Optimal packings and coverings of λDK_v with k-circuits *

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Abstract. Let λDK_v denote the complete directed multigraph with v vertices, where any two distinct vertices x and y are joined by λ arcs (x, y) and λ arcs (y, x). By a k-circuit we mean a directed cycle of length k. In this paper, we consider the problem of constructing maximal packings and minimal coverings of λDK_v with k-circuits. Using the leave-arcs graph of packing and the repeat-arcs graph of covering, we give a unified method for finding packings and coverings. Also, we completely solve the existence of optimal packings and coverings for $5 \leq k \leq 14$ and any λ.

1 Introduction

A *complete directed multigraph* of order v and index λ, denoted by λDK_v, is a directed graph with v vertices where any two distinct vertices x and y are joined by λ arcs (x, y) and λ arcs (y, x).

A *decomposition* of λDK_v into arc-disjoint k-circuits (directed cycles of length k) is a (v, k, λ)-Mendelsohn design, denoted by k-$MD_\lambda(v)$. A *packing (covering)* of λDK_v by k-circuits is a collection $\mathcal{D}$ of k-circuits of λDK_v such that any two distinct vertices x and y of λDK_v are linked by an arc from x to y in at most (at least) λ-circuits of $\mathcal{D}$. If no other such packing (covering) has more (fewer) circuits, the packing (covering) is said to be *maximum (minimum)* and is denoted by k-$PM_\lambda(v)$ (k-$CM_\lambda(v)$). The number of circuits in a k-$PM_\lambda(v)$(k-$CM_\lambda(v)$) is called the *packing*

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(covering) number and is denoted by $P_\lambda(v,k)$ $(C_\lambda(v,k))$. Let

$$S_\lambda(v,k) = \lfloor \frac{\lambda v(v-1)}{k} \rfloor \text{ and } T_\lambda(v,k) = \lceil \frac{\lambda v(v-1)}{k} \rceil,$$

where $v \geq k$ and $\lfloor x \rfloor$ $(\lceil x \rceil)$ denotes the greatest (least) integer y such that $y \leq x$ $(y \geq x)$. It is easy to see that

$$P_\lambda(v,k) \leq S_\lambda(v,k) \leq \frac{\lambda v(v-1)}{k} \leq T_\lambda(v,k) \leq C_\lambda(v,k).$$

The *leave-arcs graph* $LG_\lambda(\mathcal{D})$ of a packing $\mathcal{D}$ (of λDK_v by k-circuits) is a subgraph of λDK_v, its arcs are the supplement of $\mathcal{D}$ in λDK_v. The number of arcs in $LG_\lambda(\mathcal{D})$ is denoted by $|LG_\lambda(\mathcal{D})|$. When $\mathcal{D}$ is maximum (i.e., a k-$PM_\lambda(v)$), $|LG_\lambda(\mathcal{D})|$ is called the *leave-arcs number* and is denoted by $l_\lambda(v,k)$. Similarly, the *repeat-arcs graph* $RG_\lambda(\mathcal{D})$ of a covering $\mathcal{D}$ (of λDK_v by k-cricuits) is a subgraph of λDK_v, its arcs are the supplement of λDK_v in $\mathcal{D}$. When $\mathcal{D}$ is minimum (i.e., a k-$CM_\lambda(v)$), $|RG_\lambda(\mathcal{D})|$ is called the *repeat-arcs number* and is denoted by $r_\lambda(v,k)$. In some papers, the repeat-arcs graph was called the *excess graph.* It is not difficult to show the following proposition.

Proposition 1.

(1) In a $LG_\lambda(\mathcal{D})$ or a $RG_\lambda(\mathcal{D})$, the in-degree and the out-degree of each vertex are same. Thus, $|LG_\lambda(\mathcal{D})| \neq 1$ and $|RG_\lambda(\mathcal{D})| \neq 1$ for any $\mathcal{D}$.

(2) If there exists a k-$MD_\lambda(v)$, then $P_\lambda(v,k) = C_\lambda(v,k) = \frac{\lambda v(v-1)}{k}$ (and $l_\lambda(v,k) = r_\lambda(v,k) = 0$), otherwise

$$\begin{cases} l_\lambda(v,k) = \lambda v(v-1) - kP_\lambda(v,k) > 1, \\ r_\lambda(v,k) = kC_\lambda(v,k) - \lambda v(v-1) > 1. \end{cases}$$

(3)

$\lambda v(v-1) \equiv (\text{mod } k)$	$P_\lambda(v,k) \leq$	$l_\lambda(v,k) \geq$	$C_\lambda(v,k) \geq$	$r_\lambda(v,k) \geq$
1	$S_\lambda(v,k) - 1$	$k+1$	$T_\lambda(v,k)$	$k-1$
$2 \leq i \leq k-2$	$S_\lambda(v,k)$	i	$T_\lambda(v,k)$	$k-i$
$k-1$	$S_\lambda(v,k)$	$k-1$	$T_\lambda(v,k)+1$	$k+1$

In the last four columns of the table, the equality holds for most cases. For the case in which equality holds, we call the corresponding k-$PM_\lambda(v)$ (or k-$CM_\lambda(v)$) *optimal.* For an optimal k-$PM_\lambda(v)$ (or k-$CM_\lambda(v)$), the leave-arcs (or repeat-arcs) number $l_\lambda(v,k)$ (or $r_\lambda(v,k)$) will be briefly denoted by L_λ (or R_λ). Note that, if k even, L_λ and R_λ are all even for any λ. Thus, for given k, v and any λ, we have

$$L_\lambda + R_\lambda = \begin{cases} 2k & (k \text{ odd, and } L_\lambda = k-1 \text{ or } k+1) \\ k & (\text{else}) \end{cases} \qquad (*)$$

In researching packings and coverings, the main problem is to show whether there exists an optimal k-$PM_\lambda(v)$ (k-$CM_\lambda(v)$) for given k, λ and all $v \geq k$. Also, determine the value of $P_\lambda(v,k)$ ($C_\lambda(v,k)$) if there is no optimal one.

