On optimal solutions to the problem of gossiping in minimum time

Giulio Salerni*
Piazza A. Zamorani 4, I-00157 Rome, Italy

February 23, 2000

Abstract

We determine solutions to the problem of gossiping in minimum time (briefly: minimum time problem or MTP) which require less calls than the previously known solutions for infinitely many values of the number n of persons and optimal solutions to the MTP, i.e. solutions of the MTP which minimize the number of calls, for some values of n. We conjecture that our methods provide optimal solutions of the MTP for all n.

Keywords: Gossiping, Minimum time, Optimal solution.
AMS subject classification: 94A15.

1 INTRODUCTION

In this paper we deal with the *problem of gossiping in minimum time* (briefly: minimum time problem or MTP), which is a variant of the well known *gossip problem* (GP).

In the GP there are n persons, each knowing a piece of information unknown to the others. They communicate by telephone (conference calls are not allowed), each person can take part in only one call at a time, anybody can speak to anybody and in each call the two parties exchange all the information they know at that time. It is required to minimize the number of calls needed to transmit all the information to everyone.

On the other hand, in the MTP, under the assumption that each call takes a unit of time, it is required to minimize the time. A *round* of calls is defined to be any set consisting of exactly all the simultaneously performed calls, therefore the total time equals the number of rounds.

*E-mail: g.salerni@tin.it.

A solution of the GP or the MTP is called NODUP (*no duplication*) if in each call anybody only receives some new piece of information.

The number of calls required for the solutions of the GP (for a number $n \geq 4$ of persons) is exactly $2n - 4$. A solution is the following [1]: four persons (*chiefs*) are chosen and one of them communicates with the remaining $n - 4$ persons, then the four chiefs exchange all the information and finally one of the four chiefs transmits all the information to the other $n - 4$ persons. The total number of calls is $(n - 4) + 4 + (n - 4)$.

The time required for the solutions of the MTP is exactly $\lceil \log_2 n \rceil$ for n even and $\lceil \log_2 n \rceil + 1$ for n odd (Knödel's solution, which for even n requires exactly $\frac{n}{2} \lceil \log_2 n \rceil$ calls and is NODUP iff n is power of 2). In particular, for $n = 2^s$, a solution is obtained by partitioning the n persons into $n/2$ subsets of two persons who communicate between them at the first round and by joining, in the $s - 1$ subsequent rounds, the subsets in pairs. For odd n, a solution is obtained using $2^{\lfloor \log_2 n \rfloor}$ chiefs [3]. We observe that to obtain solutions of the MTP for odd n using chiefs, the number of chiefs must clearly be even, otherwise the number of rounds would be $> \lceil \log_2 n \rceil + 1$.

The solutions to the GP and the MTP at the same time are called *ideal* solutions [4]. They exist iff $n \leq 9$ or $n = 11, 13, 15$ [5].

Let us call *optimal* solutions of the GP and of the MTP, respectively, the solutions of the GP which minimize the time and the solutions of the MTP which minimize the number of calls.

An optimal solution of the GP is obtained [5], for $2^s < n \leq 2^{s+1}$ $(s \geq 3)$, using 2^s chiefs and the required time is exactly $2\lceil \log_2 n \rceil - 3$.

The problem of finding optimal solutions to the MTP, on the other hand, has been solved for some (infinite) values of n only: optimal solutions are known, in $2n - 3$ calls, for $n = 10$ [4] and $n = 12$ (the latter is NODUP) [6], and for even $n > 2^{\lceil \log_2 n \rceil} - 2^{\lceil \lceil \log_2 n \rceil / 2 \rceil}$ Knödel's solution is optimal [4]. Moreover, the number of calls of the optimal solutions to the MTP is $\Theta(n \log n)$ [2]. In particular, for $n = 12$ a solution is the following [6]. Partition the 12 persons into three subsets of four persons; in each subset, the four persons exchange all the information they know. Then, form three subsets of four persons — each consisting in two persons from one of the previous subsets and one person from each of the two other previous subsets — who exchange all the information. The total number of calls is $(3 \cdot 4) + (3 \cdot 3) = 21$.

For some (infinite) values of n, (NODUP) solutions to the MTP which require less calls than Knödel's solution are known: e.g., for $n = 36$ in 90 calls [6], for $n = 40$ in 104 calls (by Lenstra et al., see [7]) and for all n of the form $2^s \cdot 12$ in $\frac{n}{2} \left(\lceil \log_2 n \rceil - \frac{1}{2} \right)$ calls; for the latter a solution is the following [6]: partition the n persons into three subsets of 2^{s+2} persons who perform optimal solutions of the MTP among them (e.g. by Knödel's method) and then form 2^s subsets of 12 persons, consisting of four persons of each subset of the first stage, who exchange all the information. We present here a new solution. Let us partition the $n = 2^s \cdot 12$ persons into 2^s subsets of 12 persons each; in the first four rounds, every subset performs an optimal NODUP solution of the MTP; then, in further $s = \lceil \log_2 n \rceil - 4$ rounds, exchange of all the information is completed, by joining, at each round, the subsets in pairs.

