The Isometric Path Number of a Graph

David C. Fisher
University of Colorado
Denver, Colorado
80217-3364

Shannon L. Fitzpatrick
University of Prince Edward Island
Charlottetown, Prince Edward Island
C1A 4P3

Abstract

An isometric path is merely any shortest path between two vertices. Inspired by the game of 'Cops and Robber' and a result by Aigner & Fromme [1], we are interested in determining the minimum number of isometric paths required to cover the vertices of a graph. We find a lower bound on this number in terms of the diameter of a graph and find the exact number for trees and grid graphs.

Key words: isometric, path, cover, Cartesian product.

AMS(MOS) subject classifications: 05C38, 05C70.

1 Introduction

An isometric subgraph of a graph G is defined to be a subgraph H of G such that for all $x, y \in V(H)$, $d_H(x,y) = d_G(x,y)$. Hence, an **isometric path** is any shortest path between two vertices of a graph. A set of isometric paths is said to **cover** G if every vertex of G lies in at least one of the paths in the set. We define the **isometric path number** of G, denoted $p(G)$, to be the minimum number of isometric paths required to cover the vertices of G.

The **strong product** of a set of graphs $\{G_i : i = 1, 2, \ldots, k\}$ is the graph $\boxtimes_{i=1}^{k} G_i$ whose vertex set is the Cartesian product of the sets $\{V(G) : i = 1, 2, \ldots, k\}$ and there is an edge between $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_k)$ if a_i is either adjacent or equal to b_i in G_i for all $i = 1, 2, \ldots, k$. The

Cartesian product of $\{G_i : i = 1, 2, \ldots, k\}$ is the graph $\square_{i=1}^k G_i$ whose vertex set is the Cartesian product of the sets $\{V(G_i) : i = 1, 2, \ldots, k\}$ and there is an edge between $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_k)$ if and only if there is some j such that $a_i = b_i$ for $i \neq j$ and a_j is adjacent to b_j.

Examples of both the strong product and the Cartesian product of two paths are shown in Figure 1. The Cartesian product of two paths is also called a **complete grid graph**, or grid, for short. Specifically, an $m \times n$ grid is the Cartesian product of a path on m vertices and a path on n vertices.

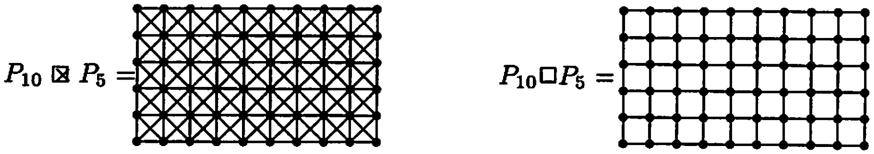

Figure 1: The strong product of two paths (left) and the Cartesian product of two paths (right).

The problem of finding the isometric path number of a graph was inspired by the game of 'Cops and Robber', introduced independently by Nowakowski & Winkler [3] and Quilliot [4]. The cop side consists of some set of k cops and the robber side consists of a single robber. The rules of the game are: given a connected graph G, the cops each choose a vertex, then the robber chooses a vertex and they then move alternately. The cops' move consists of some (possibly empty) subset of the cops moving to an adjacent vertex and the robber's move is to move to an adjacent vertex or stay at his current position. The cops win if one or more cops manage to occupy the same vertex as the robber; the robber wins if this situation never occurs. The **cop number** of G, denoted $c(G)$, is the minimum number of cops required to ensure the capture of a robber on G.

It was shown by Aigner & Fromme [1] that a single cop moving on a isometric path P can guarantee that after a finite number of moves the robber will be immediately caught if he moves onto P. Therefore, if one cop is assigned to each isometric path in an isometric path cover, then the robber will be apprehended. Hence, $c(G) \leq p(G)$ for all graphs G.

Our first result gives a lower bound on the isometric path number in terms of the diameter of a graph. We then show that for some classes of graphs, such as the strong product of paths, this bound is actually met. We also examine the relationship between the isometric path numbers of a graph and its subgraphs. In Section 3, we determine the isometric path number of a tree by finding a set of isometric paths that cover not only its vertices, but all its edges as well. Finally, in Section 4, we examine grids. We show the isometric path number of an $m \times n$ grid is exactly $\lceil \frac{2}{3}(m+n-\sqrt{m^2+n^2-mn}) \rceil$.

