On a Relationship between 2-dominating and 5-dominating sets in Graphs

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ABSTRACT. In a graph, a set D is an n-dominating set if for every vertex x, not in D, x is adjacent to at least n vertices of D. The n-domination number, $\gamma_n(G)$, is the order of a smallest n-dominating set. When this concept was first introduced by Fink and Jacobson, they asked whether there existed a function $f(n)$, such that if G is any graph with minimum degree at least n, then $\gamma_n(G) < \gamma_{f(n)}(G)$. In this paper we show that $\gamma_2(G) < \gamma_5(G)$ for all graphs with minimum degree at least 2. Further, this result is best possible in the sense that there exist infinitely many graphs G with minimum degree at least 2 having $\gamma_2(G) = \gamma_4(G)$.

1 Introduction

For the purpose of this paper, we consider only finite undirected simple graphs. For any undefined terms see [3]. For a graph G, with vertex set V and edge set E, a subset $D \subseteq V$ is a *dominating set* if every vertex in $V - D$ is adjacent to at least one vertex in D. The *domination number*, denoted $\gamma(G)$, is the order of a smallest dominating set in G. The area of domination has been around for quite some time, and is rich in research problems, as

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is evidenced by the extensive bibliography given by Hedetniemi and Laskar [4].

In this paper, we will concentrate on a generalization of the domination number of a graph. For a positive integer n, a subset D is an n-dominating set if for every vertex x, not in D, x is adjacent to at least n vertices of D. The *n-domination number*, $\gamma_n(G)$, is the order of a smallest n-dominating set in G. This concept was first introduced by Fink and Jacobson [2], in which they conjectured:

If G is a graph of minimum degree $\delta(G) \geq n$, then

$$\gamma_n(G) < \gamma_{2n+1}(G).$$

In [5] an example of a graph G was given for which

$$\gamma_n(G) = \gamma_{n^2/4}(G)$$

This still left the problem of deciding whether there existed an integer N depending on n so that

$$\gamma_n(G) < \gamma_N(G).$$

for every positive integer n and all graphs G with $\delta(G) \geq n$. The smallest such N is denoted by $f(n)$ if it exists.

It is the purpose of this paper to show the following:

Theorem 1. *If G is any graph with $\delta(G) \geq 2$, then*

$$\gamma_2(G) < \gamma_5(G).$$

That is, we show that $f(2) \leq 5$. For any positive integer $m \geq 4$, the graph $K_{4,m}$ has

$$\gamma_2(K_{4,m}) = \gamma_4(K_{4,m}) = 4.$$

This indicates that $f(2) > 4$. Hence we have $f(2) = 5$.

We will give the proof in the next section. The proof technique unfortunately doesn't seem to generalize in a "nice" fashion and thus the question of the existence of the function $f(n)$ originally posed in [1] remains open.

2 Main Result

Before proceeding with the proof of Theorem 1, we give the following useful lemmas.

Lemma 1. *Let H be a bipartite graph with partite sets X and Y. If H has a path*

$$P = u_0u_1u_2\ldots u_p$$

with $u_0 \in Y$ *and* $|N(u_0) \cap V(P)| \geq 3$, *then* H *has a path*

$$Q = v_0 v_1 v_2 \dots v_m$$

with $v_0 \in X$ *and* $|N(v_0) \cap V(Q)| \geq 2$.

Proof: Since $|N(u_0) \cap \{u_1, u_2, \dots, u_p\}| \geq 3$, it must be the case that there exists $2 < i < j \leq p$ such that $u_0 u_i$ and $u_0 u_j$ are in $E(H)$. Now since H is bipartite, u_i is in X, and

$$u_i u_{i-1} u_{i-2} \dots u_0 u_j u_{j+1} \dots u_p$$

is a path that has the desired property since $\{u_0, u_{i-1}\} \subseteq N(u_i)$. □

Lemma 2. *Let* H *be a bipartite graph with partite sets* X *and* Y. *If* H *has a path*

$$P = u_0 u_1 u_2 \dots u_p$$

with

$$u_p \in Y, \ |N(u_p) \cap V(P)| \geq 4,$$

then H *has a path*

$$Q = x_0 y_1 x_1 y_2 x_2 \dots y_m x_m$$

with $\{x_0, x_1, \dots, x_m\} \subseteq X$ *and* $\{y_1, y_2, \dots, y_m\} \subseteq Y$ *such that*

$$|N(x_0) \cap V(Q)| \geq 2, \ \textit{and} \ |N(x_m) \cap V(Q)| \geq 2.$$

Proof: Since $|N(u_p) \cap V(P)| \geq 4$, let i, j and k be such that $0 \leq i < j < k < m - 1$ with $u_p u_i$, $u_p u_j$, $u_p u_k$ and $u_p u_{p-1}$ all in $E(G)$. Then, the path

$$u_k u_{k+1} \dots u_p u_i u_{i+1} \dots u_j$$

has the required properties. □

2.1 Proof of Theorem 1

Let G be any graph with $\delta(G) \geq 2$ and D be a minimum 5-dominating set with $\gamma_5(G)$ elements. To prove the theorem, we need only to show that there is a 2-dominating set D^* with fewer elements than D.

