Some Initial Results on the Supermagicness of regular complete k-partite Graphs

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Abstract: Let G be a (p,q)-graph with p vertices and q edges. An edge-labeling assignment $L : E \to N$ is a map which assigns a positive integer to each edge in E. The induced map $L^+ : V \to N$ defined by $L^+(v) = \Sigma\ \{L(u,v) :$ for all (u,v) in E$\}$ is called the vertex sum. The edge labeling assignment is called magic if L^+ is a constant map. If L is a bijection with $L(E) = \{1,2,\ldots,q\}$ and L is magic then we say L is supermagic. B. M Stewart showed that K_5 is not supermagic and when $n \equiv 0 \pmod 4$, K_n is not supermagic. In this paper, we exhibit supermagicness for a class of regular complete k-partite graphs.

Key words and phrases: magic, supermagic, composition, regular complete k-partite graphs.

AMS 1991 subject classification: 05C78, 05C25.

1. **Introduction** Let G be a (p,q)-graph with p vertices and q edges. An edge-labeling assignment $L : E \to N$ is a map which assigns a positive integer to each edge in E. The induced map $L^+ : V \to N$ defined by $L^+(v) = \Sigma\ \{L(u,v) :$ for all (u,v) in E$\}$ is called the vertex sum. The edge labeling assignment is called magic if L^+ is a constant map. If L is a bijection with $L(E) = \{1,2,\ldots,q\}$ and L is magic then we say L is supermagic [1, 11]. Figure 1 shows a magic graph which is not supermagic and another which is a supermagic graph.

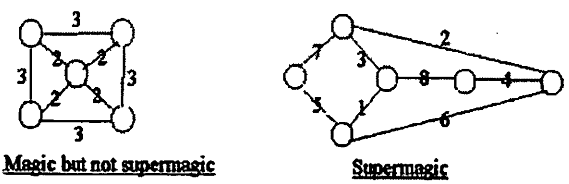

Figure 1

The concept of a magic graph was first introduced by J. Sedlacek [3, 4] in 1963 where he used label values as distinct non-negative real numbers. Jeurissen [2] called a magic labeling pseudomagic if the labels were pairwise distinct.

It is obvious that K_3, K_4 are not supermagic. B. M. Stewart [11] showed that the complete graph K_5 is not supermagic and when $k \equiv 0 \bmod 4$, K_k is not supermagic. Stewart used induction to show that K_k is supermagic when $k \equiv 2 \pmod 4$. Hartsfield and Ringel gave a short direct-construction-proof in [1]. In fact, for $k>5$, K_k is supermagic if and only if $k \neq 0 \pmod 4$ [11].

The second author, Seah and Tan, introduced another labeling concept which is a generalization of supermagic graphs [6]. A (p,q)-graph $G = (V,E)$ is said to be *edge-magic* if there exists a bijection $L : E \rightarrow \{1,2,\ldots, q\}$ such that the induced map $L^+ : V \rightarrow \{0,1,\ldots,p-1\}$ defined by $L^+(v) = \Sigma \{L(u,v) : \text{for all } (u,v) \text{ in } E\} \pmod p$ is the constant map. It is clear that any supermagic graph is an edge-magic graph, but not conversely. The complete graph K_5 is not supermagic and it was shown in [6] that it is edge-magic. Figure 2 exhibits one of the edge-magic labelings of K_5.

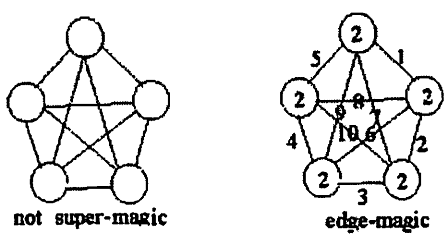

Figure 2

Given two graphs G and H. The composition of G with H, denoted as G[H], is the graph with vertex set V(G) x V(H) in which (u_1,v_1) is adjacent to (u_2,v_2) if and only if either $(u_1,u_2) \in E(G)$ or $u_1=u_2$ and $(v_1,v_2) \in E(H)$. For example, $K_3[N_2]$ where N_2 is the null graph with two vertices is shown in Figure 3.

