i-Connectivity in a Generalized Hypercube

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Abstract. Inclusive connectivity parameters for a given vertex in a graph G are measures of how close that vertex is to being a cutvertex. Thus they provide a local measure of graph vulnerability. In this paper we provide bounds on the inclusive connectivity parameters in $K_2 \times G$ and inductively extend the results to a certain generalized hypercube.

Key words: Connectivity, inclusive connectivity, cohesion, i–connectivity.

1 Introduction

Let G be a connected graph without loops or multiple edges with vertex set $V(G)$, edge set $E(G)$, (vertex) connectivity $\kappa(G)$, and edge connectivity $\lambda(G)$. If not defined here, we follow the notation found in [6].

If S is a set of vertices of G and v is a vertex of G, we denote with $G - S - v$ the graph obtained from G by deleting all vertices of S with their incident edges and the vertex v with its incident edges. If S contains vertices not in $V(G)$ then $G - S$ is the graph obtained from G by deleting all vertices of S which are in $V(G)$ along with their incident edges. The open neighborhood of v in G, $N_G(v)$, is the set of all vertices adjacent to v. Given $S \subseteq V(G)$, the subgraph of G induced by S is denoted $\langle S \rangle$. A cutvertex of G is a vertex whose deletion either increases the number of components or increases the number of isolates in G. Note that this definition permits either vertex of a K_2-component to be a cutvertex.

For $v \in V(G)$, the inclusive edge connectivity of v, $\lambda_i(v, G)$, formerly called cohesion, is the minimum number of edges whose removal yields a subgraph in which v is a cutvertex. For various results on cohesion see [10, 12, 13, 14, 15, 16, 17]. Similarly, for $v \in V(G)$, the inclusive vertex (mixed) connectivity of v, $\kappa_i(v, G)$, $(\mu_i(v, G))$ is the minimum number of vertices (graph elements) whose removal yields a subgraph in which v is a cutvertex. The parameters $\lambda_i(e, G)$, $\kappa_i(e, G)$, and $\mu_i(e, G)$ are defined similarly for any edge e of G where 'cutvertex' is replaced by 'bridge' in the preceding definitions. When the underlying graph is apparent reference to that graph may be suppressed, for instance we may use $\lambda_i(v)$ instead of $\lambda_i(v, G)$ when no confusion arises. Inclusive connectivity, also referred to as i-connectivity, has been studied in [1, 2, 3, 4, 5, 7, 8, 9]. If S is a smallest set of vertices (respectively edges, graph elements) whose removal from G makes v a cutvertex, then we call S a κ_i–set (respectively λ_i–set, μ_i–set) for $v \in V(G)$. If S is a κ_i–set (respectively λ_i–set, μ_i–set) for v in G and $G - S - v$ leaves a neighbor of v as an isolated vertex then we say that S is a neighborhood κ_i–set (respectively λ_i–set, μ_i–set) for v.

Figure 1 illustrates these parameters. Note that in this case $\kappa_i(v) = 4, \lambda_i(v) = 3$ so that $\kappa_i(v) > \lambda_i(v)$ which is in contrast to Whitney's Theorem, which gives $\kappa(G) < \lambda(G)$ for the global parameters. This example also serves to illustrate our rather peculiar definition of a cutvertex. Notice that since $\langle N(v) \rangle$ is complete, the only way to make v a cutvertex by removing vertices is to isolate it in a K_2 component. In fact it is now apparent that $\kappa_i(v) = \min\{deg(w) : w \in N(v)\} - 1$ whenever $N(v)$ induces a complete subgraph in G. For the graph in Figure 1, every κ_i–set for v is a neighborhood κ_i–set.

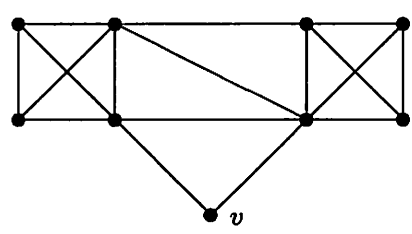


Figure 1: A graph with $\kappa_i(v) > \lambda_i(v)$.