In this paper, we will provide a method to solve the existence problem for optimal packings and optimal coverings with any λ. The method will deal with the packings and coverings together, in which the leave-arcs graph, the repeat-arcs graph and their structure will play an important role. For these (leave-arcs and repeat-arcs) graphs, we will use the notations C_t (directed cycle with length t, $\langle x_1, x_2, ..., x_t \rangle$) and DK_t (complete directed graph of order t). Since the existence problems for k-$PM_\lambda(v)$ and k-$CM_\lambda(v)$ for $k = 3$ and 4 have been complete solved for any λ [1], we focus on $k \geq 5$, and particularly $5 \leq k \leq 14$.

2 General construction for $\lambda = 1$

Denote by $DK_{n_1,n_2,...,n_h}$ the *complete symmetric multipartite directed graph* with vertex set $X = X_1 \bigcup X_2 \bigcup ... \bigcup X_h$, where $X_1, X_2, ..., X_h$ are pairwise disjoint sets with $|X_i| = n_i, 1 \leq i \leq h$, and where two vertices x and y from distinct sets X_i and X_j, respectively, are linked by exactly two arcs (x,y) and (y,x). A *holey Mendelsohn design*, briefly denoted by k-$HMD(n_1, n_2, ..., n_h)$, is a trio $(X, \{X_i; 1 \leq i \leq h\}, \mathcal{A})$ where $X = X_1 \bigcup ... \bigcup X_h$ is a v-set, each X_i is a n_i-set, $1 \leq i \leq h$, and $\mathcal{A}$ is a collection of k-circuits from X, which form an arc-disjoint decomposition of $DK_{n_1,...,n_h}$. Each X_i is called a *hole* (or *group*) of this design and the multiset $\{n_1, ..., n_h\}$ can be described by its "exponential form" $1^i 2^j 3^s ...$, which denotes i occurrences of $1, j$ occurrences of 2, etc. A k-$HMD(1^{v-u}u^1)$ can be renamed as *incomplete Mendelsohn design* and is denoted by k-$IMD(v,u)$.

Lemma 1. For even k, there exists a k-$HMD(r^1 s^1)$ if and only if $r, s \geq \frac{k}{2}$ and $2rs \equiv 0 \pmod{k}$. (Refer to [2])

Lemma 2. There exists a k-$MD(k+1)$ for any integer $k \geq 2$. There exists a k-$MD(k)$ for any positive integer $k \neq 4, 6$. (refer to [3, 4])

Lemma 3. Let k be even and $n \geq k$.

1. If a k-$MD(n)$ exists then a k-$MD(n+k)$ exists too.

2. If an optimal k-$PM(n)$ (k-$CM(n)$) exists then an optimal k-$PM(n+k)$ (k-$CM(n+k)$) exists too.

Proof. Let $X\bigcup Y\bigcup\{\infty\}$ be a $(n+k)$-set, where $|X|=n-1$ and $|Y|=k$. From Lemma 1, there exists a k-$HMD((n-1)^1k^1)=(X\bigcup Y,\{X,Y\},\mathcal{A})$ since $n-1,k\geq\frac{k}{2}$ and $2k(n-1)\equiv 0 \pmod k$. And, there exists a k-$MD(k+1)=(Y\bigcup\{\infty\},\mathcal{B})$ by Lemma 2. Let the known k-$MD(n)$ (or optimal k-$PM(n)$, k-$CM(n)$) be $(X\bigcup\{\infty\},\mathcal{C})$, then $(X\bigcup Y\bigcup\{\infty\},\mathcal{A}\bigcup\mathcal{B}\bigcup\mathcal{C})$ is a k-$MD(n+k)$ (or optimal k-$PM(n+k)$, or optimal k-$CM(n+k)$). For the last conclusion, we have

$$|\mathcal{A}|+|\mathcal{B}|+|\mathcal{C}|=\frac{2k(n-1)}{k}+\frac{(k+1)k}{k}+[\frac{n(n-1)}{k}]=[\frac{(n+k)(n+k-1)}{k}],$$

where the notation $[x]$ represents x (for MD), $\lfloor x\rfloor$ (for PM) or $\lceil x\rceil$ (for CM). □

From this Lemma, using induction, it is easy to obtain the following theorem.

Theorem 1. Let $k\geq 4$ be an even integer. If there exist optimal k-$PM(v)$ and k-$CM(v)$ for each $v\in[k+2,2k-1]$, then there exist optimal k-$PM(v)$ and k-$CM(v)$ for any $v\geq k+2$.