2 Preliminary Results

The **diameter** of a graph G, denoted $diam(G)$, is defined to be the length of the longest isometric path in G. Therefore, a single isometric path in G contains at most $diam(G)+1$ vertices. This gives us the following lower bound on the isometric path number of G:

Theorem 1 *Let G be any connected graph with vertex set V. Then*

$$p(G) \geq \left\lceil \frac{|V|}{diam(G)+1} \right\rceil .$$

For some common graph families, Theorem 1 is exact.

Theorem 2 *Let P_n, C_n, K_n, and E_n be the path, cycle, complete graph, and edgeless graph, respectively, on n vertices.*
(a) for all $n \geq 1$, $p(P_n) = 1$, $p(K_n) = \lceil n/2 \rceil$, and $p(E_n) = n$;
(b) for all $n \geq 3$, $p(C_n) = 2$;
(c) for any $k \geq 1$ and $1 \leq n_1 \leq n_2 \cdots \leq n_k$, $p(\boxtimes_{i=1}^{k} P_{n_i}) = \Pi_{i=1}^{k-1} n_i$.

Proof: The proofs of (a) and (b) are straight forward, so we leave them out. For (c), let $G = \boxtimes_{i=1}^{k} P_{n_i}$ for some $k \geq 1$ and $1 \leq n_1 \leq n_2 \cdots \leq n_k$. Then $diam(G) = \max\{n_i : i = 1, 2, \ldots, k\} - 1 = n_k - 1$. Since $|V| = \Pi_{i=1}^{k} n_i$ then by Theorem 1, $p(G) \geq \Pi_{i=1}^{k-1} n_i$. Now, let $H = \boxtimes_{i=1}^{k-1} P_{n_i}$ and $P_{n_k} = \{x_1, x_2, \ldots, x_{n_k}\}$. Then $V = \{(v, x_i) : v \in V(H),\ i = 1, 2, \ldots, n_k\}$. For every $v \in V(H)$, let $P_v = \{(v, x_1), (v, x_2), \ldots, (v, x_{n_k})\}$. This is an isometric path in G, and $\mathcal{P} = \{P_v : v \in V(H)\}$ is a set of $|V(H)| = \Pi_{i=1}^{k-1} n_i$ isometric paths that cover G. Hence, $p(G) = \Pi_{i=1}^{k-1} n_i$. □

There is also a relationship between the isometric path numbers of a graph and its isometric subgraphs.

Theorem 3 *Let G be any graph and H be any isometric subgraph of G. Then $p(H) \leq p(G)$.*

Proof: Suppose $p(G) = n$ and G can be covered by isometric paths $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$. Let H be an isometric subgraph in G and suppose $H = G \setminus S$, for some set of vertices S. For any path P in $\mathcal{P}$, the set of vertices in $P \cap H$, if nonempty, constitute a set of paths in H. Hence,

$P \cap H = Q^1 \cup Q^2 \cup \cdots \cup Q^k$ for some $k \geq 1$ where each Q^i is a path in H for $i = 1, 2, \ldots, k$. Since H is an isometric subgraph of G, there is a path in H from the last end vertex in Q^i to the first end vertex in Q^{i+1} that is isometric in both H and G. This path contains no vertices of $P \cap H$ other than its end vertices, due to the isometry of P. Hence, by adding the isometric paths joining Q^i to Q^{i+1} for each $i = 1, 2, \ldots, k-1$ to the set $\{Q^1, Q^2, \ldots, Q^k\}$ we obtain a path, Q. Since P was an isometric path in G, then Q is an isometric path in H, and Q contains all the vertices in $P \setminus S$. Therefore, we can find a set of isometric paths $\{Q_1, Q_2, \ldots, Q_n\}$, some of which may be empty, which cover all the vertices of $G \setminus S = H$. Hence, $p(H) \leq p(G)$. □

Such a relationship does not exist between the isometric path numbers of a graph and its induced subgraphs. Consider the graphs Figure 2. The graph G can be covered with two isometric paths, while the graph H, which is an induced subgraph of G, requires three.

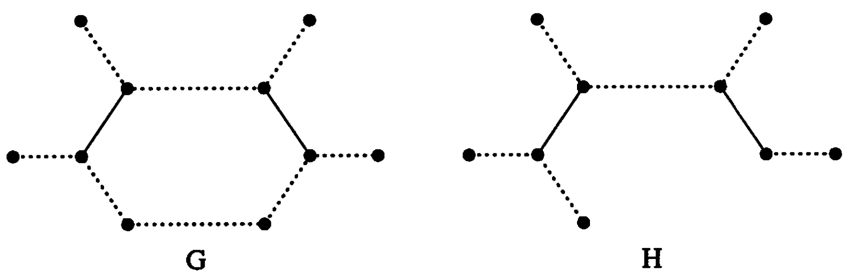

Figure 2: H is an induced subgraph of G such that $p(G) < p(H)$.