To the contrary, assume that no such D^* exists. First we observe that the maximum degree in the subgraph induced by D is at most 1 for otherwise if there was a $v \in D$ with $|N(v) \cap D| \geq 2$, then $D^* = D - \{v\}$ would contradict the assumption.

Let $T = V(G) - D$. Note that T is not empty since a 2-dominating set of order less than the number of vertices in G clearly exists. Let H

be the bipartite subgraph of G with all edges between D and T, that is, $E(H) = \{uv \in E(G): u \in D \text{ and } v \in T\}$. For each vertex of G, let

$$N_H(x) = \{y: xy \in E(H)\}.$$

Further, let

$$D_1 = \{v \in D: |N(v) \cap T| = 1\},$$

and

$$D_2 = \{v \in D: |N(v) \cap T| \geq 2\}.$$

Claim 1. *For every $x \in T$, $|N(x) \cap D_1| \leq 2$. Further, if the equality holds, the two neighbors of x in D_1 are adjacent in G.*

Proof: Note that $\Delta(< D >) \leq 1$. It is sufficient to show that x does not have two neighbors in D_1 which are not adjacent. Suppose, to the contrary x has two neighbors u and v in D such that $uv \notin E(G)$. It follows that

$$D^* = (D - \{u, v\}) \cup \{x\}$$

would contradict the assumption. □

Claim 2. *H has a path*

$$P = u_0u_1u_2 \dots u_{p-1}u_p$$

such that $u_0 \in D_2$ and $|N(u_0) \cap V(P)| \geq 2$.

Proof: Let $P = u_0u_1 \dots u_p$ be a longest path in H. Clearly,

$$N_H(u_0) \subseteq \{u_1, u_2, \dots, u_{p-1}, u_p\}.$$

If $u_0 \in D_2$, then $u_0u_1 \dots u_p$ itself is such a desired path. Suppose $u_0 \in D_1$. Since

$$|N_H(u_1) \cap D| \geq 5 \text{ and } |N_H(u_1) \cap D_1| \leq 2,$$

there are at least 3 neighbors of u_1 in D_2. If there is a neighbor w_0 of u_1 which is in $D_2 - V(P)$ then $Q = w_0u_1u_2 \dots u_p$ would also be a longest path and $N_H(w_0) \subseteq V(Q)$. Thus, Q would be a desired path. Otherwise,

$$|N_H(u_1) \cap \{u_2, u_3, \dots, u_p\}| \geq 3,$$

and then by Lemma 1, a desired path results.

Suppose $u_0 \in T$. Then

$$|N(u_0) \cap \{u_1, u_2, \dots, u_p\}| \geq 5.$$

Again, by Lemma 1, there is a desired path. □

A path $P = x_0y_1x_1y_2x_2 \dots y_px_p$ of H with x_0 and $x_p \in D$ having one of the following properties is called a W-path.

1. $|N(x_0) \cap \{y_1, y_2, \dots, y_p\}| \geq 2$ and $|N(x_p) \cap \{y_1, y_2, \dots, y_p\}| \geq 2$;
2. $|N(x_0) \cap \{y_1, y_2, \dots, y_p\}| \geq 2$ and $x_p \in D_1$ having $N(x_p) \cap \{x_0, x_1, \dots, x_{p-1}\} = \emptyset$.

Claim 3. *H has a W-path.*

Proof: By Claim 2, let

$$P = u_0 u_1 u_2 \dots u_p$$

be a path in H such that $u_0 \in D_2$ and $|N(u_0) \cap V(P)| \geq 2$ and having p as large as possible. Clearly, $N_H(u_p) \subseteq \{u_1, u_2, \dots, u_{p-1}\}$. If $u_p \in D_2$, this claim would follow immediately. If $u_p \in T$, then $|N(u_p) \cap \{u_1, u_2, \dots, u_{p-1}\}| \geq 5$. By Lemma 2, this claim would follow. Thus, we assume that $u_p \in D_1$ and $N(u_p) \cap V(P) \cap D \neq \emptyset$ since otherwise there would be a W-path. Since $\Delta(< D >) \leq 1$, we have $N(u_p) \cap (D - \{u_0, u_1, \dots, u_{p-1}\}) = \emptyset$.