Lemma 4. For each positive integer $m\geq 3$ and each odd $k\geq 3$, there exists a k-$HMD(k^m)$. (refer to [5])

Lemma 5. There exists a k-$IMD(k+u,u)$ for $1\leq u\leq k-1$ and $k=p^m$, pq, p^2q, p^3q, where p and q are distinct odd primes, and m is any positive integer. (refer to [5])

Lemma 6. Let k be as described in Lemma 5.

1. If there exists a k-$MD(n)$ for each admissible $n\in[k,3k-1]$ then there exists a k-$MD(v)$ for each admissible $v\geq k$ (For k-$MD(v)$, v is called *admissible* if $v\geq k$ and $k|v(v-1)$) .

2. If there exists an optimal k-$PM(n)$ (k-$CM(n)$) for each $n\in[k+2,3k-1]$ then there exists an optimal k-$PM(v)$ (k-$CM(v)$) for any $v\geq k+2$.

Proof. Let $v=mk+u$, where $m>0$ and $0\leq u\leq k-1$.

When $m=1$ or 2, the conclusion is obviously true. When $m\geq 3$, let the vertex set be $X\bigcup(\bigcup_{i\in Z_m}(\{i\}\times Z_k))$, where X is a u-set. There exists a k-$HMD(k^m)=(Z_m\times Z_k,\{\{i\}\times Z_k;i\in Z_m\},\mathcal{A})$ from Lemma 4. For each $i\in Z_m\backslash\{0\}$, there exists a k-$IMD(k+u,u)=(X\bigcup(\{i\}\times$

$Z_k), \{X, \{(i,0)\}, \{(i,1)\}, ..., \{(i,k-1)\}\}, \mathcal{B}_i)$, from Lemma 5 . Let the known k-$MD(k+u)$ (or optimal k-$PM(k+u)$, or optimal k-$CM(k+u)$) be $(X \bigcup(\{0\} \times Z_k), \mathcal{C})$, then $(X \bigcup(\bigcup_{i=0}^{m-1}(\{i\} \times Z_k)), \mathcal{A} \bigcup(\bigcup_{i=1}^{m-1} \mathcal{B}_i) \bigcup \mathcal{C})$ is a k-$MD(v)$ (or optimal k-$PM(v)$, or optimal k-$CM(v)$), by the equation

$$\begin{aligned} & |\mathcal{A}| + \sum_{i=1}^{m-1} |\mathcal{B}_i| + |\mathcal{C}| \\ & = m(m-1)k + (m-1)(k+2u-1) + \lceil \tfrac{(k+u)(k+u-1)}{k} \rceil \\ & = \lceil \tfrac{(mk+u)(mk+u-1)}{k} \rceil. \end{aligned}$$

□

From Theorem 1 and Lemma 6, we can obtain the following theorem, especially for $5 \le k \le 14$, the range discussed in this paper.

Theorem 2. Let $k \ge 3$ and, if k is odd, let k be as described in Lemma 5. If there exists an optimal k-$PM(v)$ (k-$CM(v)$) for each $v \in [k+2, 2k-1]$ when k even or for each $v \in [k+2, 3k-1]$ when k odd, then there exist an optimal k-$PM(v)$ (k-$CM(v)$) for any $v \ge k$.

Corollary 1. Let $5 \le k \le 14$. If there exist optimal k-$PM(v)$ and k-$CM(v)$ for $k+2 \le v \le 2k-1$ (k even) or $k+2 \le v \le 3k-1$ (k odd), then there exist optimal k-$PM(v)$ and k-$CM(v)$ for any $v \ge k$ with the exception of $(v,k) = (6,6)$.

Proof. The odd k, $5 \le k \le 14$, satisfy Lemma 5. Form [10], there exist k-$MD(v)$ for $5 \le k \le 14$ and $v \equiv 0, 1 (\text{mod } k)$, except for $(v,k) = (6,6)$. The result then follows from Theorem 2 and the fact that the existence of k-$MD(v)$ implies the existence of both optimal k-$PM(v)$ and k-$CM(v)$. Lastly, let us give a maximum 6-$PM(6)$ with four 6-circuits and a minimum 6-$CM(6)$ with six 6-circuits as follows.

6-$PM(6)$: 123456, 132654, 152463, 143625 with $LG = \langle 1,6,4,2 \rangle \bigcup \langle 3,5 \rangle$;

6-$CM(6)$: 123456, 132654, 152463, 145362, 164352, 136425 with $RG = \langle 1,3,6,4,5,2 \rangle$. □

3 Discussion for any λ

For a given integer $k \ge 3$, all integers are partitioned into k residue classes modulo k. Choose an integer from each class as the *representative* of this class and denote the representative of the class containing the integer x by $\{x\}_k$ (or simply $\{x\}$). Define all k representatives to be $0, 1, ..., k-1$ when k is even; or $0, 2, ..., k-1, k+1$ when k is odd. From Proposition 1, equality $(*)$ and definition, if the leave-arcs number of an optimal k-$PM(v)$ is L_1, then for any positive integer λ

the leave-arcs number of an optimal k-$PM_\lambda(v)$ is $L_\lambda = \{\lambda L_1\}_k$;

the repeat-arcs number of an optimal k-$CM_\lambda(v)$ is $R_\lambda = \{-\lambda L_1\}_k$.