We now require a few definitions which turn out to be useful in bounding i-connectivity values in the cartesian product of two graphs. These definitions are motivated by and are very similar to concepts used by Piazza and Ringeisen [11] to bound graph connectivity in cycle permutation graphs. Given a graph G and any $v \in V(G)$ define $K(v)$ by $K(v) = \min\{|V(C)|\}$

where the minimum is taken over all κ_i–sets S for $v \in V(G)$ and all components C of $G - S - v$ such that C contains a neighbor of v. We similarly define $L(v)$ and $M(v)$ with the minimum taken over all λ_i-sets and μ_i-sets for $v \in G$, respectively. With $N_k(v)$ we denote the least number of vertices from $N_G(v)$ in any component of $G - S - v$ when that component contains vertices of $N_G(v)$ taken over all κ_i–sets S for $v \in V(G)$. By taking the minimum over all λ_i–sets and μ_i–sets for $v \in V(G)$, respectively, $N_\lambda(v)$ and $N_\mu(v)$ are defined similarly. We denote the cartesian product of G_1 and G_2 by $G = G_1 \times G_2$ and note that $G_1 \times G_2 = G_2 \times G_1$. In illustration of the definition of cartesian product, consider the graph in Figure 2. This graph is isomorphic to both $C_5 \times C_3$ and $C_3 \times C_5$. These graphs are commonly conceptualized as consisting either of five copies of C_3 connected such that corresponding vertices in the copies form a C_5 or as three copies of C_5 with corresponding vertices joined as triangles. In particular, given a graph G, $K_2 \times G$ can be conveniently viewed as consisting of two copies of G with corresponding vertices adjacent. In what follows we denote the two copies of G in $K_2 \times G$ by G and G' and for each $u \in V(G)$ the corresponding vertex of G' by u'.

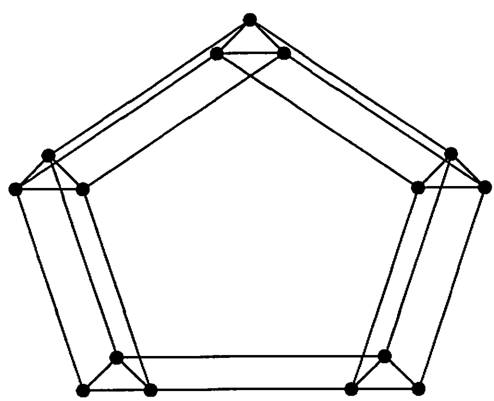


Figure 2: The graph $C_5 \times C_3$.

Studying connectivity in cartesian products of graphs is interesting not only because of the richness of the results but also because this is a common method for expansion of practical networks. For instance, the hypercube network configuration can be defined recursively by $Q_0 = K_1$ and $Q_n = Q_{n-1} \times K_2$ for $n \geq 1$. We now turn our attention to bounding inclusive connectivity in $K_2 \times G$ for an arbitrary graph G. Toward that end the next lemma is useful.

Lemma 1 *Let G be a connected graph and $v \in V(G)$. Then separating v' from some $w \in N_G(v)$ into different components of $K_2 \times G$ requires a separating set of vertices of size at least* $\min\{deg_G(v)+1, \kappa_i(v,G)+N_k(v)+1\}$.

Proof: Let S^* be a set of vertices that separates v' from $w \in N_G(v)$ in $(K_2 \times G) - S^*$. That is, in the graph $(K_2 \times G) - S^*$, v' and w are in different components. Notice that necessarily $v \in S^*$ and $w' \in S^*$. It could occur that $N_G(v) - \{w\} \subseteq S^*$ in which case we immediately obtain $|S^*| \geq \deg_G(v) + 1$. Now, if some vertex in the set $N_G(v) - \{w\} - S^*$ is not reachable from w in $G - S^*$ then at least $\kappa_i(v,G) + 1$ internally disjoint paths of G between w and that vertex must have been severed with the removal of S^* from $K_2 \times G$. Thus, in this case, $|S^*| \geq \kappa_i(v,G) + N_k(v) + 1$ due to the existence of paths of the form $xx'v'$ in $K_2 \times G$ where x is a vertex of $N_G(v)$ which is in the same component of $(K_2 \times G) - S^*$ as is w. The remaining alternative is if every vertex from $N_G(v) - \{w\} - S^*$ is reachable from w in $G - S^*$. Then for any $u \in N_G(v) - \{w\}$ reachable from w in $(K_2 \times G) - S^*, u' \in S^*$. Thus S^* must contain at least $|N_G(v) - \{w\}|$ vertices of G' and additionally S^* must contain the vertices w' and v for a total of at least $\deg_G(v) + 1$ vertices. The result is thereby established. □