Note that $u_{p-1} \in T$. By Claim 1, and since P is not a W-path,

$$N(u_{p-1}) \cap (D_1 - \{u_0, u_1, u_2, \dots, u_p\}) = \emptyset.$$

If there is a vertex $w_p \in N(u_{p-1}) \cap (D_2 - \{u_0, u_1, \dots, u_{p-1}\})$, the path

$$u_0 u_1 \dots u_{p-1} w_p$$

is also a longest path in H with $|N(u_0) \cap \{u_0, u_1, \dots, u_{p-1}\}| \geq 2$. Then $N_H(w_p) \subseteq \{u_1, u_2, \dots, u_{p-1}\}$. Since $w_p \in D_2$, $|N_H(w_p)| \geq 2$. Thus the claim holds.

Therefore $|N(u_{p-1}) \cap \{u_0, u_1, \dots, u_{p-2}\}| \geq 4$ since D is a 5-dominating set of G. By Lemma 2, H has a W-path. □

Claim 4. *There is a W-path $x_0 y_1 x_1 \dots y_p x_p$ such that $|N(y) \cap \{x_0, x_1, \dots, x_p\}| \leq 3$ for each $y \in T - \{y_1, y_2, \dots, y_p\}$.*

Proof: To the contrary, we assume that the claim is not true. Let $P = u_0 u_1 \dots u_p$ be a W-path of H with minimum length. Since the claim fails, there is a vertex $y \in T - V(P)$ such that $|N(y) \cap V(P)| \geq 4$.

Let $0 \leq i_1 < i_2 < i_3 < i_4 \leq p$ be such that $N(y) \supseteq \{u_{i_1}, u_{i_2}, u_{i_3}, u_{i_4}\}$. Then

$$P^* = u_{i_1} u_{i_1+1} \dots u_{i_2-1} u_{i_2} y u_{i_3} u_{i_3+1} \dots u_{i_4}$$

is a W-path and by assumption is not shorter than P. Note that $u_{i_1} y \in E(H)$ and $u_{i_4} y \in E(H)$. Thus by the minimality of the length of P, we have $i_1 = 0$ and $i_4 = p$ and $i_3 = i_2 + 2$. In particular, we have shown that H has a shortest W-path

$$P = x_0 y_1 x_1 y_2 x_2 \dots y_p x_p$$

such that $x_0 y_k \in E(H)$ and $x_p y_k \in E(H)$ for some k with $1 < k < p$.

For convenience, let

$$X_1 = \{x_0, x_1, \dots x_{k-1}\},$$
$$X_2 = \{x_k, x_{k+1}, \dots, x_p\},$$
$$X = \{x_0, x_1, \dots, x_p\},$$
$$Y = \{y_1, y_2, \dots, y_p\}.$$

By the assumption, there is a vertex $y \in T - Y$ such that $|N(y) \cap X| \geq 4$. Since P was chosen to be a path of minimum length with the given properties, then clearly

$$|N(y) \cap X_i| \leq 2 \text{ for each } i = 1, 2,$$

for otherwise a shorter W-path with the given properties from x_0 to x_p would be immediate. Consequently, $|N(y) \cap X_1| = 2$ and $|N(y) \cap X_2| = 2$. Further, the neighbors of y in X_i would have to be consecutive for each $i = 1, 2$ for otherwise again a shorter W-path would result. Assume that

$$N(y) \cap X_1 = \{x_s, x_{s+1}\}$$

and

$$N(y) \cap X_2 = \{x_t, x_{t+1}\}.$$

Then the path

$$P^* = x_0 y_1 x_1 \dots x_s y x_{s+1} y_{s+2} \dots x_t$$

is a W-path and is shorter than P, a contradiction. □

Let $P = x_0 y_1 x_1 \dots y_p x_p$ be a W-path such that $|N(y) \cap X| \leq 3$ for every vertex $y \in T - Y$, where

$$X = \{x_0, x_1, \dots, x_p\}$$

and

$$Y = \{y_1, y_2, \dots, y_p\}.$$

Thus $|N(y) \cap (D - X)| \geq 2$ for every $y \in T - Y$. Let $D^* = (D - X) \cup Y$. It is readily seen that D^* is a 2-dominating set of G with fewer elements than D, a contradiction. □

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