The Metric Dimension and Metric Independence of a Graph

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Abstract

A vertex x of a graph G resolves two vertices u and v of G if the distance from x to u does not equal the distance from x to v. A set S of vertices of G is a resolving set for G if every two distinct vertices of G are resolved by some vertex of S. The minimum cardinality of a resolving set for G is called the metric dimension of G. The problem of finding the metric dimension of a graph is formulated as an integer programming problem. It is shown how a relaxation of this problem leads to a linear programming problem and hence to a fractional version of the metric dimension of a graph. The linear programming dual of this problem is considered and the solution to the corresponding integer programming problem is called the metric independence of the graph. It is shown that the problem of deciding whether, for a given graph G, the metric dimension of G equals its metric independence is NP-complete. Trees with equal metric dimension and metric independence are characterized. The metric independence number is established for various classes of graphs.

Key words: metric dimension, metric independence, NP-completeness, integer programming, linear programming.
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1 Introduction

Let G be a graph. A vertex x of G is said to *resolve* two vertices u and v of G if the distance $d(x, u)$ from x to u does not equal the distance $d(x, v)$ from x to v. A set

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S of vertices of G is said to be a *resolving set* for G if for every two distinct vertices u and v, there is a vertex x of S that resolves u and v. The minimum cardinality of a resolving set for G is called the *metric dimension* of G and is denoted by $dim(G)$. A minimum resolving set is called a *metric basis* for G. Harary and Melter [3] and independently Slater in [6] and [7] introduced this concept. Slater referred to the metric dimension of a graph as its location number and motivated the study of this invariant by its application to the placement of a minimum number of sonar/loran detecting devices in a network so that the position of every vertex in the network can be uniquely described in terms of its distances to the devices in the set. It was noted in [2] that the problem of finding the metric dimension of a graph is NP-hard. Khuller, Raghavachari and Rosenfeld [4] gave another construction that shows that the metric dimension of a graph is NP-hard. Their interest in this invariant was motivated by navigation of robots in a graph space. A resolving set for a graph corresponds to the presence of distinctively labeled "landmark" nodes in the graph. It is assumed that a robot navigating a graph can sense the distance to each of the landmarks and hence uniquely determine its location in the graph. They also gave approximation algorithms for this invariant and established properties of graphs with metric dimension 2. Motivated by a problem from Pharmaceutical Chemistry, this problem received renewed attention in [1]. In the same paper this problem was formulated as an integer programming problem.

We give another integer programming formulation of the problem, consider its linear programming relaxation and the corresponding duals. A large collection of graph problems can be formulated as integer programming problems. Their corresponding linear programming versions can be solved (efficiently) using linear programming methods and are often useful for establishing bounds for classical invariants and may lead to meaningful fractional versions of the problem. Their duals lead to other potentially interesting invariants. A considerable amount of work has been done on fractional graph theory and indeed an entire text has recently been devoted to the topic [5].

2 A Fractional Version of the Metric Dimension Problem and its Dual

This section is devoted to the formulation of a fractional version of the metric dimension of a graph and both its fractional and integer dual.

Let G be a connected graph of order n. Suppose V is the vertex set of G and V_p the collection of all $\binom{n}{2}$ pairs of vertices of G. Let $R(G)$ denote the bipartite graph with partite sets V and V_p such that x in V is joined to a pair $\{u, v\}$ in V_p if and only if x resolves u and v in G. We call $R(G)$ the *resolving graph* of G. Figure 1 illustrates this construction for the graph obtained from a 3-cycle by joining a pendant vertex to one of its vertices.