Theorem 2 *Let G be any connected graph and $v \in V(G)$. Then*
$\min\{deg_G(v),\ \kappa_i(v,G) + N_k(v), 2\kappa_i(v,G) + 1\} \leq \kappa_i(v, K_2 \times G)$
$\leq \min\{deg_G(v),\ 2\kappa_i(v,G) + 1,\ \kappa_i(v,G) + K(v)\}$.

Proof: First we establish the lower bound. Let S be a κ_i–set for v in $K_2 \times G$. If there exists a vertex $w \in N_G(v)$ with w and v' in different components of $(K_2 \times G) - S - v$, we obtain the result by Lemma 1. The only other possibility is that S separates vertex u from vertex w, where both u and w are vertices of $N_G(v)$. Let this be the case. Notice then that a κ_i–set for v in G must be a part of S. We require some notation. Let C_1 denote the component of $G - S - v$ containing vertex u and C_2 the component containing w. The subgraphs of G' induced by the vertices of G' corresponding to C_1 and C_2 are represented as C_1' and C_2' respectively. Without loss of generality, let $|V(C_1)| \leq |V(C_2)|$. Now, if there is some $x \in V(C_1)$ where x is a neighbor of v in G and $x' \notin S$ then in the graph $(K_2 \times G) - S - v$ there can be no path between x' and any vertex in $V(C_2')$ which is in $G' - S - v$. Then either (a) two neighbors of v' in G' must be separated in $G' - S - v'$ giving $\kappa_i(v, K_2 \times G) \geq 2\kappa_i(v,G) + 1$ or (b) all vertices of C_2' which are neighbors of v' in G' are in S giving $\kappa_i(v, K_2 \times G) \geq \kappa_i(v,G) + N_k(v)$. Thus the lower bound is seen to hold.

To verify the upper bound we show that any one of the three numbers suffice for the size of a set of vertices whose removal from $K_2 \times G$ makes v a cutvertex. In this respect it is clear that $\deg_G(v)$ suffices.

To establish $2\kappa_i(v,G)+1$ as an upper bound let S be any κ_i–set for v in G and S' be those vertices of G' corresponding to S. If there are two neighbors of v, u and w say, in different components of $G-S-v$, we claim that $S^* = S \cup S' \cup \{v'\}$ is a set of vertices from $V(K_2 \times G)$ with the property that u and w, are separated in $(K_2 \times G) - S^* - v$. Every u–w path in $(K_2 \times G) - S^* - v$ contains a vertex of G'. Because moving from one component of $G-S-v$ over to the corresponding component of $G' - S' - v'$ yields no new u–w paths, we conclude that u and w are in different components of $(K_2 \times G) - S^* - v$. This upper bound $2\kappa_i(v,G)+1$ is now seen to be valid since if the removal of S and v isolates a neighbor of v then this same neighbor can be isolated in $K_2 \times G$ with the removal in G' of $|S|+1$ vertices followed by the removal of v.

To verify $\kappa_i(v,G)+K(v)$ as an upper bound let S be any κ_i–set for v in G and C be any component of $G-S-v$ which contains a neighbor of v. Let C' be the subgraph of G' corresponding to C. Assume S separates neighbors of v in $G-S-v$. Then there is a neighbor of v, say w, such that $w \in V(C)$ and w is separated from some other neighbor of v, say u. Then $S^* = S \cup V(C')$ separates u and w in $(K_2 \times G) - S^* - v$. To realize that this is true notice that there are no u–w paths in $(K_2 \times G) - S^* - v$ consisting entirely of vertices from $V(G)$ (since S is a κ_i–set for v in G) and no vertex of $G' - S^*$ is adjacent to w or any vertex in the same component, C, as w in $(K_2 \times G) - S^* - v$.