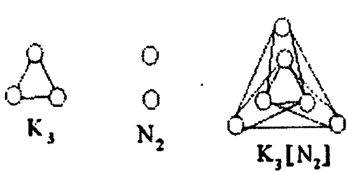

Figure 3

A regular complete k-partite graph $K_k[n]$ is isomorphic to the composition of a complete graph K_k with null graph N_n with n vertices. For a regular complete k-partite graph $K_k[n]$, we have $|V(K_k[n])|= nk$ and $|E([K_k[n]])| = n^2 \times k(k-1)/2$. Let $V(K_k[n]) = \{u_1,u_2,\ldots,u_{nk}\}$ and $E(K_k[n]) = \{(u\alpha , u\beta) : \alpha < \beta \}$ where $\alpha , \beta = 1,2,\ldots,nk$.

In this paper we want to know for what k and n the complete k-partite graph $K_k[n]$ is supermagic. We will use the following result which was established in 1998 by the second author, W.C. Shiu and P. Lam [9] to solve the problem.

THEOREM. If G is an r-regular supermagic graph, then G $[N_n]$ is an rn-regular supermagic graph for n>2.

By the result of Stewart and the above theorem we see that to solve the problem of supermagicness of a regular complete k-partite graph $K_k[n]$ we only have to consider the following cases:

Case 1. $K_k[n]$ for k=3,5 and n >1.

Case 2. $K_k[n]$ for k=0 (mod 4) and n even.

It is clear that $K_k[2t+1]$ is not supermagic for all t>0.

Case 3. $K_k[2]$ for k>5.

We prove the following result.

THEOREM. The regular complete k-partite graph $K_k[n]$ is supermagic if and only if n and k are not of the form

1. k=2 and n=2 or
2. $k \equiv 0 \pmod 4$ and n is odd.

2. Notations. Let G=(V,E) be a graph and S be a set. If f: E → S is a mapping then a *labeling matrix* for the labeling f of G is a matrix whose rows and columns are named by the vertices of G and the (u,v)-entry is f(u,v) if $(u,v) \in E$, and is 0 otherwise. Thus G is supermagic if and only if there exists a labeling f: E(G) → {1,2,…,q} such that the row sums and the column sums of the labeling matrix of G associated with f are equal.

The labeling matrix of $K_k[n]$ is of the form (Figure 4):

$$\left(\begin{array}{c|c|c} 0 & & B_{ij}^{T} \\ \hline & 0 & \\ \hline B_{ij} & & 0 \end{array}\right)$$

Figure 4

where each entry B_{ij} is an nxn submatrix and B_{ij}^{T} is the transpose of B_{ij} and O is the nxn zero matrix. Thus if $L : E \rightarrow \{1,2,3,\ldots,q\}$ is a supermagic labeling on $K_k[n]$ then for each vertex u, we have $L^{+}(u) = q(q+1)/p$.

We illustrate these concepts with three examples.

Example 1. $K_3[3]$ has the following supermagic labeling matrix. For simplicity we will omit the zero submatrices and the transpose matrices (Figure 5).

Supermagic labeling matrix of $K_3[3]$

	u1	u2	u3	u4	u5	u6	u7	u8	u9
u1									
u2									
u3									
u4	1	23	13						
u5	27	3	15						
u6	2	26	16						
u7	25	4	11	21	9	14			
u8	5	22	12	7	20	18			
u9	24	6	17	19	10	8			

Figure 5

We can further simplify the labeling matrix as follows (Figure 6):

	u1	u2	u3	u4	u5	u6	u7	u8	u9
u4	1	23	13						
u5	27	3	15						
u6	2	26	16						
u7	25	4	11	21	9	14			
u8	5	22	12	7	20	18			
u9	24	6	17	19	10	8			

Figure 6

Example 2. A supermagic labeling matrix of $K_3[2]$ (Figure 7).

	u_1	u_2	u_3	u_4
u_3	1	8		
u_4	11	6		
u_5	2	9	10	5
u_6	12	3	7	4

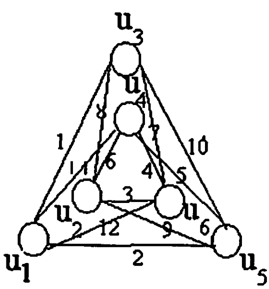

Figure 7

Example 3. A supermagic labeling matrix of $K_5[2]$ (Figure 8).