The smallest cardinality of a subset S of V such that the neighbourhood $N(S)$ of S in $R(G)$ is V_p is thus the metric dimension of G. Suppose $V = \{v_1, v_2, \ldots, v_n\}$

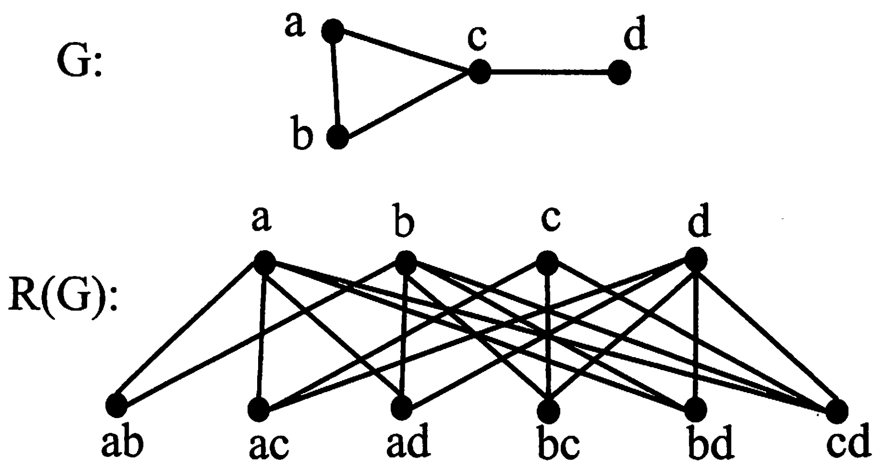

Figure 1: A Graph and its Resolving Graph

and $V_p = \{s_1, s_2, \ldots, s_{\binom{n}{2}}\}$. Let $A = (a_{ij})$ be the $\binom{n}{2} \times n$ matrix with

$$a_{ij} = \begin{cases} 1 & \text{if } s_i v_j \in E(R(G)) \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq i \leq \binom{n}{2}$ and $1 \leq j \leq n$.

The integer programming formulation of the metric dimension is given by:
minimize $\quad f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \ldots + x_n$
subject to the constraints

$$Ax \geq [1]_{\binom{n}{2}}$$

and

$$x \geq [0]_n$$

where $x = [x_1, x_2, \ldots, x_n]^T$, $[1]_k$ is the $k \times 1$ matrix all of whose entries are 1, $[0]_n$ is the $n \times 1$ matrix all of whose entries are 0 and $x_i \in \{0, 1\}$ for $1 \leq i \leq n$.

If we relax the condition that $x_i \in \{0, 1\}$ for every i and require only that $x_i \geq 0$ for all i, then we obtain the following linear programming problem:
minimize $\quad f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \ldots + x_n$
subject to the constraints

$$Ax \geq [1]_{\binom{n}{2}}$$

and

$$x \geq [0]_n.$$

In terms of the resolving graph $R(G)$ of G, solving this linear programming problem amounts to assigning nonnegative weights to the vertices in V so that for each vertex in V_p the sum of the weights in its neighbourhood is at least 1 and such that the sum of the weights of the vertices in V is as small as possible. The smallest value for f is called the *fractional dimension* of G and is denoted by $frdim(G)$.

The dual of this linear programming problem is given by:
maximize $f(y_1, y_2, \ldots, y_{\binom{n}{2}}) = y_1 + y_2 + \ldots + y_{\binom{n}{2}}$
subject to the constraints

$$A^T y \leq [1]_n$$

and

$$y \geq [0]_{\binom{n}{2}}$$

where $y = [y_1, y_2, \ldots, y_{\binom{n}{2}}]^T$.

For the resolving graph $R(G)$ of G this amounts to assigning nonnegative weights to the vertices of V_p so that for each vertex in V the sum of the weights in its neighbourhood is at most 1 and subject to this such that the sum of the weights of the vertices in V_p is as large as possible.

