facts from Section 3, we are now able to show that $f(j) = g(j)$. To show this, we use the graph in Figure 2 and consider two cases, when $c = 0$ and $c > 0$. With the assumption that $f(j) < g(j)$, we shall finally get a contradiction in each case.

Proposition 4 $f(j) = g(j)$.

Proof (1) Suppose $c = 0$. Let G be an $F(j)$-graph containing a hamilton cycle H with an odd triangle ABC such that B is adjacent to C on H. Using the notations in Section 3, we find that $2(a+b)+5 = f(j) < \frac{5}{2}(j-1) < g(j)$ and hence $|X| = a+b+2 < \frac{5}{4}(j-1)$. Since the hamilton cycle H has no even triangle, $|N(B)\cap X|+|N(C)\cap X| = |(N(B)\cup N(C))\cap X| \leq |X| < \frac{5}{4}(j-1)$. Thus $|N(B)\cap Y| + |N(C)\cap Y| > 2(j-1) - 4 - \frac{5}{4}(j-1) = \frac{3}{4}(j-1) - 4$. We may assume that $|N(B)\cap Y| > \frac{3}{8}(j-1) - 2$. Hence there are at most $(a+b) - \frac{3}{8}(j-1) + 2 < \frac{5}{4}(j-1) - \frac{3}{8}(j-1) = \frac{7}{8}(j-1)$ vertices in Y not adjacent to B. By looking at the vertex u_1, we have $N(u_1)\cap N(B)\cap Y \neq \emptyset$ or $N(u_1)\cap X \neq \emptyset$. If $N(u_1)\cap N(B)\cap Y \neq \emptyset$, then there exists an even triangle with respect to H. Assume that there exists $x \in X$ such that u_1 is adjacent to x. If x is in V, then G is not an $F(j)$-graph. Thus u_1 must be adjacent to some u_i in X for some odd integer $i \geq 5$. If there exists $u_s \in U$ with $1 \leq s \leq i$ such that u_s is adjacent to v_k in X, then G is not an $F(j)$-graph. If $u_s \in U$ with $1 \leq s \leq i$ is adjacent to u_t in $X, t > i$, and there exists $u_m \in U$ with $1 \leq m \leq t$ such that u_m is adjacent to v_k in X, then G is not an $F(j)$-graph. Without loss of generality, we may assume that u_1 is adjacent to u_i for some odd integer $i \geq 5$ and no $u_s \in U$ with $1 \leq s \leq i$ is adjacent to any vertex in X except on the path from u_1 to u_i along the hamilton cycle H. It should be noted that for the path from u_1 to u_i along the cycle H, we find that u_i can be adjacent to at most one vertex of each pair of the form $\{u_{2k-1}, u_{2k}\}$ with $1 \leq k \leq \frac{1}{2}(i-1)$. Similarly u_{i-1} can be adjacent to at most one vertex of each pair of the form $\{u_{2k}, u_{2k+1}\}$ with $1 \leq k \leq \frac{1}{2}(i-1)$. Moreover, $N(u_i) \cap N(u_{i-1}) \cap (Y \setminus \{u_1, u_2, \ldots, u_i\}) = \emptyset$ because, otherwise, we will have an even triangle. Thus $|N(u_i)| + |N(u_{i-1})| = 2(j-1) \leq i + \frac{2a+1-i}{2} + b + 3$. Therefore $\frac{i+1}{2} \geq 2(j-1) - (a+b+3) \geq 2(j-1) - \frac{5}{4}(j-1) = \frac{3}{4}(j-1)$. In this case we have $i \geq \frac{3}{2}(j-1) - 1$ and there must exist $u_s \in U$ with $1 \leq s \leq i$ which is adjacent to v_1. This is a contradiction.

(2) We may now suppose $c > 0$ and there is no triangle which has $c = 0$ relative to any hamilton cycle in G.

In order to get a contradiction in this case, we first consider the following facts:

1. If x, y, z are 3 consecutive vertices along the hamilton cycle H, then $N(x) \cap N(y) = N(y) \cap N(z) = \emptyset$ and $|N(x) \cap N(z)| \geq \frac{1}{2}(j-1)$.

2. If x, y, z are 3 vertices in G such that x, y, z form a path in G, then $|N(x) \cap N(y)| + |N(y) \cap N(z)| + |N(x) \cap N(z)| \geq \frac{1}{2}(j-1)$.

Since G is an $F(j)$-graph of order $f(j) < g(j)$, any hamilton cycle of G cannot contain an even triangle. We may assume that ABC is an odd triangle with respect to H such that c is minimized. By this assumption, we have $N(B) \cap N(C) \cap W = \emptyset$.

(2.1) Suppose u_1 is not adjacent to any vertex in X. Then $|N(B) \cap Y| < \frac{1}{4}(j-1)$, because otherwise $N(u_1) \cap N(B) \neq \emptyset$ or u_1 must be adjacent to consecutive vertices in W. Similarly, $|N(C) \cap Y| < \frac{1}{4}(j-1)$. We now consider the path $u_1 - B - C$, we have

$$|N(u_1) \cap N(B)| + |N(B) \cap N(C)| + |N(u_1) \cap N(C)| \geq \frac{1}{2}(j-1).$$

But $N(u_1) \cap N(B) = \emptyset$ and $A \in N(B) \cap N(C)$. Furthermore, since $N(B) \cap N(C) \cap X = N(B) \cap N(C) \cap W = N(u_1) \cap N(C) \cap X = \emptyset$, we have $|N(B) \cap N(C) \cap Y| + |N(u_1) \cap N(C) \cap Y| + |N(u_1) \cap N(C) \cap W| \geq \frac{1}{2}(j-1)$. Since $|N(C) \cap Y| < \frac{1}{4}(j-1)$ and $N(u_1) \cap N(B) = \emptyset$, we have $|N(B) \cap N(C) \cap Y| + |N(u_1) \cap N(C) \cap Y| < \frac{1}{4}(j-1)$. Therefore $|N(u_1) \cap N(C) \cap W| > \frac{1}{4}(j-1)$.