	u_1	u_2						
u_3	1	39						
u_4	40	2						
u_5	3	37	10	31				
u_6	38	4	32	9				
u_7	5	35	11	29	26	16		
u_8	36	6	30	12	15	25		
u_9	7	33	13	27	24	17	22	21
u_{10}	34	8	28	14	18	23	20	19

Figure 8

3. Supermagicness of $K_3[2t+1]$

The proof of $K_3[2t+1]$ is supermagic is done by induction. First we show $K_3[3]$ is supermagic, then we extend a supermagic labeling matrix of $K_3[3]$ to a supermagic labeling matrix of $K_3[7]$. We then explain how to extend a supermagic labeling matrix of $K_3[2m+1]$ to a supermagic labeling matrix of $K_3[2m+3]$.

i. A supermagic labeling matrix of $K_3[3]$ is shown in Figure 9.

	u_1	u_2	u_3	u_4	u_5	u_6
u_4	1	23	13			
u_5	27	3	15			
u_6	2	26	16			
u_7	5	22	12	7	20	18
u_8	25	4	11	21	9	14
u_9	24	6	17	19	10	8

Figure 9

The sum of each vertex is $q(q+1)/p=84$.

ii. We extend to a supermagic labeling matrix $K_3[5]$ as follows:

a. Embed B_{11} and $[B_{21}\ B_{22}]$ of the labeling matrix of $K_3[3]$ into the labeling matrix of $K_3[5]$ in the position indicated as follows (Figure 10):

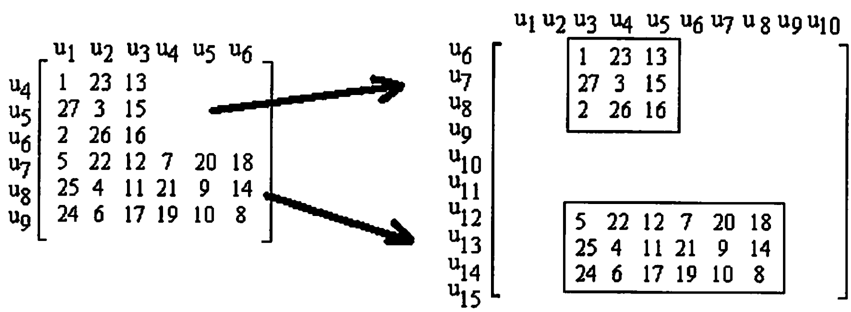

Figure 10

b. Calculate the increment value d where $d=3(5^2-3^2)/2=24$. Then we change the value of two blocks of matrices by adding two matrices $d(1_{3x3})$ and $d(1_{3x6})$ respectively, i.e. $B_{11} + d(1_{3x3})$ and $[B_{21}\ B_{22}] + d(1_{3x6})$ (Figure 11).

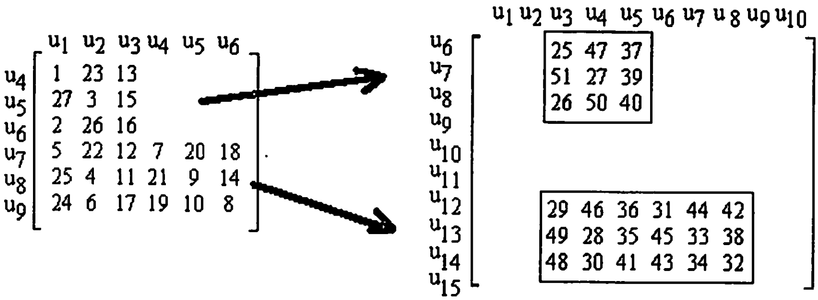

Figure 11

c. Arrange {1,2,...,d=24} and {q-d+1=75-24+1=57,...,75=q} in the remaining empty slots as follows (Figure 12):

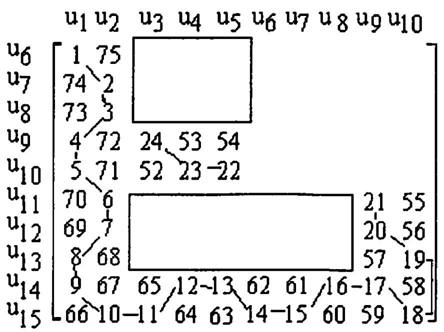

Figure 12

For each "x" along the "Snake 1-2-3-...-24" we assign the mirror image by "q+1-x", then we have the mirror-image of the "Snake 57-58-...-75".