Finally, if S is a neighborhood κ_i–set isolating v in a K_2–component with vertex u then $K(v) = 1$ and that same vertex can be isolated in a K_2–component of $K_2 \times G$ with the removal of $|S|+1$ vertices. □

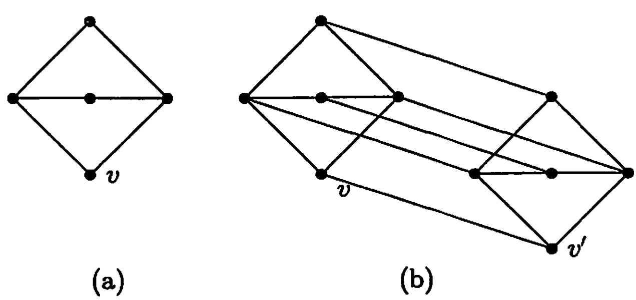


Figure 3: The graph $G \times K_2$ for Example 1.

The following examples show that the bounds on $\kappa_i(v, G \times K_2)$ provided by Theorem 2 are sharp.

Example 1. The graph, G, shown in Figure 3(a) is an example of a graph G used in Figure 3 (b) to form $K_2 \times G$ where $\kappa_i(v,G) + N_k(v) = 3$, $2\kappa_i(v,G)+1 = 5$, $K(v) = 1$ and $\deg_G(v) = 2$. Therefore the bounds of

Theorem 2 are $2 \leq \kappa_i(v, K_2 \times G) \leq 2$ giving $\kappa_i(v, K_2 \times G) = 2 = deg_G(v)$.

Example 2. With the graph in Figure 4(a) showing G and the graph in Figure 4(b) depicting $K_2 \times G$ we have $\kappa_i(v, G) + N_k(v) = 2$, $2\kappa_i(v, G) + 1 = 3$, $K(v) = 2$ and $\deg_G(v) = 3$. Therefore $2 \leq \kappa_i(v, K_2 \times G) \leq 3$ and in actuality $\kappa_i(v, K_2 \times G) = \kappa_i(v, G) + N_k(v) = 2$.

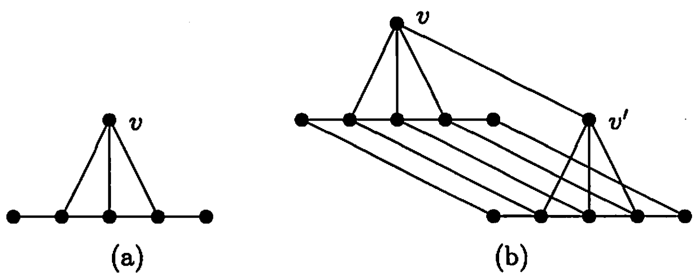


Figure 4: The graph $G \times K_2$ for Example 2.

Example 3. The graph in Figure 5(a) illustrates a graph G and Figure 5(b) shows $K_2 \times G$ in which $\kappa_i(v, G) + N_k(v) = 2$, $2\kappa_i(v, G) + 1 = 3$, $K(v) = 3$ and $\deg_G(v) = 5$. Therefore $2 \leq \kappa_i(v, K_2 \times G) \leq 3$ and in actuality $\kappa_i(v, K_2 \times G) = 2\kappa_i(v, G) + 1 = 3$.