The removal of an edge can either increase or decrease the isometric path number, as well. For example, the isometric path number of a cycle is two. If we remove any edge of a cycle we obtain a path which has isometric path number one. However, if we take the graph G in Figure 2 and remove the edge joining the two vertices of degree two, the resulting graph has isometric path number three.

The problem of finding the minimum number of paths (not necessarily isometric) that cover all the *edges* of a graph, G, was considered by Harary & Schwenk [2]. They called this the **unrestricted path number** of G, denoted $\pi^*(G)$. Since any edge cover of G induces a vertex cover, $\pi^*(G)$ provides an upper bound for $p(G)$ in those cases where isometric paths provided the optimum edge cover. Trees are an obvious class of graphs to consider for this, since every path in a tree is necessarily an isometric path.

The unrestricted path number of a tree was determined in [2]. The isometric path number of a tree follows as a corollary.

Theorem 4 (Harary & Schwenk [2]) *If T is any tree and $\ell(T)$ is the number of leaves in T then $\pi^*(T) = \lceil \ell(T)/2 \rceil$.*

Corollary 5 *If T is any tree then $p(T) = \lceil \ell(T)/2 \rceil$.*

Proof: Any path in a tree, T, may contain at most two leaves. Therefore, if T has $\ell(T)$ leaves then $p(T) \geq \lceil \ell(T)/2 \rceil$. Furthermore, $p(T) \leq \pi^*(T) = \lceil \ell(T)/2 \rceil$. Hence, $p(T) = \lceil \ell(T)/2 \rceil$. □

3 Grids

We now determine the isometric path number of a grid. We denote an $n \times m$ grid by $G_{m,n}$ and label the vertices of $G_{m,n}$ as coordinates on the grid. Hence, $V(G_{m,n}) = \{(i,j) | i = 1,2,\ldots,m, j = 1,\ldots,n\}$ and the distance between any pair of vertices (a,b) and (c,d) in $G_{m,n}$ is given by $d((a,b),(c,d)) = |c-a| + |d-b|$. We also wish to assign a direction to every isometric path, P, in $G_{m,n}$. The **first end point** of P will be that endpoint of P with the minimum first coordinate, or if the first coordinates are the same then with minimum second coordinate. Hence, $P = \{(a_0,b_0),(a_1,b_1),\ldots,(a_n,b_n)\}$ implies that either $a_0 \leq a_n$ or $a_0 = a_n$ and $b_0 \leq b_n$. Let $P[(a_i,b_i),(a_j,b_j)]$ denote the subpath of P from (a_i,b_i) to (a_j,b_j) for any $i \leq j$.

Lemma 6 *Suppose $P = \{(a_0,b_0),\ldots,(a_k,b_k)\}$ is an isometric path in $G_{m,n}$, then*
(a) $a_i \leq a_{i+1}$ for all $i = 0,1,\ldots,k-1$;
(b) if $b_0 \leq b_k$ then $b_i \leq b_{i+1}$ for all $i = 0,1,\ldots,k-1$;
(c) if $b_0 > b_k$ then $b_i \geq b_{i+1}$ for all $i = 0,1,\ldots,k-1$.

Proof: (a) Let $P = \{(a_0,b_0),(a_1,b_1),\ldots,(a_k,b_k)\}$ be a path in $G_{m,n}$. Now suppose that for some i, $a_i > a_{i+1}$. Hence, $a_{i+1} = a_i - 1$ and $b_{i+1} = b_i$. Since P is an isometric path, each subpath of P is also isometric. Hence,

$$\begin{aligned} d((a_0,b_0),(a_k,b_k)) &= d((a_0,b_0),(a_i,b_i)) + d((a_i,b_i),(a_{i+1},b_{i+1})) + \\ &\quad d((a_{i+1},b_{i+1}),(a_k,b_k)) \\ &= |a_i - a_0| + |b_i - b_0| + 1 + |a_k - a_{i+1}| + |b_k - b_{i+1}| \\ &\geq |a_i - a_0 + a_k - a_i + 1| + |b_i - b_0 + b_k - b_i| + 1 \\ &= d((a_0,b_0),(a_k,b_k)) + 2. \end{aligned}$$

This contradiction implies that $a_i \leq a_{i+1}$ for all $i = 0,1,\ldots,k$.