d. Interchange the diagonal pair {66,67} from the left-bottom corner and the diagonal pair {58,59} from the right-bottom corner of the Snake 57-58-...-75. We obtain the following (Figure 13):

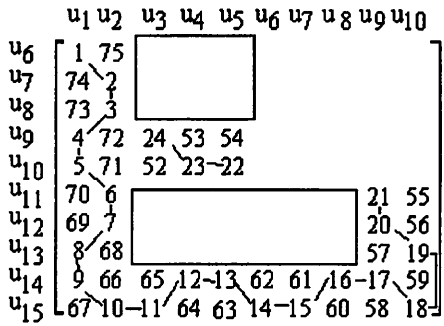

Figure 13

i. The following is the supermagic labeling matrix of $K_3[5]$ (Figure 14):

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}
u_1															
u_2															
u_3															
u_4															
u_5															
u_6	1	75	25	47	37										
u_7	74	2	51	27	39										
u_8	73	3	26	50	40										
u_9	4	72	24	53	54										
u_{10}	5	71	52	23	22										
u_{11}	70	6	29	46	36	31	44	42	21	55					
u_{12}	69	7	49	28	35	45	33	38	20	56					
u_{13}	8	68	48	30	41	43	34	32	57	19					
u_{14}	9	66	65	12	13	62	61	16	17	59					
u_{15}	67	10	11	64	63	14	15	60	58	18					

Figure 14

Having thus explained these specific lines of construction, the general method of contructing a labeling matrix of $K_3[2m+3]$ from $K_3[2m+1]$ may now be formulated:

1. Suppose we already have a labeling matrix of $K_3[2m+1]$. We embed B_{11} and $[B_{21}\ B_{22}]$ of the labeling matrix of $K_3[2m+1]$ into the labeling matrix of $K_3[2m+3]$ in the position indicated as above.

2. Each value in the blocks add the increment value d, which is given by the formula $d=[3(2m+3)^2-3(2m+1)^2]/2 = 12(m+1)$.

3. Arrange {1,2,...,d=12m+1} and {q-d+1,...,q} in the remaining empty slots as follows. We create two intertwining snakes: Snake 1-2-...d and Snake q-d+1-...q. Each x in Snake 1-2-...-d is corresponding to the mirror image "q+1-x" (Figure 15).

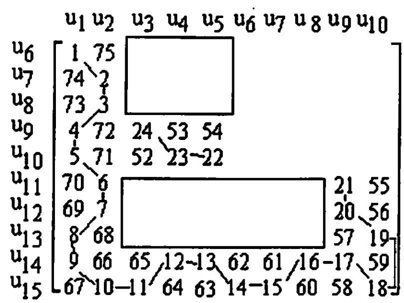

Figure 15

4. Interchange the diagonal pair {c,d} from the left-bottom corner and the diagonal pair {e,f} from the right-bottom corner where c,d,e,f are not in the "Snake 1-2-...-d". We obtain the supermagic labeling matrix of $K_3[2m+3]$.

The reason why this construction has worked is that:

For each row of labeling matrix of $K_3[2m+3]$, its sum is:

$2(q+1)$ + (the row sum of labeling matrix of $K_3[2m+1]$) + $(4m+2)[12(m+1)]$

$= 2[3(2m+1)^2+1] + (2m+1)[3(2m+1)2+1] + (48m^2 + 72m + 24)$

Simplifying the sum, we obtain $24m^3+108m^2+164m+84$, which can be expressed as $(2m+3)[3(2m+3)^2+1] = q(q+1)/p$.

Thus $K_3[2m+3]$ is supermagic.

Using the above scheme, the reader can construct the supermagic labeling matrix of $K_3[7]$.

Example 3. The supermagic labelling matrix of $K_3[7]$ (Figure 16).