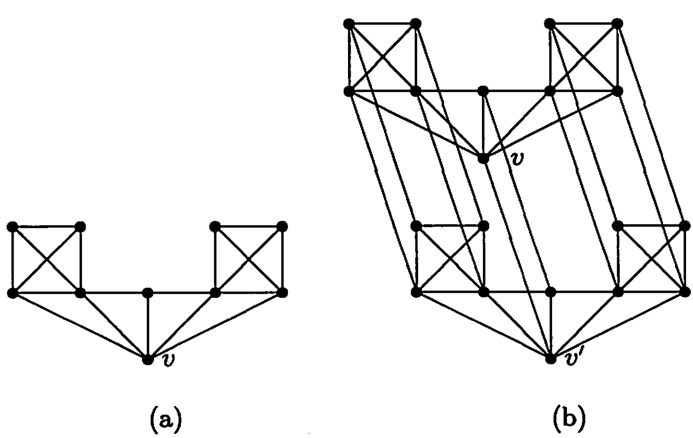
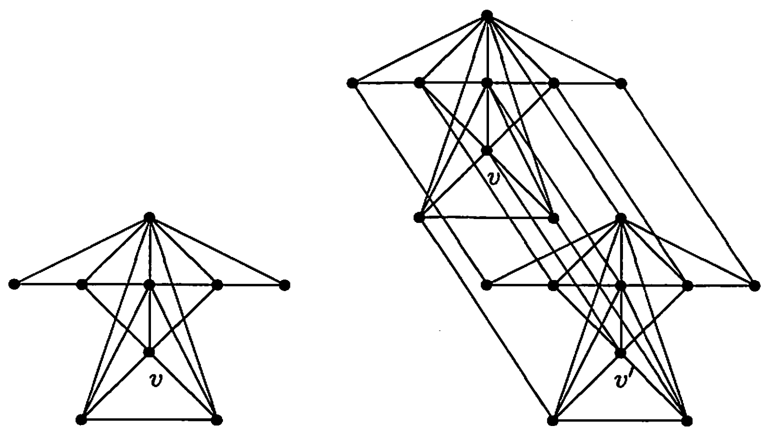


Figure 5: The graph $G \times K_2$ for Example 3.

Example 4. Consider the graph G in Figure 6(a). A κ_i–set for v in G consists of the two vertices directly above v. The parameters of interest for G are $\kappa_i(v,G)+N_k(v)=3$, $2\kappa_i(v,G)+1=5$, $K(v)=2$ and $\deg_G(v)=5$. The product $G \times K_2$ is shown in Figure 6(b). Theorem 2 gives the bounds $3 \leq \kappa_i(v,K_2 \times G) \leq 4$ and in actuality $\kappa_i(v,K_2 \times G)=\kappa_i(v,G)+K(v)=4$.



(a) (b)

Figure 6: The graph $G \times K_2$ for Example 4.

We now point out a direct result of Theorem 2, providing easily checked conditions which enable the value of $\kappa_i(v,K_2 \times G)$ to be determined exactly. Notice for these graphs the removal of the neighbors of the vertex v' in $V(G')$ leaves v' isolated when v is then removed.

Corollary 3 *If $deg_G(v) \leq \kappa_i(v,G)+1$ then $\kappa_i(v,K_2 \times G)=deg_G(v)$.*

The next corollary of Theorem 2 is easy to verify upon considering that it is always the case that $\kappa_i(v,G)+K(v) \geq \kappa_i(v,G)+1$.

Corollary 4 *If $deg_G(v) \geq \kappa_i(v,G)+1$ and $K(v)=1$ then $\kappa_i(v,K_2 \times G)=\kappa_i(v,G)+1=\kappa_i(v,G)+K(v)$.*

Since $N_k(v) \geq 1$ and $\kappa_i(v,G) \geq 1$ we also have the following result.

Corollary 5 *For any graph G,*

$$min\{deg_G(v), \kappa_i(v,G)+1\} \leq \kappa_i(v,K_2 \times G).$$

Essentially the same proof used in showing Theorem 2 establishes the following result. Notice however that our upper bound has no term dependent upon $\lambda_i(v,G)$, i.e., we would expect based on the previous theorem

$2\lambda_i(v,G)+1$ as an upper bound. That this is not the case in general is demonstrated with the graph G from Example 3 where $2\lambda_i(v,G)+1=3$ yet $\lambda_i(v,K_2\times G)=4$. This difference arises since the removal of vertices (from κ_i–sets) may delete many associated edges. In particular, when trying to make v a cutvertex in $K_2\times G$ by removing vertices we may wish to delete the vertex v' (the vertex of G' corresponding to v).

Theorem 6 *Let G be any connected graph and $v\in V(G)$. Then* $\min\{deg_G(v),\ \lambda_i(v,G)+N_\lambda(v)\}\le \lambda_i(v,K_2\times G)\le \min\{deg_G(v),\ \lambda_i(v,G)+L(v)\}$.