Parts (b) and (c) can be shown similarly. □

For any path $P = \{(a_0, b_0), \ldots, (a_k, b_k)\}$ if (a), (b) and (c) hold then P is an isometric path. Therefore, we may categorize each isometric path in $G_{m,n}$ as one of two types. Call the path **rising** if $b_0 \leq b_k$ and **sinking** if $b_0 > b_k$. Note that rising paths include those in which $b_i = b_{i+1}$ for all $i = 0, \ldots, k-1$.

Suppose $P = \{(a_0, b_0), \ldots, (a_k, b_k)\}$ is a path with end points $(1, 1)$ and (m, n). Then P cuts the grid into two components, one which lies "above" the path and one which lies "below". Therefore, if Q is a path with vertices in both components of the grid, as determined by P, then Q must intersect P at some vertex.

Theorem 7 *For all integers $m, n \geq 1$,*

$$p(G_{m,n}) \geq \left\lceil \frac{2}{3}\left(m + n - \sqrt{m^2 + n^2 - mn}\right) \right\rceil.$$

Proof: Suppose $p = p(G_{m,n})$ where $1 \leq m \leq n$. Obviously, m isometric paths are sufficient to cover all the vertices of $G_{m,n}$. Hence, $p \leq m$. Let $\mathcal{P}$ be a set of p isometric paths which cover all the vertices of $G_{m,n}$. The set $\mathcal{P}$ can be partitioned into a set of rising paths and a set of sinking paths, denoted $\mathcal{R}$ and

$\mathcal{S}$, respectively. Suppose $p = r + s$ where $r = |\mathcal{R}|$ and $s = |\mathcal{S}|$.

We first consider the paths in $\mathcal{R}$. Without loss of generality, we can assume that each path in $\mathcal{R}$ has end points $(1, 1)$ and (m, n). (If this is not the case, the path can be extended to contain these vertices by adding a rising path from $(1, 1)$ to the first end point of the path and a rising path from the last end point to (m, n)). We wish to show that the maximum number of vertices covered by the paths in $\mathcal{R}$ is $(m+n)r - r^2$. We proceed by induction on m.

Suppose $G = G_{1,n}$ for some $n \geq 1$. Then G is a path, and obviously one rising path covers at most n vertices in G. If $G = G_{2,n}$ then one rising path on G covers at most $n + 1$ vertices and two rising paths cover at most $2n$ vertices. Hence, the result holds for $m = 1$ and $m = 2$.

Assume that for all k such that $1 \leq k < m$, the maximum number of vertices covered by a set of r rising paths in $G = G_{k,n}$, where $r < k$, is $(k + n)r - r^2$. Also, assume that for any set of r rising paths in G there is another set of rising paths $\{P_1', \ldots, P_r'\}$, which covers the same vertices as the original set, but has the additional property that each P_i' has first end point $(i, 1)$ and last end point $(k, n - i + 1)$ for all $i = 1, \ldots, r$.

Let $G = G_{m,n}$ and let $\mathcal{R}$ be a set of r rising paths in G. Let $(1, \bar{b})$ be the vertex covered by $\mathcal{R}$ such that no vertex $(1, j)$ is covered by $\mathcal{R}$ for any

$j > \bar{b}$. Similarly, let $(\underline{a}, n)$ be the vertex covered by $\mathcal{R}$ such that no vertex (i, n) is covered by $\mathcal{R}$ for any $i < \underline{a}$.

Suppose no single path in $\mathcal{R}$ contains both $(1, \bar{b})$ and $(\underline{a}, n)$. Then $\underline{a} > 1$ and $n > \bar{b}$. Let P and Q be distinct paths in $\mathcal{R}$ containing $(1, \bar{b})$ and $(\underline{a}, n)$, respectively. The paths P and Q must intersect on the subpaths $P[(1, \bar{b}), (m, n)]$ and $Q[(1, 1), (\underline{a}, n)]$.

Let the vertex (x, y) be on both $P[(1, \bar{b}), (m, n)]$ and $Q[(1, 1), (\underline{a}, n)]$. We wish to consider new paths $P^* = P[(1, 1), (x, y)] \cup Q[(x, y), (m, n)]$ and $Q^* = Q[(1, 1), (x, y)] \cup P[(x, y), (m, n)]$. These paths are both rising and, hence, isometric. Replace P and Q with P^* and Q^* in $\mathcal{R}$. The new set, $\mathcal{R}^*$, of rising paths is the same size and covers the same vertices as the set $\mathcal{R}$.