Let $\overline{\lambda} = \frac{k}{gcd(k,L_1)}$, then $L_{\overline{\lambda}} = \{\overline{\lambda} L_1\}_k = 0$. $R_{\overline{\lambda}} = \{-\overline{\lambda} L_1\}_k = 0$ and $\lambda = \overline{\lambda}$ is the minimum positive integer satisfying $L_\lambda = R_\lambda = 0$. It is easy to see that

$$L_{\overline{\lambda}} = L_{\overline{\lambda}-1} - R_1 = 0 \text{ and } R_{\overline{\lambda}} = R_{\overline{\lambda}-1} - L_1 = 0;$$
$$L_\lambda = L_{\lambda'} \text{ and } R_\lambda = R_{\lambda'} \text{ for } \lambda \equiv \lambda' \pmod{\overline{\lambda}}.$$

Proposition 2. Take positive integers $k \geq 3$ and $v \geq k$. Let $\overline{\lambda} = \frac{k}{gcd(k,L_1)}$, then, for $1 \leq \lambda \leq \overline{\lambda}$ and $1 \leq i < \lambda$,

(1) when k even

$$\begin{cases} L_\lambda = L_i + L_{\lambda-i} \iff L_i < R_{\lambda-i}, \\ L_\lambda = L_i - R_{\lambda-i} \iff L_i \geq R_{\lambda-i}; \end{cases}$$
$$\begin{cases} R_\lambda = R_i + R_{\lambda-i} \iff R_i < L_{\lambda-i}, \\ R_\lambda = R_i - L_{\lambda-i} \iff R_i \geq L_{\lambda-i}. \end{cases}$$

(2) when k odd

$$\begin{cases} L_\lambda = L_i + L_{\lambda-i} \iff 2 \leq L_i \leq R_{\lambda-i} - 1 \leq k-3 \text{ or} \\ \qquad 3 \leq L_i = R_{\lambda-i} + 1 \leq k-1 \text{ or } (L_i, R_{\lambda-i}) = (2, k+1), \\ L_\lambda = L_i - R_{\lambda-i} \iff L_i \geq R_{\lambda-i} \text{ and } L_i \neq R_{\lambda-i} + 1; \end{cases}$$
$$\begin{cases} R_\lambda = R_i + R_{\lambda-i} \iff 2 \leq R_i \leq L_{\lambda-i} - 1 \leq k-3 \text{ or} \\ \qquad 3 \leq R_i = L_{\lambda-i} + 1 \leq k-1 \text{ or } (R_i, L_{\lambda-i}) = (2, k+1), \\ R_\lambda = R_i - L_{\lambda-i} \iff R_i \geq L_{\lambda-i} \text{ and } R_i \neq L_{\lambda-i} + 1; \end{cases}$$

Proof. Firstly, by the definition of $\{x\}_k$, we have:

when k even, $\{x\}_k + \{y\}_k = \{x+y\}_k \iff \{x\}_k + \{y\}_k < k$, and

$$\{x\}_k - \{y\}_k = \{x-y\}_k \iff \{x\}_k \geq \{y\}_k;$$

when k odd, $\{x\}_k + \{y\}_k = \{x+y\}_k \iff \{x\}_k + \{y\}_k \leq k-1$ or $\{x\}_k + \{y\}_k = k+1$, and

$$\{x\}_k - \{y\}_k = \{x-y\}_k \iff \{x\}_k \geq \{y\}_k \text{ and } \{x\}_k \neq \{y\}_k + 1.$$

Since $L_i = \{iL_1\}, L_{\lambda-i} = \{(\lambda-i)L_1\}$ and $L_\lambda = \{\lambda L_1\} = \{iL_1 + (\lambda-i)L_1\}$, the equivalent condition of the equality $L_\lambda = L_i + L_{\lambda-i}$ will be transform into

$$\begin{cases} L_i < k - L_{\lambda-i} = R_{\lambda-i} \text{ when } k \text{ even;} \\ L_i \le k - L_{\lambda-i} - 1 = R_{\lambda-i} - 1 (3 \le R_{\lambda-i} \le k-2) \text{ or} \\ L_i = k - L_{\lambda-i} + 1 = \begin{cases} R_{\lambda-i} + 1 & (2 \le R_{\lambda-i} \le k-2) \\ 2 & (R_{\lambda-i} = k+1) \end{cases} \quad \text{when } k \text{ odd;} \end{cases}$$

Since $L_i = \{iL_1\}, R_{\lambda-i} = \{-(\lambda-i)L_1\}$ and $L_\lambda = \{\lambda L_1\} = \{iL_1 - (-(\lambda - i)L_1)\}$, the equivalent condition of the equality $L_\lambda = L_i - R_{\lambda-i}$ will be transform into

$$\begin{cases} L_i \ge R_{\lambda-i} \text{ when } k \text{ even;} \\ L_i \ge R_{\lambda-i} \text{ and } L_i \ne R_{\lambda-i} + 1 \text{ when } k \text{ odd.} \end{cases}$$

The proof of the equivalent conditions for R_λ is similar. □

Theorem 3.