Let $\{w_{i_1}, w_{i_2}, \ldots, w_{i_t}\} = N(u_1) \cap N(C) \cap W$, with $i_1 < i_2 < \ldots < i_t$. It is clear that $\{i_1, i_2, \ldots, i_t\}$ contains no consecutive integers in W. We now construct i_t pairs of vertices in W such that the k^{th} pair is $\{w_{i_k}, w_{i_k+1}\}$ or $\{w_{i_k-1}, w_{i_k}\}$, according as i_k is odd or even, respectively. Since $t > \frac{1}{4}(j-1)$, v_1 must be adjacent to at least one vertex in a pair, say k^{th} pair. But one vertex in the pair was adjacent to u_1 and C, so v_1 must be adjacent to the other vertex in the pair. Thus G is not an $F(j)$-graph.

(2.2) Now suppose u_1 is adjacent to some vertice $u_i \in X$. Then $i \geq 5$ and of course i is odd. Without loss of generality, we may assume that no $u_s \in U$ with $1 \leq s \leq i$ is adjacent to any vertex in X except those vertices from u_1 to u_s along the hamilton cycle.

By looking at $N(u_i)$ and $N(u_{i-1})$, we first claim that $|(N(u_i) \cup N(u_{i-1})) \cap W| \geq c + \frac{1}{2}(j-1)$.

For suppose that $|(N(u_i) \cup N(u_{i-1})) \cap W| \leq c + \frac{1}{2}(j-1) - 1$. By counting all possible vertices that are either adjacent to u_i or u_{i-1} and $N(u_i) \cap N(u_{i-1}) = \emptyset$, we have $a + b + c + \frac{1}{2}(j-1) + 3 + \frac{i+1}{2} \geq 2(j-1)$. Thus

$$\begin{aligned} \tfrac{i+1}{2} &> \tfrac{3}{2}(j-1) - (a+b+c+3) \\ &\geq \tfrac{3}{2}(j-1) - \tfrac{5}{4}(j-1) = \tfrac{1}{4}(j-1). \end{aligned}$$

Therefore, $i \geq \frac{1}{2}(j-1)$. In this case v_1 must be adjacent to at least one vertex $u_s \in U$ with $1 \leq s \leq i$. Thus G is not an $F(j)$-graph.

Thus, by contradiction we may assume that $|(N(u_i) \cup N(u_{i-1})) \cap W| \geq c + \frac{1}{2}(j-1)$. Hence there are at least $\frac{1}{2}(j-1)$ disjoint pairs of consecutive vertices in W, all of which are either adjacent to u_i or u_{i-1}. Let $\{w_{i_1}, w_{i_1+1}, w_{i_2}, w_{i_2+1}, \ldots, w_{i_t}, w_{i_t+1}\}$ be $t := \frac{1}{2}(j-1)$ disjoint pairs of consecutive vertices with $i_1 < i_2 < \ldots < i_t$. By removing $w_{i_1}, w_{i_1+1}, w_{i_t}$ and w_{i_t+1} from the set and considering the path from w_{i_1+2} to w_{i_t-1} along the hamilton cycle, we find that one of w_{i_1}, w_{i_1+1} is odd, one of w_{i_t}, w_{i_t+1} is even and the length of the path from w_{i_1+2} to w_{i_t-1} is at least $j-5$. If $j \geq 9$, then $j - 5 \geq \frac{1}{2}(j-1)$. Thus v_1 must be adjacent to at least one vertex on the path from w_{i_1+2} to w_{i_t-1}. It is easy to see that no matter whether v_1 is adjacent to w_k for an odd or even k in the interval $i_1 + 2 \leq k \leq i_t - 1$, it turns out that G can not be an $F(j)$-graph. ∎

Thus we have completed the proof of Theorem 1. It can be easily seen that Theorem 2 is a consequence of Theorem 1 together with the fact that any $2r$-regular graph must contain a 2-factor [5] and for given integer n and odd integer j such that $n \geq f(j)$, there exists a $(j-1)$-regular, triangle-free and hamilton graph G on n vertices.

Acknowledgement

I would like to thank Professor Roger B. Eggleton for drawing my attention to the subject.

References

[1] P. Erdös and T. Gallai, *Solution of a problem of Dirac,* Theory of Graphs and its applications: Proceedings of the symposium,

Smolenice, June 1963. Publishing House of the Czechoslavian Academy of Science, Prague (1964) 167-168.

[2] D. Hanson, P. Wang and G. MacGillivray, *A note on minimum graphs with girth pair* $(4, 2\ell + 1)$,J. Graph Theory **18** (1994) 325-327.

[3] B. Jackson, *Hamiltonian cycles in regular 2-connected graphs,* J. Combin. Theory Ser.B, **29** (1980) 27-46.

[4] K.R. Parthasarathy, "Basic Graph Theory", Tata McGraw-Hill Publishing Com. Ltd., New Delhi, 1994.

[5] J. Peterson, *Die Theorie der regularen Graphen,* Acta Math. **15** (1891) 193-220.

[6] W.D. Wallis, *The smallest regular graphs without one-factors.* Ars Combinatoria **11** (1981) 295-300.