The corresponding integer programming problem asks for an assignment of 0's and 1's to the vertices in V_p such that the sum of the weights of the neighbours of every vertex in V is at most 1 and such that the sum of the weights of the vertices in V_p is as large as possible. This integer programming problem, which corresponds to the dual of the fractional form of the metric dimension problem, is equivalent to finding the largest collection of pairs of vertices of G no two of which are resolved by the same vertex. We call this maximum the *metric independence number* of G, denoted by $mi(G)$. A collection of pairs of vertices of G, no two of which are resolved by the same vertex is called an *independently resolved collection of pairs.* The *fractional metric independence number* of G is defined in the expected manner and is denoted by $frmi(G)$. Clearly $dim(G) \geq frdim(G)$ and $frmi(G) \geq mi(G)$. It follows from the Duality Theorem for linear programming that $frdim(G) = frmi(G)$. We thus obtain the following string of inequalities:

$$dim(G) \geq frdim(G) = frmi(G) \geq mi(G).$$

In the next section we focus on the problem of determining for which graphs G the metric dimension equals the metric independence.

3 Graphs with Equal Metric Dimension and Metric Independence Number

We begin by showing that the problem of deciding whether the metric dimension of a graph equals its metric independence number is NP-complete. Consider the following decision problem:

Metric Independence Equals Metric Dimension (MIMD)
Instance: A connected graph G.

Question: Is $dim(G) = mi(G)$?

Theorem 3.1 *MIMD is NP-complete.*

Proof. We begin by showing that MIMD is in NP. Let G be a graph of order n. Construct the resolving graph $R(G)$ of G. A nondeterministic algorithm need only guess an ordered pair (S_V, S_p) where $S_V \subseteq V$, $S_p \subseteq V_p$, $|S_V| = |S_p| \leq p/2$ and check whether the neighbourhood $N(S_V)$ of S_V in $R(G)$ is V_p and if the elements of S_p are pairwise independently resolved, i.e., whether no two of them have a common neighbour in $R(G)$. If G is a yes-instance of MIMD, then there is an ordered pair (S_V, S_p) where $S_V \subseteq V$, $S_p \subseteq V_p$, $|S_V| = |S_p| \leq p/2$ and such that in $R(G)$, $N(S_V) = V_p$ and $N(x) \cap N(y) = \emptyset$ for all $x, y \in S_p$ with $x \neq y$. If G is a no-instance of MIMD, then $dim(G) > mi(G)$ and consequently no such pair (S_V, S_p) exists. Note $R(G)$ can be constructed in $O(n^3)$ time. Checking if $N(S_V) = V_p$ and if $N(x) \cap N(y) = \emptyset$ for all $x, y \in S_p$ with $x \neq y$ can be done in $O(n^2)$ time. Hence MIMD is in NP.

To show that MIMD is NP-complete, we show that 3-SAT can be polynomially transformed to MIMD. The ideas employed here are similar to those used in [4]. Consider an instance I of 3-SAT with n variables and m clauses. Let the variables be $x_1, x_2, \ldots, x_n$ and the clauses $C_1, C_2, \ldots, C_m$. For each variable x_i construct a 6-cycle $T_i, a_i^1, b_i^1, F_i, b_i^2, a_i^2, T_i$ as shown in Figure 2. This is called the gadget for x_i. The only vertices in this gadget that will be joined to other vertices in the graph are T_i and F_i.

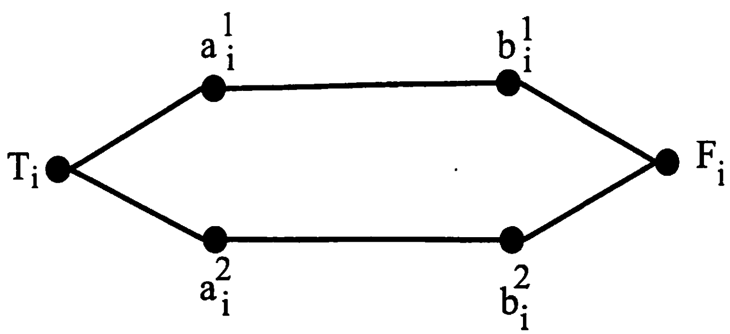

Figure 2: A 6-cycle Corresponding to a Variable

For a clause C_j, consisting of the literals y_j^1, y_j^2 and y_j^3, construct a gadget isomorphic to the star $K_{1,4}$ and labelled as in Figure 3.