If a single path, say P, contains both $(1, \bar{b})$ and $(\underline{a}, n)$, then $P^* = P$ and $\mathcal{R}^* = \mathcal{R}$.

In either case, $\mathcal{R}^*$ is a set of r paths which cover the same vertices as the paths in $\mathcal{R}$. Furthermore, the path P^* in $\mathcal{R}^*$ contains all the vertices covered by $\mathcal{R}$ which have either 1 as a first coordinate or n as a second coordinate. Therefore, the vertices covered by $\mathcal{R}^* \setminus P^*$ but not by P^* lie in the set $\{(i, j) | 1 < i \leq m, 1 \leq j < n\}$. The subgraph induced by these vertices is the graph $G_{m-1,n-1}$.

Therefore, by the induction hypothesis, the maximum number of vertices of the subgraph covered by the $r - 1$ paths in $\mathcal{R}^* \setminus P^*$ is $(m - 1 + n - 1)(r - 1) - (r - 1)^2 = (m + n)(r - 1) - r^2 + 1$. Since P^* covers $m + n - 1$ vertices, the total number of vertices covered by the paths in $\mathcal{R}$ is at most $(m + n - 1) + (m + n)(r - 1) - r^2 + 1 = (m + n)r - r^2$.

Also by induction, the vertices covered by any set of $r - 1$ paths in $G_{m-1,n-1}$ can also be covered by a set of $r - 1$ paths which have $\{(i, 1) : i = 1, 2, \ldots, r-1\}$ as the set of all first end points of the paths and $\{(m-1, n-i) : i = 1, 2, \ldots, r - 1\}$ as the set of all final endpoints. This was, however, assuming that the grid was labeled so that the first coordinates ranged from 1 to $m - 1$ and the second coordinates ranged from 1 to $n - 1$. Since the $(m - 1) \times (n - 1)$ subgrid of $G_{m,n}$ has coordinates from 2 to m and 1 to $n - 1$, it follows that there is a set of $r - 1$ paths, say $\{P_2', P_3', \ldots, P_r'\}$ which cover the same vertices as $\mathcal{R}^* \setminus P^*$ such that P_i' has end coordinates $(i, 1)$ and $(m, n - i + 1)$ for $i = 2, 3, \ldots, r$. Since P^* has end points $(1, 1)$ and (n, n), if we let $P_1' = P^*$, then $\{P_1', P_2', \ldots, P_r'\}$ a set of r paths which have the end points required for the induction.

Hence, for any $1 \leq m \leq n$, the maximum number of vertices covered by a set, $\mathcal{R}$, of r rising paths in $G_{m,n}$ is $(m + n)r - r^2$. Furthermore, by induction, there is a set of r rising paths which cover the same vertices as $\mathcal{R}$ and can be ordered $\{P_1, P_2, \ldots, P_r\}$ such that P_i has end vertices $(i, 1)$ and $(m, n - i + 1)$ for all $i = 1, \ldots, r$. Call this set $\mathcal{R}'$.

Similarly, we can find a set of sinking paths, $\mathcal{S}' = \{Q_1, \ldots, Q_s\}$, such that each Q_j has end vertices (j, n) and (m, j) for all $j = 1, \ldots, s$, and the set $\mathcal{S}'$ covers the same vertices as the original set, $\mathcal{S}$. Hence, the number of vertices covered by $\mathcal{S}'$ alone is at most $(m+n)s - s^2$. However, each path in $\mathcal{S}'$ must intersect every path in $\mathcal{R}'$. To demonstrate, consider a rising path, P_r with end vertices $(i, 1)$ and $(m, n-i+1)$ and a sinking path, P_s with end vertices (j, n) and (m, j), where $i \leq r$ and $j \leq s$. If P_r contains either (j, n) or (m, j), then P_r and P_s obviously intersect. Otherwise, P_r cuts the grid into two components, one containing (j, n) and the other containing (m, j). Hence, P_s must intersect P_r at least once.

Therefore, for each $j = 1, \ldots, k$, there are r vertices in Q_j which have already been counted in the set of vertices covered by the rising paths. Therefore, for $j = 1, \ldots, s$, the path Q_j contributes at most $|Q_j| - r$ new vertices to our total. Hence, the number of vertices covered by $\mathcal{S}'$ but not by $\mathcal{R}'$ is at most $(m+n)s - s^2 - rs$.