(1) When k is even, for any $\lambda \in [2, \overline{\lambda}]$, there is an $i \in [1, \lambda - 1]$ such that $L_\lambda = L_i + L_{\lambda-i}$ or $L_\lambda = L_i - R_{\lambda-i}$. And there is an $i \in [1, \lambda-1]$ such that $R_\lambda = R_i + R_{\lambda-i}$ or $R_\lambda = R_i - L_{\lambda-i}$.

(2) When k is odd, for any $\lambda \in [3, \overline{\lambda}]$, the same conclusion as (1) holds. If $\lambda = 2$, then the conclusion still holds except for the cases $L_1 = k-1$ (no representation for L_2) and $L_1 = k+1$ (no representation for R_2).

Proof. When k is even, by Proposition 2, the conclusion is obvious. In fact, we have a stronger conclusion: "$L_\lambda = L_i + L_{\lambda-i}$or $L_\lambda = L_i - R_{\lambda-i}$ (and $R_\lambda = R_i + R_{\lambda-i}$ or$R_\lambda = R_i - L_{\lambda-i}$) holds for any $\lambda \in [2, \overline{\lambda}]$ and any $i \in [1, \lambda - 1]$", because $L_i < R_{\lambda-i}$ and $L_i \ge R_{\lambda-i}$ ($R_i < L_{\lambda-i}$ and $R_i \ge L_{\lambda-i}$) are incompatible.

When k is odd, by Proposition 2, it is not difficult to verify that, for any λ and given i, both $L_\lambda = L_i + L_{\lambda-i}$ and $L_\lambda = L_i - R_{\lambda-i}$ do not hold only if

$$\begin{aligned} &\text{“}L_{\lambda-i} = k-1, R_{\lambda-i} = k+1, 3 \le L_i \le k-1\text{” or} \\ &\text{“}L_{\lambda-i} = k+1, R_{\lambda-i} = k-1, 2 \le L_i \le k-2\text{”} \qquad (**) \end{aligned}$$

By the minimization of $\overline{\lambda}$, $L_1, L_2, ..., L_{\overline{\lambda}-1}$ are mutually distinct, so the statement (**) holds for at most two values of i. Therefore, the conclusion (for L_λ) holds when $\lambda \ge 4$. Furthermore, it is impossible that $\{L_1, L_2\} = \{k-1, k+1\}$ when $k > 3$. Hence, the conclusion (for L_λ) holds when $\lambda = 3$. For the remaining case $\lambda = 2$, the statement (**) holds only if "$L_1 = k-1$

and $R_1 = k+1$". Similarly, we can prove the conclusion for R_λ. □

Remarks. The exceptional cases of Theorem 3 happen only for k odd and $v(v-1) \equiv \pm 1 \pmod{k}$. For $5 \le k \le 14$ these values are:

$k = 5, v \equiv 3 \pmod 5$, $L_1 = 6, R_1 = 4$, no representation for R_2;

$k = 7, v \equiv 3, 5 \pmod 7$, $L_1 = 6, R_1 = 8$, no representation for L_2;

$k = 11, v \equiv 4, 8 \pmod{11}$, $L_1 = 12, R_1 = 10$, no representation for R_2;

$k = 13, v \equiv 4, 10 \pmod{13}$, $L_1 = 12, R_1 = 14$, no representation for L_2.

Theorem 4. Take positive integers k, v and λ, μ. Let X be a v-set.

1. Suppose there exist both a k-$PM_\lambda(v) = (X, \mathcal{D})$ (with leave-arcs graph $LG_\lambda(\mathcal{D})$) and a k-$PM_\mu(v) = (X, \mathcal{E})$ (with leave-arcs graph $LG_\mu(\mathcal{E})$). If $|LG_\lambda(\mathcal{D})| + |LG_\mu(\mathcal{E})| = L_{\lambda+\mu}(v,k)$, then there exists an optimal k-$PM_{\lambda+\mu}(v)$ and its leave-arcs graph is just $LG_\lambda(\mathcal{D}) \bigcup LG_\mu(\mathcal{E})$.

2. Suppose there exist both a k-$CM_\lambda(v) = (X, \mathcal{D})$ (with repeat-arcs graph $RG_\lambda(\mathcal{D})$) and a k-$CM_\mu(v) = (X, \mathcal{E})$ (with repeat-arcs graph $RG_\mu(\mathcal{E})$). If $|RG_\lambda(\mathcal{D})| + |RG_\mu(\mathcal{E})| = R_{\lambda+\mu}(v,k)$, then there exists an optimal k-$CM_{\lambda+\mu}(v)$ and its repeat-arcs graph is just $RG_\lambda(\mathcal{D}) \bigcup RG_\mu(\mathcal{E})$.

3. Suppose there exist both a k-$PM_\lambda(v) = (X, \mathcal{D})$ (with leave-arcs graph $LG_\lambda(\mathcal{D})$) and a k-$CM_\mu(v) = (X, \mathcal{E})$ (with repeat-arcs graph $RG_\mu(\mathcal{E})$). If $RG_\mu(\mathcal{E}) \subset LG_\lambda(\mathcal{D})$ and $|LG_\lambda(\mathcal{D})| - |RG_\mu(\mathcal{E})| = L_{\lambda+\mu}(v,k)$, then there exists an optimal k-$PM_{\lambda+\mu}(v)$ and its leave-arcs graph is just $LG_\lambda(\mathcal{D}) \backslash RG_\mu(\mathcal{E})$.