The edges between the gadgets for the clauses and the gadgets for the variables are now added as follows: If x_i appears in C_j we add the edges $T_i c_j^1$, $F_i c_j^1$ and $F_i c_j^3$. If $\bar{x}_i$ appears in C_j we add the edges $T_i c_j^1$, $F_i c_j^1$ and $T_i c_j^3$. For all k such that neither x_k nor $\bar{x}_k$ appears in C_j join T_k and F_k both to each of c_j^1 and c_j^3. The resulting graph G_I has $6n + 5m$ vertices.

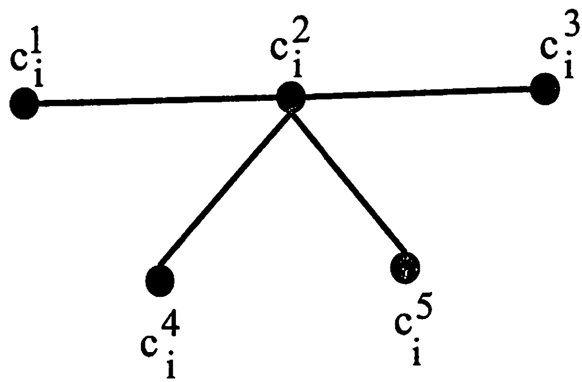

Figure 3: A $K_{1,4}$ corresponding to a Clause

We will show that I is satisfiable if and only if $dim(G_I) \geq m + n$. Observe that for each variable x_i, the pair $\{a_i^1, a_i^2\}$ is not resolved by a vertex of G_I that does not also belong to the gadget for x_i. Moreover, for each clause C_j, the pair $\{c_j^4, c_j^5\}$ is not resolved by a vertex of G_I other than c_j^4 and c_j^5. Hence $\{\{a_i^1, a_i^2\} | 1 \leq i \leq n\} \cup \{\{c_j^4, c_j^5\} | 1 \leq j \leq m\}$ is a collection of $m + n$ pairwise independently resolved pairs of vertices. So $mi(G_I) \geq n + m$. Since $dim(G_I) \geq mi(G_I)$, $dim(G_I) \geq m + n$.

We now show that if I is satisfiable, then $dim(G_I) = m + n$. Consider an assignment of truth-values to the $x_i's$ such that each clause of I is satisfied. Let B be the collection of $m + n$ vertices of G_I obtained by placing c_j^4 into B for $1 \leq j \leq m$ and for each variable x_i we place a_i^1 into B if x_i has value true and b_i^1 otherwise. We now show that B is a resolving set for G_I.

Let $\{u, v\}$ be a pair of vertices of G_I. We consider four cases.

Case 1 *One of u and v, say u belongs to the gadget for C_j for some j and v does not belongs to the same gadget.* Then c_j^4 resolves $\{u, v\}$ as $d(u, c_j^4) < d(v, c_j^4)$.

Case 2 *u belongs to the gadget for x_i and v belongs to the gadget for x_i for $i \neq j$* . Then if the element of B in $\{a_i^1, b_i^1\}$ does not resolve $\{u, v\}$, necessarily the element of B in $\{a_j^1, b_j^1\}$ resolves $\{u, v\}$.

Case 3 *u and v both belong to the gadget for x_i for some i.* We may assume $a_i^1 \in B$. (If $b_i^1 \in B$ the argument is similar.) Since $a_i^1 \in B$, a_i^1 resolves $\{u, v\}$ or one of u and v is closer to c_j^4 than the other for any j. So c_j^4 resolves $\{u, v\}$ in this case.

Case 4 *u and v both belong to a gadget for C_j for some j.* Any such pair containing c_j^4 or c_j^2 is resolved by c_j^4. If the pair contains c_j^5 and one of c_j^1 or c_j^3, then it is resolved by a_i^1 or b_i^1 for any $i(1 \leq i \leq n)$. So it remains to show that the pair $\{c_j^1, c_j^3\}$ is resolved by some vertex in B. Since some literal of C_j is assigned value true, we may assume x_i or $\bar{x}_i$ belongs to C_j for some i and has value true. Suppose $x_i \in C_j$ and has truth value true (if $\bar{x}_i$ is in C_j with truth value true, the argument is similar). Since x_i has been assigned value true, a_i^1 belongs to B.