Therefore, the total number of vertices covered by $\mathcal{R}' \cup \mathcal{S}'$, and thus by $\mathcal{P} = \mathcal{R} \cup \mathcal{S}$, is at most $((m+n)r - r^2) + ((m+n)s - s^2) - rs = (m+n)p - p^2 + rs$, where $s + r = p$ and $rs = s(p-s) = sp - s^2$. We maximize rs at $s = p/2$ and, hence, at most $(m+n)p - \frac{3}{4}p^2$ vertices in $G_{m,n}$ are covered by $\mathcal{P}$. If p paths are sufficient to cover all the vertices of $G_{m,n}$, then $mn \leq (m+n)p - \frac{3}{4}p^2$. Hence,

$$\frac{3}{4}p^2 - (m+n)p + mn \leq 0$$

$$p \geq \frac{2}{3}\left(m + n - \sqrt{m^2 + n^2 - mn}\right)$$

Hence, $p(G_{m,n}) \geq \left\lceil \frac{2}{3}\left(m + n - \sqrt{m^2 + n^2 - mn}\right)\right\rceil$. □

Now, suppose we have a set of r rising path and s sinking paths which cover the vertices of $G_{m,n}$. Define this cover to be **normal** if the rising paths cover the vertices

$$\{(r,1), (r-1,2), \ldots, (1,r), (m-r+1,n), (m-r+2,n-1), \ldots, (m,n-r+1)\}$$

and the sinking paths cover the vertices

$$\{(m-s+1,1), (m-s+2,2), \ldots, (m,s), (1,n-s+1), (2,n-s+2), \ldots, (s,n)\}.$$

Theorem 8 *Let r, s, m, n be integers such that $m, n \geq 1$ and $r, s \geq 0$. If*

$$(m-r-s)(n-r-s) \leq rs$$

then r rising paths and s sinking paths are sufficient to cover all the vertices of $G_{m,n}$.

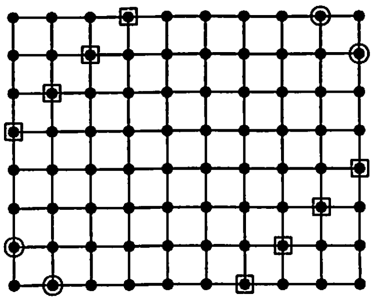

Figure 3: A cover of $G_{10,8}$ with two rising paths and four sinking paths is normal if the rising paths cover the circled vertices and the sinking paths cover the boxed vertices.

Proof: Suppose $r + s \geq \min(m, n)$. Then for $1 \leq k \leq r$, let the k^{th} rising path on $G_{m,n}$ go from $(1, k)$ to $(m - k + 1, k)$ to $(m - k + 1, n)$. And for $1 \leq \ell \leq s$, let the ℓ^{th} sinking path go from $(1, n-\ell+1)$ to $(m-\ell+1, n-\ell+1)$ to $(m' - \ell + 1, 1)$. Hence, in this case there is a normal cover of $G_{m,n}$ with r rising and s sinking paths. (See Figure 4 for an example.)

Figure 4: A Normal Cover of $G_{m,n}$ when $r + s \geq n$. This shows a normal cover of $G_{11,7}$ with 3 rising paths (on left) and 5 sinking paths (on right).

Now, suppose $r + s < \min(m, n)$. Without loss of generality, assume $r \leq s$ and $n \leq m$ (if $r > s$, flip the graph to interchange the bottom and top; if $m < n$, flip along the $i = j$ diagonal). Also assume that $r \leq n$, since there is an obvious cover of $G_{m,n}$ otherwise.

Then

$$(m - r - s)(n - r - s) \leq rs$$

$$(n - r - s)^2 \leq s^2$$

$$n \leq r + 2s.$$

$$r + 2s - n \geq 0$$

Note that for $r+s < \min(m,n)$, the inequality $(m-r-s)(n-r-s) \leq rs$ is not satisfied for $r = 0$. Hence, we may assume $1 \leq r \leq s$. Since $r+s < \min(m,n) = n$, then $n > 2$. Hence, we need only consider the cases $3 \leq n \leq m$.

We proceed by induction on n. Let $G = G_{m,3}$. Then $r = s = 1$ are the only possible values that need to be tested. Hence, $(m-r-s)(n-r-s) = (m-2)(3-2) = m-2$. Therefore, $(m-r-s)(n-r-s) \leq rs$ only if $m \leq 3$. Since $m \geq n$, then $m = 3$. Therefore, $m = n = 3$ and $r = s = 1$ are the only values which satisfy the inequality, and in fact there is a normal cover of $G_{3,3}$ with one rising and one sinking path (see Figure 5).

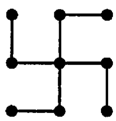

Figure 5: A normal cover of $G_{3,3}$ with one rising path and one sinking path.