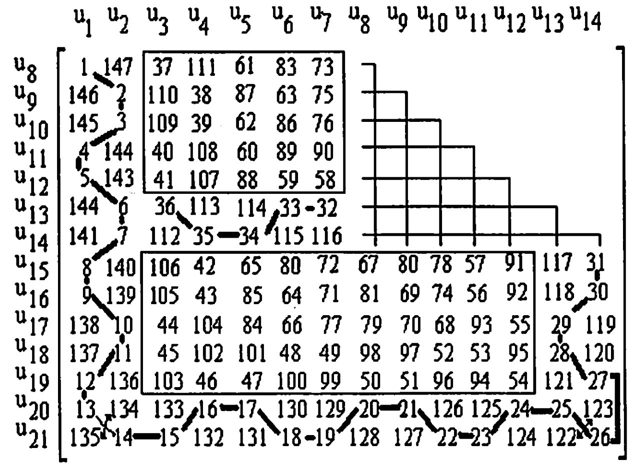

Figure 16

4. Supermagicness of $K_5[2t+1]$

The labeling scheme of $K_5[2t+1]$ is similar to $K_3[2t+1]$. The only difference is in step 5 in which we interchange only one diagonal pair.

1. First we exhibit a supermagic labeling matrix for $K_5[3]$ (Figure 17).

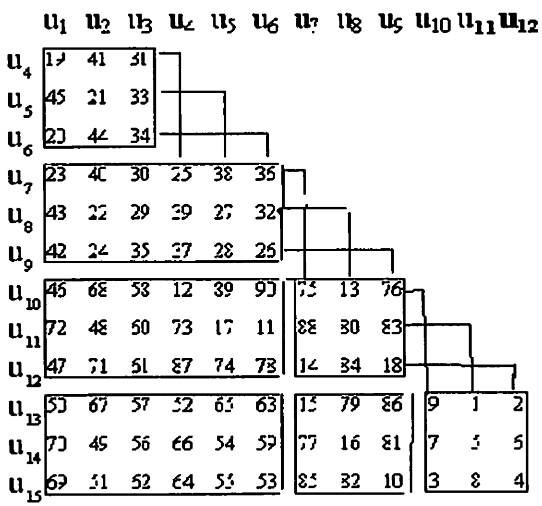

Figure 17

2. We embed the block matrices B_{11}, $[B_{21}\ B_{22}]$, $[B_{31}\ B_{32}]$, B_{33}, $[B_{41}\ B_{42}]$, B_{43}, B_{44} of the labeling matrix of $K_5[3]$ into the labeling matrix of $K_5[5]$

in the position indicated as follows (Figure 18):

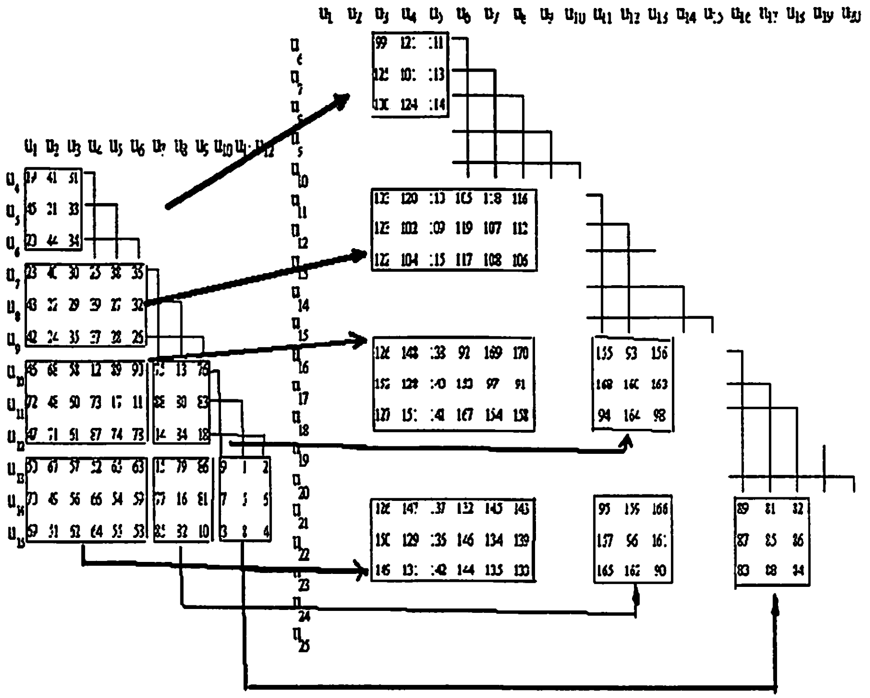

Figure 18

3. Each value in the blocks is increased by the increment value d, which is given by the formula:

$d=[10(2m+3)^2-10(2m+1)^2]/2 = 40(m+1)=80.$

See Figure 19.