In this case $d(c_j^1, a_i^1) < d(c_j^3, a_i^1)$. So a_i^1 resolves the pair $\{c_j^1, c_j^3\}$.

From these four cases we see that $dim(G_I) \leq m + n$. So $dim(G_I) = m + n$.

Suppose now that $dim(G_I) = m + n$. We show that I is satisfiable. Let B be any basis for G_I. Note that a basis for G_I must contain at least one of c_j^4 and c_j^5 for every j $(1 \leq j \leq m)$ and at least one vertex from $\{a_i^1, a_i^2, b_i^1, b_i^2\}$. Since $dim(G_I) = m + n$, it follows that B contains exactly one vertex from each gadget for each clause and one from the gadget for each variable. We now assign truth-values to the variables of I as follows: If a_i^1 or a_i^2 is in B set x_i to be true; otherwise x_i is false. We now show that each clause C_j contains a literal that is true. Note that c_j^1 and c_j^3 are the same distance from c_j^4 and c_j^5, neither of these two vertices resolves the pair $\{c_j^1, c_j^3\}$. Moreover, for $j \neq k$, both c_k^4 and c_k^5 are at distance 4 from both c_j^1 and c_j^3. So $\{c_j^1, c_j^3\}$ is resolved by a vertex in a gadget for x_l for some l.

If x_p is a variable that does not occur in C_j, none of the vertices $\{a_p^1, a_p^2, b_p^1, b_p^2\}$ resolves $\{c_j^1, c_j^3\}$. Thus the only vertices that can resolve the pair $\{c_j^1, c_j^3\}$ belong to a gadget for x_q where x_q is a variable that occurs in C_j. By the way G_I was constructed, a vertex in a gadget for x_q will resolve the vertices in the pair$\{c_j^1, c_j^3\}$ only if one of the following occurs.

1. x_q occurs as a positive literal in C_j and either a_q^1 or a_q^2 is in B; in this case x_q is set to true.
2. x_q occurs as a negative literal in C_j and either b_q^1 or b_q^2 is in B; in this case x_q is set to false.

In either case, x_q is assigned a truth value so that C_j contains a literal with truth value true. This completes the proof. □

Since the problem of deciding, for a given graph G, whether its metric dimension equals its metric independence is NP-complete it is natural to search for classes of graphs for which these two parameters are equal for every graph in the class. In particular we wish to determine which trees have equal metric dimension and metric independence. Before answering this question we define a few useful terms, state a formula for the metric dimension of a tree and derive an expression for the metric independence of a tree.

A vertex of degree at least 3 in a graph G will be called a *major vertex* of G. Any leaf u of G is said to be a *terminal vertex of a major vertex* v of G if $d(u, v) < d(u, w)$ for every other major vertex w of G. The *terminal degree* $ter(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of G is an *bounding major vertex of* G if it has positive terminal degree. Let $\sigma(G)$ denote the sum of the terminal degrees of the major vertices of G and let $bm(G)$ denote the number of bounding major vertices of G. The following was established in [1].

Theorem 3.2 *If G is a graph, then $dim(G) \geq \sigma(G) - bm(G)$, and if G is a tree that is not a path, then this inequality becomes an equality. Moreover, $dim(P_n) =$ 1.*

Let w be a bounding major vertex of G and suppose v is a terminal vertex of w. Then every vertex of degree 1 or 2 on the $v-w$ path is called an *exterior vertex* of w. A vertex is called *exterior* if it is an exterior vertex of some bounding major vertex of G. All other vertices are called *interior vertices* of G. It is useful to observe that if $\{u, v\}$ is a pair of vertices in a graph, then both u and v resolve this pair. Let v be a vertex of a tree T. Suppose $T_1, T_2, \ldots, T_k$ are the components of $T-v$. Then the k subgraphs induced by the vertices in $V(T_i) \cup \{v\}$ are called the *branches of T at v*. We now establish a formula for finding the metric dimension of a tree.