In the case of mixed i-connectivity the methods employed in Theorem 2 can be modified easily to yield the next assertion.

Theorem 7 *Let G be any connected graph and $v\in V(G)$. Then* $\min\{deg_G(v),\ \mu_i(v,G)+N_\mu(v)\}\le \mu_i(v,K_2\times G)\le \min\{deg_G(v),\ 2\mu_i(v,G)+1,\ \mu_i(v,G)+M(v)\}$.

In order to extend Theorem 2, we define a a generalized hypercube recursively in terms of products of K_2 and an arbitrary graph G. More formally, let the generalized hypercubes be defined by $G_1=G\times K_2$, and for $n\ge 1$, $G_{n+1}=G_n\times K_2$. Notice that this gives the definition for hypercubes when $G=K_2$. Also notice that for any G, $G_1=G\times K_2=G\times Q_1$ and in general $G_n=G\times Q_n$, where Q_n is the hypercube with degree n. Using this definition we have the following result.

Theorem 8 *Let G be any connected graph and $v\in V(G)$. Then* $\min\{deg_G(v)\ +n-1,\ \kappa_i(v,G)+n\}\le\kappa_i(v,G_n)\le$ $\min\{deg_G(v)+n-1,\ 2^n(\kappa_i(v,G)+1)-1,\ \kappa_i(v,G)+(2^n-1)K(v)\}$.

Proof: We proceed by induction on n. The base step, when $n=1$ follows from Theorem 2 and Corollary 5. Assume the theorem is true for n. Consider the case for $n+1$.

Let G_{n+1} be considered as two copies of G_n where the second copy is denoted by G'_n (and vertices in this copy also are denoted using primes) with the corresponding vertices adjacent. Let S be a κ_i–set for $v\in V(G_{n+1})=V(G_n\times K_2)$. This means that either the removal of S leaves v in a K_2 so that the removal of v isolates a vertex, or there exist two neighbors of v, say u and w, where u and w are in different components of $G_{n+1}-S-v$.

Consider first the case when v is isolated in a K_2 with v'. Then since each vertex in $N_{G_n}(v)$ was removed, $|S|\ge \deg_{G_n}(v)=\deg_G(v)+n$. If v is isolated in a K_2 with $w\in V(G_n)$ then necessarily $N_{G_n}(v)-\{w\}$ and v' must be contained in S. This also yields $|S|\ge\deg_{G_n}(v)=\deg_G(v)+n$. If the separated vertices u,w satisfy $u,w\in N_{G_n}(v)$ then because $V(G_n)\cap S$ must be a κ_i–set for v in G_n, $|S \cap V(G_n)| \geq \min\{\deg_G(v)+n-1, \kappa_i(v,G)+n\}$. However, in G_{n+1} there exists at least one additional path from u to w so that $|S| \geq |S \cap V(G_n)| + 1 \geq \min\{\deg_G(v)+n, \kappa_i(v,G)+n+1\}$. If $w \in N_{G_n}(v)$ is separated from v' in G_{n+1} then for all $x \in N_{G_n}(v) - S - \{w\}$, x must be reachable from w (otherwise we have already condisered this case). For any such x, $x' \in S$. This means for any $x \in N_{G_n}(v)$ either $x \in S$ or $x' \in S$ implying $|S| \geq \deg_{G_n}(v) = \deg_G(v) + n$. The case when $N_{G_n}(v) - S - \{w\} = \phi$ has already been considered. Hence the lower bound holds by induction.

Consider now the upper bound. This is shown by demonstrating that the removal of vertex sets of the appropriate sizes will result in v being a cutvertex in G_{n+1}. Clearly the removal of all neighbors of v' in $G_{n+1}-v$ will cause v' to be isolated upon the removal of v. However, $|N_{G_{n+1}-v}(v')| = |N_{G_n}(v)| = \deg_G(v) + n$.