4. Suppose there exist both a k-$CM_\lambda(v) = (X, \mathcal{D})$ (with repeat-arcs graph $RG_\lambda(\mathcal{D})$) and a k-$PM_\mu(v) = (X, \mathcal{E})$ (with leave-arcs graph $LG_\mu(\mathcal{E})$). If $LG_\mu(\mathcal{E}) \subset RG_\lambda(\mathcal{D})$ and $|RG_\lambda(\mathcal{D})| - |LG_\mu(\mathcal{E})| = R_{\lambda+\mu}(v,k)$, then there exists an optimal k-$CM_{\lambda+\mu}(v)$ and its repeat-arcs graph is just $RG_\lambda(\mathcal{D}) \backslash LG_\mu(\mathcal{E})$.

Proof. Given k and v. Here, for brevity, we denote the graphs λDK_v by λDK, k-$PM_\lambda(v)$ by PM_λ, k-$CM_\lambda(v)$ by CM_λ, $LG_\lambda(\mathcal{D})$ by LG_λ, $RG_\lambda(\mathcal{D})$ by RG_λ. Firstly, we have the following facts:

(*1) $\lambda DK = PM_\lambda \bigcup LG_\lambda = CM_\lambda \backslash RG_\lambda$;

(*2) If there are a family $\mathcal{A}$ of k-circuits in λDK and a directed subgraph G of λDK such that $\lambda DK = \mathcal{A} \bigcup G$ (or $\mathcal{A} \backslash G$) and $|G| = L_\lambda$ (or R_λ), then $\mathcal{A}$ forms an optimal k-$PM_\lambda(v)$ (or an optimal k-$CM_\lambda(v)$) and G is just its leare-arcs (or repeat-arcs) graph.

Now, let us prove the conclusions 1 and 4 (2 and 3 are similar)

1. $(\lambda+\mu)DK = (\lambda DK)\bigcup(\mu DK) = (PM_\lambda \bigcup LG_\lambda)\bigcup(PM_\mu \bigcup LG_\mu) = (PM_\lambda \bigcup PM_\mu)\bigcup(LG_\lambda \bigcup LG_\mu)$ by (*1). Furthermore, by (*2) and $|LG_\lambda \bigcup LG_\mu| = |LG_\lambda| + |LG_\mu| = L_{\lambda+\mu}$, $PM_\lambda \bigcup PM_\mu$ forms an optimal k-$PM_{\lambda+\mu}(v)$ and its leave-arcs graph is just $LG_\lambda \bigcup LG_\mu$.

4. $(\lambda+\mu)DK = (\lambda DK)\bigcup(\mu DK) = (CM_\lambda \backslash RG_\lambda)\bigcup(PM_\mu \bigcup LG_\mu) = (CM_\lambda \bigcup PM_\mu)\backslash(RG_\lambda \backslash LG_\mu)$ by (*1). Furthermore, by (*2) and $|RG_\lambda \backslash LG_\mu| = |RG_\lambda| - |LG_\mu| = R_{\lambda+\mu}$, $CM_\lambda \bigcup PM_\mu$ forms an optimal k-$CM_{\lambda+\mu}(v)$ and its repeat-arcs graph is just $RG_\lambda \backslash LG_\mu$. □

By Theorem 3, Theorem 4 and the statement before Proposition 2, in order to obtain all optimal k-$PM_\lambda(v)$ and k-$CM_\lambda(v)$ for given k ($5 \le k \le 14$), v and any λ, we only need to complete the following works:

(1) construct an optimal k-$PM(v)$ with suitable LG;

(2) construct an optimal k-$CM(v)$ with suitable RG;

(3) construct an optimal k-$PM_2(v)$ (or k-$CM_2(v)$) with suitable LG_2 (or RG_2) for those orders v pointed in the earlier Remarks following the proof of Theorem 3.

Here, a suitable (leave-arcs or repeat-arcs) graph is important for our method, especially when using Theorem 4(3)(4) (i.e., subtraction). For example, let $L_i = 4, R_{\lambda-i} = 2$ and $L_\lambda = L_i - R_{\lambda-i}$ be used. If $LG_i = C_2 \bigcup C_2$ then the subtraction can be completed, but if $LG_i = C_4$ then the subtraction can not be done since $RG_{\lambda-i} = C_2$. For this reason, even though the construction of many packings and coverings have been known (e.g., in [1, 7, 8, 9]), but most of them can not used since their (leave-arcs or repeat-arcs) graphs were unsuitable. Here, the symbol C_2 denotes a directed cycle of length 2. Below, we will use the following symbols to denote the leave-arcs graph LG and the repeat-arcs graph RG:

C_m or $\langle x_1, x_2, ..., x_m \rangle$ (a directed cycle of length m),

DK_m (a complete directed graph of order m),

$G \bigcup H$ (the union of the graphs G and H, being joint or disjoint depends on the given structure),

mG (the union of m graphs which are isomorphic to G).