For induction purposes, assume there is a normal cover of $G_{m',n'}$ with r' rising and s' sinking paths if $(m'-r'-s')(n'-r'-s') \leq r's'$, and $3 \leq n' < n$. We now cover part of $G_{m,n}$ as follows:

- For $1 \leq k \leq r$, the k^{th} rising path goes from vertex $(k,1)$ to $(k, n-s-k+1)$ to $(n-s+k, n-s-k+1)$.
- For $1 \leq j \leq r+2s-n$, the j^{th} sinking path goes from vertex $(1, n-j+1)$ to $(r+s-j+1, n-j+1)$. For $r+2s-n < j \leq s$, the j^{th} sinking path goes from vertex $(1, n-j+1)$ to $(r+s-j+1, n-j+1)$, then to $(r+s-j+1, s-j+1)$ and finally to $(m, s-j+1)$.

This set of paths covers all the vertices in the first $n-s$ columns and the first $n-r-s$ rows of $G_{m,n}$. This leaves an $(m-(n-s)) \times (n-(n-r-s))$ grid to cover. However, the r rising and the first $r+2s-n$ sinking paths described above enter this "subgrid" in such a way that a normal cover of $G_{m,n}$ can be completed if there is a normal cover of the $(m-n+s) \times (r+s)$ grid with r rising paths and $r+2s-n$ sinking paths (see Figure 6).

Let $m' = m-n+s$, $n' = r+s$, $r' = r$ and $s' = r+2s-n$. Then

$$\begin{aligned}(m'-r'-s')(n-r'-s') - r's' &= (m-2r-s)(n-r-s) - r(r+2s-n) \\ &= (m-r-s)(n-r-s) - rs \\ &\leq 0.\end{aligned}$$

Since $n' = r+s < n$, then, by induction, there is a normal cover of $G_{m',n'}$. Therefore, there is a normal cover of $G_{m,n}$ with r rising and s sinking paths whenever $(m-r-s)(n-r-s) \leq rs$. □

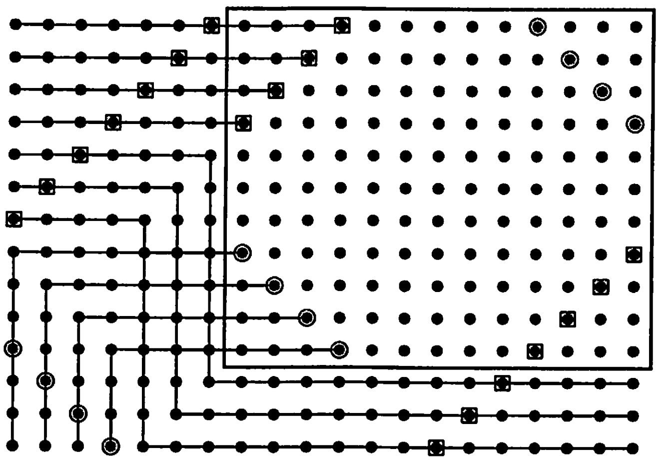

Figure 6: The problem of covering $G_{20,14}$ with 4 rising paths and 7 sinking paths is reduced to the problem to finding a normal cover for $G_{13,11}$ with 4 rising paths and 4 sinking paths.

Figure 7 illustrates the normal cover of $G_{20,14}$ given by the construction in the proof of Theorem 8. This requires a normal cover of $G_{13,11}$ with four rising and four sinking paths which in turn requires a normal cover of $G_{6,8}$ with four rising paths and one sinking path which in turn requires a normal cover of $G_{5,6}$ with three rising paths and one sinking path which in turn requires a normal cover of $G_{4,4}$ with two rising path and one sinking paths which in turn requires a normal cover of $G_{3,2}$ with one rising and one sinking path. This last cover uses the construction in Figure 4 resulting in an edge on the right side that is in both a rising and a sinking path.

We now show that the construction given in Theorem 6 gives the desired upper bound on $p(G_{m,n})$.