Theorem 3.3 *Let T be a tree that is not a path and $\mathcal{T}$ the collection of bounding major vertices of T. Then*

$$mi(T) = \sum_{v \in \mathcal{T}} \lfloor \frac{ter(v)}{2} \rfloor.$$

Moreover, $mi(P_n) = 1$.

Proof. It is not difficult to see that $mi(P_n) = 1$. Suppose then that T is a tree with at least one bounding major vertex. We first show that $mi(T) \leq \sum_{v \in \mathcal{T}} \lfloor \frac{ter(v)}{2} \rfloor$. Let $v_1, v_2, \ldots, v_k$ be the bounding major vertices of T. For each i, let $v_{i,1}, v_{i,2}, \ldots, v_{i,j_i}$ be the exterior neighbours of v_i. If j_i is even let

$$\mathcal{C}_i = \{\{v_{i,1}, v_{i,2}\}, \{v_{i,3}, v_{i,4}\}, \ldots, \{v_{i,j_i-1}, v_{i,j_i}\}\}$$

and if j_i is odd let

$$\mathcal{C}_i = \{\{v_{i,1}, v_{i,2}\}, \{v_{i,3}, v_{i,4}\}, \ldots, \{v_{i,j_i-2}, v_{i,j_i-1}\}\}.$$

Then $|\mathcal{C}_i| = \lfloor \frac{ter(v_i)}{2} \rfloor$. (If $j_i = 1$, then $\mathcal{C}_i = \emptyset$.) A vertex v resolves a pair $\{v_{i,k}, v_{i,k+1}\}$ if and only if v belongs to the component of $T - v_i$ containing $v_{i,k}$ or $v_{i,k+1}$. Hence $\bigcup_{i=1}^{k} \mathcal{C}_i$ is a collection of independently resolved pairs of vertices of T. So $mi(T) \geq \sum_{v \in \mathcal{T}} \lfloor \frac{ter(v)}{2} \rfloor$.

We now show that $mi(T) \leq \sum_{v \in \mathcal{T}} \lfloor \frac{ter(v)}{2} \rfloor$. Among all collections of $mi(T)$ pairwise independently resolved pairs of vertices of T, let $\mathcal{C}$ be one that minimizes the number of pairs of vertices that contain an interior vertex of T. We show that no element of $\mathcal{C}$ contains an interior vertex of T. Suppose that some pair $\{x, y\} \in \mathcal{C}$ is such that x or y is an interior vertex, say x. Then either there exists a branch of T at x that does not contain y but contains a vertex of degree at least 3 or every branch of T at x that does not contain y is a path. Since x is an interior vertex, there are, in the latter case, at least two branches of T at x not containing y that are paths.

Consider the first case. In this case, no vertex of a branch of T at x that does not contain y belongs to any element of $\mathcal{C}$, otherwise such a vertex would simultaneously resolve the pair to which it belongs and the pair $\{x, y\}$, which is not possible. Let w be a bounding major vertex with terminal degree at least 3 in a branch of T at x that does not contain y. Let u and v be two exterior vertices adjacent with w. Let $\mathcal{C}' = (\mathcal{C} - \{\{x, y\}\}) \cup \{\{u, v\}\}$. Then $\mathcal{C}'$ is a collection of

$mi(T)$ independently resolved pairs of vertices of T with fewer pairs of vertices containing an interior vertex of T, contrary to our choice of $\mathcal{C}$.