Furthermore, by Theorem 2, the works (1), (2), (3) above-mentioned only need to do for $v \in [k+2, 2k-1]$ if k even or $v \in [k+2, 3k-1]$ if k odd. In the next sections, we will complete these constructions listed in the table. The order v and the nonempty L_i, R_i in the table show our task. For example, the second row for $k = 5$ means that we need to construct 5-$PM(v)$, 5-$CM(v)$ and 5-$CM_2(v)$ for $v = 8, 13$. The orders v, $v(v-1) \equiv 0 \pmod{k}$, are omited from the table since there exists a k-$MD(v)$ for these orders(see [10]).

k (odd)	v $[k+2, 3k-1]$	L_1	R_1	L_2	R_2	$\overline{\lambda}$
5	7 9 12 14	2	3			5
	8 13	6	4		3	5
7	9 13 16 20	2	5			7
	10 12 17 19	6	8	5		7
	11 18	5	2			7
9	11 17 20 26	2	7			9
	12 16 21 25	6	3			3
	13 15 22 24	3	6			3
	14 23	2	7			9
11	13 21 24 32	2	9			11
	14 20 25 31	6	5			11
	15 19 26 30	12	10		9	11
	16 18 27 29	9	2			11
	17 28	8	3			11
13	15 25 28 38	2	11			13
	16 24 29 37	6	7			13
	17 23 30 36	12	14	11		13
	18 22 31 35	7	6			13
	19 21 32 34	4	9			13
	20 33	3	10			13

k (even)	v $[k+2, 2k-1]$	L_1	R_1	$\overline{\lambda}$
6	8 11	2	4	3
8	10 15	2	6	4
	11 14	6	2	4
	12 13	4	4	2
10	12 19	2	8	5
	13 18	6	4	5
	14 17	2	8	5
12	14 23	2	10	6
	15 22	6	6	2
	17 20	8	4	3
	18 19	6	6	2
14	16 27	2	12	7
	17 26	6	8	7
	18 25	12	2	7
	19 24	6	8	7
	20 23	2	12	7

Below, for brevity, we will use some notations about differences, which were firstly introduced in [6]. Consider the construction of k-circuits from the points $Z_n \bigcup\{\infty_1, ..., \infty_t\}$, where $n \geq k$. Regard the numbers $x_1, x_2, ...$ in Z_n and $\infty_1, \infty_2, ...$ as *points*, and regard the non-zero numbers d_1, d_2, ... in $Z_n^* = Z_n \backslash \{0\}$ as *differences*. Call the ordered tuple $[d_1, ..., d_s]$ a *difference-path* (briefly, DP), if the corresponding point-tuple $(x, x+d_1, x+d_1+d_2, ..., x+d_1+...+d_s)$ consists of distinct points from Z_n, where $x \in Z_n$ and the addition is module n. If $d_1 + ... + d_s \equiv 0 \pmod n$, then the DP, $[d_1, ..., d_s]$ is called *difference-cycle* (briefly, DC). A DC $[d_1, ..., d_k]$ represents a family of k-circuits $\langle x, x+d_1, ..., x+d_1+...+d_{k-1}\rangle$ with $x \in Z_n$ if $d_1, ..., d_k$ are distinct. A DC $[d_1, ..., d_k]$, where $d_i = d_{i+s}$ $(1 \leq i \leq k-s)$ for some s, $s|k$ and $d_1, ..., d_s$ are distinct, can be denoted by $[d_1, ..., d_s]_{\frac{k}{s}}$. It is not difficult to see that $[d_1, .., d_s]_{\frac{k}{s}}$ forms a DC if and only if $gcd(d_1 + ... + d_s, n) = \frac{ns}{k}$, $\frac{k}{s}|n$ and $d_1, d_1+d_2, ..., d_1+d_2+...+d_s$ are not congruent module $\frac{ns}{k}$. A DC $[d_1, ..., d_s]_{\frac{k}{s}}$ represents $\frac{ns}{k}$ k-circuits: $\langle x, x+d_1, x+d_1+d_2, ..., x+d_1+...+d_s, ...\rangle, 0 \leq x \leq \frac{ns}{k} - 1$. Especially, when $s = 1, [d]_k$ represents $\frac{n}{k}$ k-circuits: $\langle x, x+d, x+2d, \ldots, x+(k-1)d\rangle$, $0 \leq x \leq \frac{n}{k} - 1$. And, when $s = 1$ and $n = 2k$, the two k-circuits represented by $[d]_k$ can be denoted by $[d]_k^0$ (for $x = 0$) and $[d]_k^1$ (for $x = 1$). Let $D = (d_1, ..., d_s)$ and $0 < d_1 < ... < d_s \leq \frac{n}{2}$. Define

$$A(D) = [d_1, -d_2, d_3, -d_4, ..., (-1)^{s-1}d_s],$$
$$-A(D) = [-d_1, d_2, -d_3, d_4, ..., (-1)^s d_s],$$
$$A^{-1}(D) = [(-1)^{s-1}d_s, ..., -d_4, d_3, -d_2, d_1],$$
$$-A^{-1}(D) = [(-1)^s d_s, ..., d_4, -d_3, d_2, -d_1].$$