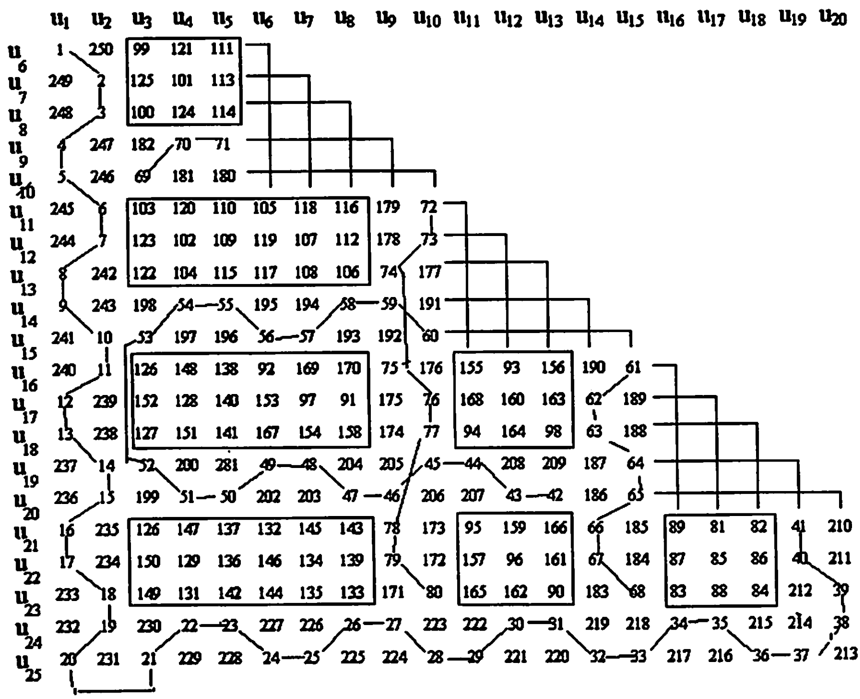

Figure 19

4. Arrange {1,2,…,d=80} and {q-d+1=250-80+1=171,…,250=q} in the remaining empty slots as follows. We create two intertwining snakes (Figure 20).

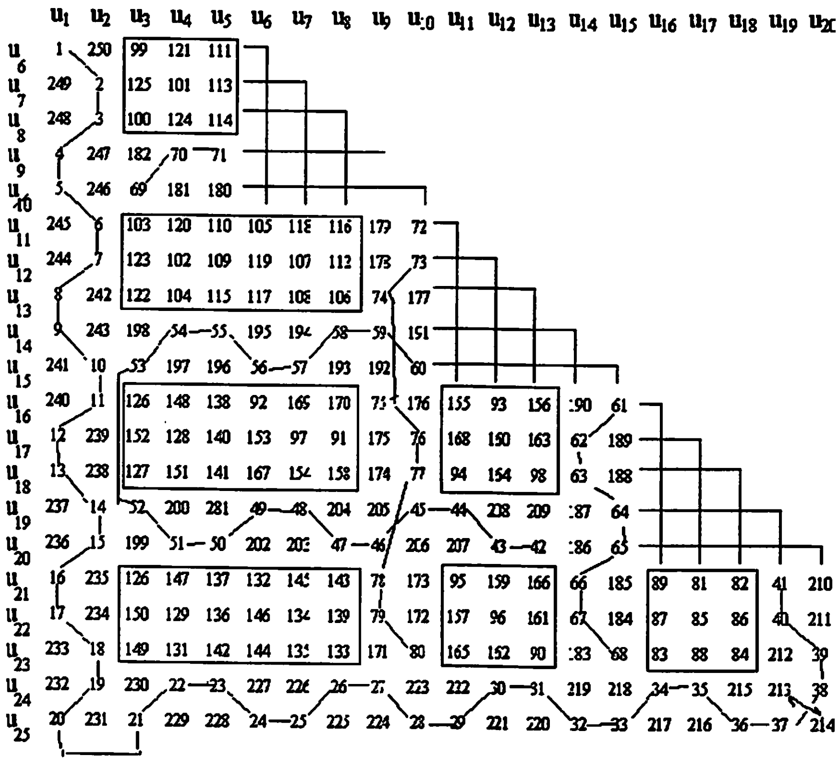

Figure 20

5. Interchange the diagonal pair {213,214} from the right-bottom corner. We obtain the supermagic labeling matrix of $K_5[5]$.