In the second case, x has degree at least 3 and every branch of T at x that does not contain y is a path. Again no vertex from a branch of T at x that does not contain y belongs to any element of $\mathcal{C}$ otherwise such a vertex would simultaneously resolve the pair to which it belongs as well as the pair $\{x, y\}$, which is not possible. Let u and v be two exterior vertices adjacent with x. Then $\mathcal{C}' = (\mathcal{C} - \{\{x, y\}\}) \cup \{\{u, v\}\}$ is a collection of $mi(T)$ independently resolved pairs of vertices with fewer pairs of vertices that contain an interior vertex of T, contrary to our choice of $\mathcal{C}$.

Hence we may assume that every pair $\{x, y\} \in \mathcal{C}$ is such that x and y are both exterior vertices of T. We now show that $\mathcal{C}$ can be chosen in such a way that every pair $\{x, y\}$ in $\mathcal{C}$ has the property that x and y are exterior vertices of the same bounding major vertex. Among all collections of $mi(T)$ independently resolved pairs of vertices of T, such that each pair in the collection consists of exterior vertices, let $\mathcal{C}$ be one for which the number of pairs for which both vertices in the pair are exterior vertices of the same bounding major vertex is maximized. Suppose there is a pair $\{x, y\}$ such that x is an exterior vertex of a bounding major vertex w and y is an exterior vertex of a bounding major vertex v where $v \neq w$. Since v and w are both bounding major vertices, they have degree at least 3. Suppose first that there is a branch of T at w not containing x or y that contains no vertex belonging to a pair of $\mathcal{C}$. If such a branch contains a vertex of degree at least 3, then there is a bounding major vertex u in this branch whose terminal degree is at least 2. Let $\{x', y'\}$ be two exterior vertices adjacent with u. Set $\mathcal{C}' = (\mathcal{C} - \{\{x, y\}\}) \cup \{\{x', y'\}\}$. If such a branch is a path, let x' be the vertex adjacent with w and on a $w - x$ path (possibly $x' = x$) and let y' be a vertex adjacent with w in a branch of T at w that is a path and does not contain a vertex belonging to a pair of $\mathcal{C}$. Set $\mathcal{C}' = (\mathcal{C} - \{\{x, y\}\}) \cup \{\{x', y'\}\}$. In either case $\mathcal{C}'$ is a collection of $mi(T)$ independently resolved pairs of exterior vertices of T with more pairs that are exterior vertices of the same bounding major vertex than $\mathcal{C}$. This contradicts our choice of $\mathcal{C}$. Similarly we may assume that every branch of T at v that does not contain x or y must contain a vertex that belongs to the some element of $\mathcal{C}$.

Let $u \neq x, y$ be a vertex in a branch of T at w that belongs to a pair of $\mathcal{C}$. Then u is not in the same branch of T at v as y or the same branch of T at w as x. As u resolves the pair of vertices to which it belongs but not the pair $\{x, y\}$, $d(u, x) = d(u, y)$. Since both the $u - x$ path and $u - y$ path in T contain w, $d(w, x) = d(w, y)$. Since v is on the $w - y$ path, $d(v, y) < d(w, y)$. Let $u' \neq x, y$ be a vertex in a branch of T at v that belongs to a pair of vertices of $\mathcal{C}$ Again, as u' resolves the pair to which it belongs but not the pair $\{x, y\}$, $d(u', x) = d(u', y)$. As in the previous case, $d(v, x) = d(v, y)$. But then $d(v, x) = d(v, w) + d(w, x) > d(w, x) = d(w, y) > d(v, y) = d(v, x)$, which is not possible. Hence every pair of $\mathcal{C}$ is a pair of exterior vertices of the same bounding major vertex. Therefore, $mi(T) \leq \sum_{v \in T} \lfloor \frac{ter(v)}{2} \rfloor$. □

Corollary 3.4 *Let T be a tree. Then $dim(T) = mi(T)$ if and only if the terminal* *degree of every major vertex of T is at most* 2.