Denote the n persons by the integers $1, \ldots, n$. To study the solutions of the MTP or the GP round by round, we introduce the *gossiping representation matrix.*

Definition 1 The gossiping *representation matrix* (after the r-th round of calls) is the $n \times n$ square matrix, $A^{(r)} = \left(a_{hk}^{(r)}\right)$, where

$$a_{hk}^{(r)} := \begin{cases} 1 & \text{if, after the } r\text{-th round,} \\ & k \text{ knows } h\text{'s initial piece of information} \\ 0 & \text{if, after the } r\text{-th round,} \\ & k \text{ does not know } h\text{'s initial piece of information.} \end{cases}$$

At the start $a_{hk}^{(0)} = \delta_{hk}$ and at the end $a_{hk}^{(T)} = 1$, where T is the final time. To update the matrix at each round, we can write just the entries equal to 1; in this way we see the *information* known by everybody (provided by the matrix *columns*) and the *persons* who know everyone's initial piece of information (provided by the matrix *rows*). A solution is NODUP iff, before any call between h and k, $(a_{ih}, a_{ik}) \neq (1,1)$, $i = 1, \ldots, n$.

In section 2, we determine solutions to the MTP which require less calls than the previously known solutions for infinitely many values of n and optimal solutions to the MTP for some values of n. In section 3, we conjecture that our methods provide optimal solutions to the MTP for all n.

2 SOLUTIONS OF THE MTP

We observe that

1. the GP [1],
2. the MTP (for odd n, $n = 2^s$, $n = 2^s \cdot 12$) [3] [6],
3. the problem of finding ideal solutions (for $n \neq 6$) [5] and
4. the problem of finding optimal solutions to the GP [5]

have been solved by partitions of the n person set into suitable subsets (see section 1). We believe this is the most efficient method to reduce the number of calls in solutions to the MTP.

Denote by $C(n)$ the number of calls in an optimal solution to the MTP for n persons.

2.1 Even n

For even n, to determine solutions of the MTP by partitioning the n persons into subsets, it is necessary that each subset contains an even number of persons and either the n persons are partitioned into two subsets of $n/2$ persons or the number of subsets is ≥ 3, otherwise the total number of rounds would be $> \lceil \log_2 n \rceil$.

Proposition 1 $C(18) \leq 38$.

Proof: We present a solution of the MTP which requires exactly 38 calls (instead of 45 as in Knödel's solution). Partition the 18 persons into three subsets of six persons each. Every subset performs an ideal solution. Next, form two new subsets of 12 and 6 persons, consisting, respectively, of four persons of each subset of the first stage and of two persons of each subset of the first stage. The persons in each subset of the second stage exchange all the information in two rounds and 9 and 5 calls, respectively, as it is easily seen (e.g. by means of the relevant representation matrix). □

Proposition 2 $C(20) \leq 43$.

Proof: We present a solution in 43 calls (instead of 50 as in Knödel's solution). Partition the 20 persons into three subsets of eight, six and six persons: $\{1,\ldots,8\}$, $\{9,\ldots,14\}$, $\{15,\ldots,20\}$. Each subset performs an ideal solution. Then, form two new subsets of 12 and 8 persons: $\{1,\ldots,4,9,\ldots,12,15,\ldots,18\}$, $\{5,\ldots,8,13,14,19,20\}$. The persons in each subset of the second stage exchange all the information in two rounds and 9 and 6 calls, respectively. □

Proposition 3 $C(22) \leq 49$.

Proof: We present a solution in 49 calls (instead of 55). Partition the 22 persons into three subsets of eight, eight and six persons: $\{1,\ldots,8\}$, $\{9,\ldots,16\}$, $\{17,\ldots,22\}$. Each subset performs an ideal solution. Then, form two new subsets of 16 and 6 persons: $\{1,\ldots,6,9,\ldots,14,17,\ldots 20\}$, $\{7,8,15,16,21,22\}$. The persons in each subset of the second stage exchange all the information in two rounds and, respectively, 12 calls (as it is proved, for instance, by means of the relevant representation matrix) and 5 calls. □

In all the three cases ($n = 18, 20$ and 22) the *second* stage can be performed in other ways.

Remark 1 By applying this method to increasing values of n, i.e. partitioning the n persons into three suitable subsets of n_i persons, n_i even, $2^{\lceil \log_2 n \rceil - 3} + 2 \leq n_i \leq 2^{\lceil \log_2 n \rceil - 2}$, solutions of the MTP which require less calls than Knödel's solution are obtained for all even n, $18 \leq n \leq 2^{\lceil \log_2 n \rceil - 2} \cdot 3$ (e.g., for $n = 34$, by three subsets of 12, 12 and 10 persons, we obtain a solution in 85 calls instead of 102 as in Knödel's solution). The bounds on the n_i's are again due to the bound on the total time ($\lceil \log_2 n \rceil$).