Let S be a κ_i–set for v in $V(G_n)$ and S' be the corresponding set in G'_n. If $S \cup \{v\}$ separates u and w where $u, w \in N_{G_n}(v)$, then we claim $S \cup S' \cup \{v'\} \cup \{v\}$ separates u and w in G_{n+1}. Notice that there are no u–w paths in $G_{n+1} - S - v$ that do not leave G_n. However any path leaving the component of G_n containing u can only reach vertices in the corresponding component of G'_n and hence cannot reach w. If the removal of $S \cup \{v\}$ from G_n isolates $w \in N_{G_n}(v)$ then the edge ww' (isomorphic to K_2) is isolated in $G_{n+1} - (S \cup S' \cup \{v'\} \cup \{v\})$. Hence $|S \cup S' \cup \{v'\}|$ is an upper bound. However, by the inductive hypothesis $|S| \leq 2^n(\kappa_i(v,G)+1) - 1$ so that $|S \cup S' \cup \{v'\}| = |S| + |S'| + 1 \leq 2(2^n(\kappa_i(v,G)+1) - 1) + 1 = 2^{n+1}(\kappa_i(v,G)+1) - 2 + 1 = 2^{n+1}(\kappa_i(v,G)+1) - 1$.

Finally to see that $\kappa_i(v,G) + (n+1)K(v)$ is an upper bound, recall that $G_{n+1} = G \times Q_{n+1}$ so that there are 2^{n+1} copies of G and $v \in V(G)$ is connected in an $(n+1)$–hypercube to n other copies of v. In the original graph, G, we know that for any κ_i–set in G there exists a component of $G - S - v$, C, where C contains a neighbor of v and $|C| = K(v)$. Let $C_1, C_2, \ldots, C_{n+1}$ be the corrresponding sets in the $n+1$ copies of G in G_{n+1} in which v has neighbors. Let $S^* = S \cup V(C_1) \cup \ldots \cup V(C_{n+1})$. We claim that S^* is a κ_i–set for G_{n+1}.

If, in the original graph, G, neighbors of v, u and w, are separated in $G - S - v$, then there exists a neighbor of v, say w, such that $w \in V(C)$ and w is not reachable from u. Notice that in $G_{n+1} - S^* - v$ there are no u–w paths because there is no u–w path consisting entirely of vertices from $V(G)$ and no path from w to some other copy of G in G_{n+1} since all of the components corresponding to C (which contains w) have been removed.

Finally if S is a neighborhood κ_i–set isolating v in a K_2 component with the other vertex w then $S^* = S \cup \{w_1\} \cup \ldots \cup \{w_{n+1}\}$, $|V(C)| = 1 = K(v)$ and the same K_2 is isolated in $G_{n+1} - S^*$. For this case $|S^*| = \kappa_i(v,G) + n + 1 = \kappa_i(v,G) + (n+1)K(v)$. □

The following result about the generalized hypercube follows directly from Theorem 8.

Corollary 9 *For the $n+1$-dimensional hypercube $\kappa_i(v, Q_{n+1}) = n$.*

The following two results can be obtained with conditions relating the various parameters of the original graph G.

Corollary 10 *If for any graph G, $deg_G(v) \le \kappa_i(v, G)$ then $\kappa_i(v, G_n) = deg_G(v) + n - 1$.*

Corollary 11 *If the only way to make v a cutvertex is to isolate a neighbor of v, that is, if a minimum κ_i-set for v is a neighborhood κ_i-set for v then $K(v) = 1$ and hence $\min\{deg_G(v) + n - 1, \kappa_i(v, G) + n\} \le \kappa_i(v, G_n) \le \min\{deg_G(v) + n - 1, 2^n(\kappa_i(v, G) + 1) - 1\}$*

Corollary 12 *If $v \in V(G)$ has a κ_i neighborhood set and $deg_G(v) \le \kappa_i(v, G) + 1$ then $\kappa_i(v, G_n) = deg_G(v) + n - 1$.*

We note as a final observation the following corollary to Theorem 8.

Corollary 13 *For sufficiently large n, $\kappa_i(v, G_n)$ is of order n.*

As a simple example of this notice that for the graph G used in Example 3, $\kappa_i(v, G_1) = 3 \neq \deg_G(v)+1-1 = 5$, but $\kappa_i(v, G_2) = 6 = \deg_G(v)+2-1$.

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