Theorem 9 *For all integers $m, n \geq 1$,*

$$p(G_{m,n}) \leq \left\lceil \frac{2}{3}\left(m+n-\sqrt{m^2+n^2-mn}\right)\right\rceil .$$

Proof: Suppose $p = \left\lceil \frac{2}{3}\left(m+n-\sqrt{m^2-mn+n^2}\right)\right\rceil$. Then

$$\frac{2}{3}\left(m+n-\sqrt{m^2+n^2-mn}\right) \leq p < 1+\frac{2}{3}\left(m+n-\sqrt{m^2+n^2-mn}\right).$$

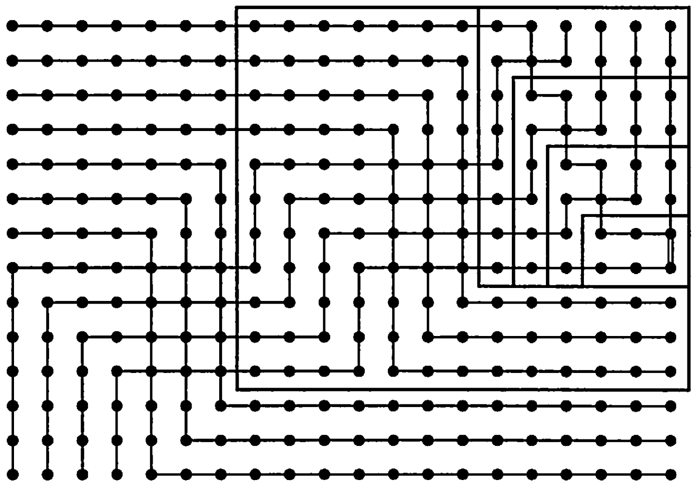

Figure 7: A normal cover of $G_{20,14}$ with four rising paths and seven sinking paths.

Since $m, n \geq 1$, then $\sqrt{m^2 + n^2 - mn} \geq 1$. Therefore,

$$\frac{2}{3}\left(m+n-\sqrt{m^2+n^2-mn}\right) \leq p < \frac{2}{3}\left(m+n+\sqrt{m^2+n^2-mn}\right).$$

This means that p lies between the roots of the quadratic equation $\frac{3}{4}x^2 - (m+n)x + mn$. Therefore,

$$\begin{aligned}
\frac{3}{4}p^2 - (m+n)p + mn &\leq 0 \\
p^2 - (m+n)p + mn &\leq \frac{p^2}{4} \\
(m-p)(n-p) &\leq \frac{p^2}{4}
\end{aligned}$$

If $m \geq n$, we have

$$\begin{aligned}
p &< 1 + \frac{2}{3}\left(m+n-\sqrt{m^2-mn+n^2}\right) \\
&< 1 + \frac{2}{3}\left(m+n-\sqrt{m^2-mn+n^2/4}\right) \\
&= 1 + \frac{2}{3}\left(m+n-(m-n/2)\right) \\
&= 1+n.
\end{aligned}$$

Since p is integer, $p \leq n$ if $n \leq m$. Similarly, $p \leq m$ if $m \leq n$. Therefore, $p \leq \min(m, n)$.

Suppose p is even. If we let $r = \frac{p}{2}$ and $s = \frac{p}{2}$ then

$$\begin{aligned}(m-r-s)(n-r-s) &= (m-p)(n-p)\\ &\leq \frac{p^2}{4}\\ &= rs\end{aligned}$$

and

$$r + s = p \leq \min(m, n).$$

Hence, by Theorem 8, $G_{m,n}$ can be covered with $p/2$ rising paths and $p/2$ sinking paths.

Suppose p is odd. Note that the left side of $(m-p)(n-p) \leq p^2/4$ is integer, while the right side is an integer plus a quarter. Therefore, $(m-p)(n-p) \leq (p^2-1)/4$. So, if we let $r = (p-1)/2$ and $s = (p+1)/2$ then

$$\begin{aligned}(m-r-s)(n-r-s) &= (m-p)(n-p)\\ &\leq \frac{p^2-1}{4}\\ &= rs\end{aligned}$$

and

$$r + s = p \leq \min(m, n).$$

Hence, by Theorem 8, $G_{m,n}$ can be covered with $(p-1)/2$ rising paths and $(p+1)/2$ sinking paths. Therefore, $G_{m,n}$ can always be covered with

$$r + s = p = \left\lceil \frac{2}{3}\left(m + n - \sqrt{m^2 - mn + n^2}\right) \right\rceil$$

isometric paths, and

$$p(G_{m,n}) \leq \left\lceil \frac{2}{3}\left(m + n - \sqrt{m^2 - mn + n^2}\right) \right\rceil.$$

□

Hence, Theorem 7 and Theorem 9 together give us:

Corollary 10 *If $G_{m,n}$ is an $m \times n$ grid for some integers $m, n \geq 2$ then*

$$p(G_{m,n}) = \left\lceil \frac{2}{3}\left(m + n - \sqrt{m^2 - mn + n^2}\right) \right\rceil.$$

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