Remark 2 For some values of n, our method provides NODUP solutions of the MTP (e.g., for $n = 44$, in 118 calls instead of 132, using three subsets of 16, 16 and 12 persons).

Remark 3 We have tried to partition the n persons in four or more subsets too, but, in this case, it seems that breaking the set into too many parts causes an increase in the number of calls. Only by partitioning the n persons in exactly four subsets, we have obtained solutions in which the number of calls *equals* that of solutions with three subsets.

2.2 Odd n

For odd n, problems 1 [1], 2 [3], 3 [5] and 4 [5] recalled at the beginning of section 2 have been solved using chiefs, as recalled in section 1, and, in particular, the problem of finding optimal solutions to the GP — which is the *symmetric* of the problem of finding optimal solutions to the MTP — has been solved, for $2^s < n \leq 2^{s+1}$ ($s \geq 3$), using 2^s chiefs [5], which is the *largest* possible number of chiefs; now we use, to determine solutions of the MTP, $(n+1)/2$ chiefs (if 4 divides $n+1$) or $(n+3)/2$ chiefs (if 4 does not divide $n+1$), which is the *smallest* possible number of chiefs.

Proposition 4 *For* $n = 17, 19, 21$ *and* 23, $C(n) = 2n - 3$.

Proof: We recall (see section 1) that $C(10) = 2n - 3 = 17$ and $C(12) = 2n - 3 = 21$.

For $n = 17, 19$, using 10 chiefs optimal solutions are obtained, since they contain exactly $(n-10)+17+(n-10) = 2n-3$ calls, the smallest possible number (in fact, a solution in $2n-4$ calls requires, at least, 7 rounds [5]).

As above, for $n = 21, 23$, with 12 chiefs in $(n-12)+21+(n-12) = 2n-3$ calls.

For $n = 17, 19$ we can use 12 chiefs, too. □

Remark 4 By applying this method to increasing values of n, i.e. using $(n+1)/2$ chiefs (if 4 divides $n+1$) or $(n+3)/2$ chiefs (if 4 does not divide $n+1$), a solution of the MTP which requires less calls than Knödel's solution is obtained for all odd n, $33 \leq n < 2^{\lceil \log_2 n \rceil} - 4$: it is sufficient that the chiefs communicate among them using Knödel's method, even though this is not, in general, an optimal solution, as it is seen in the following Table 1, which shows solutions of the MTP for odd n, $33 \leq n \leq 59$. The *calls* column shows the number of calls less than that of Knödel's solution.

We observe that for $n = 25, 27$ this solution requires a number of calls equal to that of Knödel's solution and that for odd $n > 2^{\lceil \log_2 n \rceil} - 4$ coincides with Knödel's solution.

Table 1: Solutions of the MTP for odd n, $33 \leq n \leq 59$.

n	*calls*	*method*
33, 35	−14	18 chiefs communicating among them with our method
37, 39	−13	20 chiefs as above
41, 43	−11	22 chiefs as above
45, 47	−10	24 chiefs as above
49, 51	−3	26 chiefs communicating among them with Knödel's method
53, 55	−2	28 chiefs as above
57, 59	−1	30 chiefs as above

3 OPEN PROBLEMS

It still remains to establish if our methods will, in general, provide optimal solutions to the MTP.

Up to now, we found no counterexamples to the following conjectures:

Conjecture 1 The method in Remark 1 provides optimal solutions of the MTP for even n, $18 \leq n \leq 2^{\lceil \log_2 n \rceil - 2} \cdot 3$ and for the remaining even n (≥ 14) Knödel's solution is optimal.

Conjecture 2 Using $(n+1)/2$ chiefs (if 4 divides $n+1$) or $(n+3)/2$ chiefs (if 4 does not divide $n+1$), who perform an optimal solution of the MTP among them, optimal solutions to the MTP for all odd n are obtained.

ACKNOWLEDGEMENT
The author would like to thank Professor M.J. de Resmini for her valuable remarks.

References

[1] B. Baker and R. Shostak, Gossips and telephones, *Discrete Math.* **2** (1972) 191–193.

[2] A. Czumaj, L. Gąsieniec and A. Pelc, Time and cost trade-offs in gossiping, *SIAM J. Discrete Math.* **11** (1998) 400–413.

[3] W. Knödel, New gossips and telephones, *Discrete Math.* **13** (1975) 95.

[4] R. Labahn, Mixed telephone problems, *J. Combin. Math. Combin. Comput.* **7** (1990) 33–51.

[5] R. Labahn, Kernels of minimum size gossip schemes, *Discrete Math.* **143** (1995) 99–139.

[6] Á. Seress, Gossiping old ladies, *Discrete Math.* **46** (1983) 75–81.

[7] D.B. West, Gossiping without duplicate transmissions, *SIAM J. Algebraic Discrete Methods* **3** (1982) 418–419.