We see that $d=10(2m+3)^2-10(2m+1)^2]/2 = 40(m+1)$. Thus the row sum for each row of $K_5[2m+3]$ is given by:

$$4(q+1) + (\text{row sum of } K_5[2m+1]) + 4(2m+1)$$

$$=$$

$$4[10(2m+3)^2+1]+[2(2m+1)][10(2m+1)^2+1]+160(2m^2+3m+1)$$

$$= 160m^3+720m^2+1084m+546$$

$$= 2(2m+3)[10(2m+3)^2+1]$$

$$= q(q+1)/p$$

Example 5. A supermagic labeling matrix of $K_5[7]$ (Figure 21).

	u_{1}	u_{2}	u_{3}	u_{4}	u_{5}	u_{6}	u_{7}	u_{8}	u_{9}	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}	u_{16}	u_{17}	u_{18}	u_{19}	u_{20}	u_{21}	u_{22}	u_{23}	u_{24}	u_{25}	u_{26}	u_{27}	u_{28}	u_{29}
u_{8}	1	490	121	370	219	241	231																						
u_{9}	489	2	369	122	245	221	233																						
u_{10}	488	3	368	123	220	244	234																						
u_{11}	4	487	124	367	302	190	191																						
u_{12}	5	486	125	366	189	301	300																						
u_{13}	485	6	390	102	103	387	386																						
u_{14}	484	7	101	389	388	104	105																						
u_{15}	8	483	365	126	223	240	230	225	238	236	299	192	106	385															
u_{16}	9	482	364	127	243	222	229	239	227	232	298	193	107	384															
u_{17}	481	10	128	363	242	224	235	237	228	226	194	297	383	108															
u_{18}	480	11	129	362	318	174	175	315	314	178	179	311	382	109															
u_{19}	12	479	361	130	173	317	316	176	177	313	312	180	110	381															
u_{20}	13	478	414	78	79	411	410	82	83	407	406	86	87	403															
u_{21}	477	14	77	413	412	80	81	409	408	84	85	405	404	88															
u_{22}	476	15	360	131	246	268	258	212	289	290	195	296	111	380	275	213	276	310	181	402	89								
u_{23}	16	475	132	267	272	248	260	273	217	211	295	196	379	112	288	280	283	182	309	90	401								
u_{24}	17	474	133	258	247	271	261	287	274	278	294	194	378	113	214	284	218	183	308	91	400								
u_{25}	473	18	357	134	172	320	321	169	168	324	325	165	114	377	164	328	329	307	184	399	92								
u_{26}	472	19	356	135	319	171	170	322	323	167	166	326	115	376	327	163	162	306	185	398	93								
u_{27}	20	471	76	416	417	73	72	420	421	69	68	424	425	65	64	428	429	61	60	94	397								
u_{28}	21	470	415	75	74	418	419	71	70	422	423	67	66	426	426	63	62	430	431	95	396								
u_{29}	469	22	136	355	250	267	257	252	265	263	198	293	375	116	215	279	286	186	305	395	96	209	201	202	161	330	432	59	
u_{30}	468	23	137	254	270	249	256	266	254	259	199	292	374	117	277	216	281	187	304	394	97	207	205	206	160	331	433	58	
u_{31}	24	467	353	138	269	251	262	264	255	253	291	200	118	373	285	282	210	303	188	98	393	203	208	204	332	159	57	43	
u_{32}	25	466	252	139	240	142	143	347	346	146	147	343	119	372	342	150	151	339	338	99	392	154	155	335	333	158	56	43	
u_{33}	465	26	140	251	141	349	348	144	145	345	344	148	371	120	149	341	340	152	153	391	100	337	336	156	157	334	436	55	
u_{34}	464	27	462	30	31	459	458	34	35	455	454	38	39	451	450	42	43	447	446	46	47	443	442	50	51	439	437	54	
u_{35}	28	463	29	461	460	32	33	457	456	36	37	453	452	40	41	449	448	44	45	445	444	48	49	441	440	52	53	43	

Figure 21

Conclusion

We have proved in this paper the following results:

1. $K_3[2]$, $K_5[2]$ are supermagic.

2. $K_3[2t+1]$, $K_5[2t+1]$ are supermagic for all t.

In a subsequent article we will complete to solve the remaining cases.

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