Proof This follows immediately from the previous two theorems. □

4 The Metric Independence Number of Special Classes of Graphs

In the previous section we determined the metric independence number of trees. It is not difficult to see that the metric independence number of the complete graph K_n is $\lfloor n/2 \rfloor$. For the complete bipartite graph $K_{n,m}$ with $n \geq 2$ or $m \geq 2$ it is $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor$. More generally, if $G = K_{n_1,n_2,\ldots,n_k}$ is a complete k-partite graph where $n_1 \leq n_2 \leq \ldots \leq n_k$, then $mi(G) = \sum_{i=1}^{k} \lfloor \frac{n_i}{2} \rfloor$ if $n_1 \geq 2$ and $mi(G) = \lfloor \frac{l}{2} \rfloor + \sum_{i=l+1}^{k} \lfloor \frac{n_i}{2} \rfloor$ if $n_1 = n_2 = \ldots = n_l = 1$ and $n_{l+1} \geq 2$. For the cycle C_n where $n \geq 3$ it is 2 if $n = 4$ and is 1 otherwise.

It was shown in [1] that $dim(Q_n) \leq n$. In [4] it is claimed (without proof) that the metric dimension of the n-cube is at least n. The next result shows that the metric independence number of the n-cube is always 2.

Theorem 4.1 *For all* $n \geq 2$,

$$mi(Q_n) = 2.$$

Proof It is not difficult to see that $mi(Q_n) \geq 2$. Take for example the two pairs of vertices that are diametrically opposite on any 4-cycle in Q_n. Then these pairs of vertices are not resolved by the same vertex and are thus metrically independent.

To see that $mi(Q_n) \leq 2$ we use a geometric argument. Recall that Q_n can be described as the graph whose vertex set consists of all 2^n n-tuples of 0's and 1's and its edges consist of those pairs of n-tuples that differ in exactly one position. The distance between two vertices in Q_n therefore equals the number of positions in which their n-tuples differ. This is also called the Hamming distance between the two points. The vertices of Q_n also describe a collection of points in $\mathbb{R}^n$. Since the Euclidean distance between two vertices of Q_n is just the squareroot of the Hamming distance between the two points, it follows that $d_{Q_n}(u,v) = d_{Q_n}(u,w)$ if and only if $d_{\mathbb{R}^n}(u,v) = d_{\mathbb{R}^n}(u,w)$.

Suppose now that $mi(Q_n) \geq 3$. Let $\{a,b\}, \{u,v\}$ and $\{x,y\}$ be three independently resolved pairs of vertices of Q_n. Since a resolves the pair $\{a,b\}$ and not the pair $\{u,v\}$, $d(a,u) = d(a,v)$. Similarly $d(b,u) = d(b,v)$ and $d(u,a) = d(u,b)$. Hence, $d(a,u) = d(a,v) = d(b,u) = d(b,v)$. By the same argument $d(a,x) = d(a,y) = d(b,x) = d(b,y)$ and $d(x,u) = d(x,v) = d(y,u) = d(y,v)$. The set of all points in $\mathbb{R}^n$ that are the same distance from two points forms an $n-1$ dimensional hyperplane. Let A (B and C) be the set of points in $\mathbb{R}^n$ that are the same distance from the vertices in $\{a,b\}$ ($\{u,v\}$ and $\{x,y\}$, respectively). If P is a point in $\mathbb{R}^n$, then either P is in A or P resolves $\{a,b\}$ in which case P necessarily does not resolve $\{u,v\}$ and hence P belongs to B. Similarly every point of $\mathbb{R}^n$ is either in A or C. So $A \cup B = A \cup C = \mathbb{R}^n$. Hence $\mathbb{R}^n = (A \cup B) \cap (A \cup C) = A \cup (B \cap C)$. However, $B \cap C$ is only a $n-2$ dimensional hyperplane in $\mathbb{R}^n$ and A is an $n-1$

dimensional hyperplane. As it is not possible for the union of an $n-1$ dimensional hyperplane and an $n-2$ dimensional hyperplane to equal $\mathbb{R}^n$ we have a contradiction. Hence $mi(Q_n) \leq 2$. □

It remains an open problem to determine whether the metric independence number of a graph can be found in polynomial time.

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