It is easy to verify that these ordered tuples are all DP. And, if $D = (d_1, ..., d_s)$, $D' = (d'_1, ..., d'_t), 0 < d_1 < ... < d_s \leq \frac{n}{2}$, $0 < d'_1 < ... < d'_t < \frac{n}{2}$ and $d_s \neq d'_t$ then

$$[A(D), (-1)^{s+t}A^{-1}(D')] = [d_1, -d_2, ..., (-1)^{s-1}d_s, (-1)^{s-1}d'_t, ..., (-1)^{s+t}d'_1]$$

is DP too. If D is a interval $[m, m+s] = (m, m+1, .., m+s)$, $m > 0, m+s \leq \frac{n}{2}$, then the DP, $A(D) = A([m, m+s])$, can be denoted by $A[m, m+s]$. The symbols $\infty_1, \infty_2, ..., \infty_t$ and numbers in a k-circuit (bracketed by $\langle\ \rangle$) represent points. And the symbols (except $\infty_1, ..., \infty_t$ and $*$)

in a difference-cycle (bracketed by [ ]) represent differences. When a DC is $[\infty_1,...,\infty_2,...,...,\infty_t,...](t \geq 2)$, the following statement is appended after the DC: "with starters $(a_1, a_2, ..., a_t)$". The statement means that the corresponding original k-circuit is $\langle \infty_1, a_1, ..., \infty_2, a_2, ..., ..., \infty_t, a_t, ...\rangle$. When a symbol $*$ appears between ∞_i and ∞_j in a DC, it means that there is only one point (no difference) between ∞_i and ∞_j and the point is just the corresponding $*$. For example, when $n = 13$, the DC $[\infty_1, A[1,4], \infty_2, A^{-1}[2,4], \infty_3, *]$ with starters $(0, 3, -3)$ corresponds a family of k-circuits $\langle \infty_1, 0, 1, -1, 2, -2, \infty_2, 3, -6, 4, 6, \infty_3, -3\rangle$ developed modulo 13.

4 Constructions for $L_1 = 2$

Lemma 7. Given integers $k \geq 5$ and $v \geq \lceil \frac{3k-3}{2} \rceil$, $v(v-1) \equiv 2 \pmod k$. If there exists an optimal k-$PM(v) = (X \bigcup \{\infty_1, \infty_2\}, \mathcal{B})$ with the leave-arcs graph $LG = \langle \infty_1, \infty_2 \rangle$ and a k-circuit $C \in \mathcal{B}$ such that C contains

$$\begin{cases} \text{none of } \ \infty_1 \text{ and } \infty_2 & (k \geq 6) \\ \infty_1, \text{ but not } \infty_2 & (k = 5) \end{cases},$$

then there exist optimal k-$PM_\lambda(v)$ and k-$CM_\lambda(v)$ for any $\lambda \in [1, \overline{\lambda}]$, where $\overline{\lambda} = \frac{k}{gcd(2,k)}$.

Proof. Let $C = \langle c_1, c_2, ..., c_k \rangle \in \mathcal{B}$, where $c_1, ..., c_k \in X, |X| = v - 2$.

Case k even (k $\geq$ 6): Taking $x_1, x_2, ..., x_{\frac{k}{2}-3}$ from $X \backslash C$ (by $k \geq 6$ and $v \geq \frac{3k-2}{2} = \lceil \frac{3k-3}{2} \rceil$) and replacing the k-circuit C from $\mathcal{B}$ by the k-circuits $A = \langle c_1, c_2, ..., c_{\frac{k}{2}+1}, \infty_1, \infty_2, x_1, x_2, ..., x_{\frac{k}{2}-3}\rangle$ and $B = \langle c_{\frac{k}{2}+1}, ..., c_k, c_1, x_{\frac{k}{2}-3}, ..., x_2, x_1, \infty_2, \infty_1 \rangle$, we can obtain a k-$CM(v) = (X \bigcup \{\infty_1, \infty_2\}, (\mathcal{B} \backslash C) \bigcup \{A, B\})$ with the repeat-arcs graph

$$RG = \langle \infty_1, c_{\frac{k}{2}+1}\rangle \bigcup \langle \infty_2, x_1 \rangle \bigcup \langle c_1, x_{\frac{k}{2}-3}\rangle \bigcup \Big(\bigcup_{i=1}^{\frac{k}{2}-4} \langle x_i, x_{i+1}\rangle\Big).$$

Furthermore, since $\overline{\lambda} = \frac{k}{2}, L_\lambda = 2\lambda, R_\lambda = k - 2\lambda (1 \leq \lambda < \frac{k}{2})$, we have

$$L_\lambda = L_{\lambda-1} + L_1 \ \text{ and } \ R_\lambda = R_{\lambda-1} - L_1 \ \ (2 \leq \lambda < \tfrac{k}{2}).$$

By Theorem 4 and the structure of LG and RG, all optimal k-$PM_\lambda(v)$ and k-$CM_\lambda(v)$ can be obtained.

Case k odd ($k \geq 7$): Taking $x_1, x_2, ..., x_{\frac{k-7}{2}}$ from $X \backslash C$ (by $k \geq 7$ and $v \geq \frac{3k-3}{2}$) and replacing the k-circuit C from $\mathcal{B}$ by the k-circuits $A = \langle c_1, c_2, ..., c_{\frac{k+1}{2}}, \infty_1, \infty_2, c_k, x_1, x_2, ..., x_{\frac{k-7}{2}}\rangle$ and $B = \langle c_{\frac{k+1}{2}}, ..., c_k, c_1, x_{\frac{k-7}{2}},$

$\ldots, x_2, x_1, \infty_2, \infty_1\rangle$, we can obtain a k-$CM(v) = (X \bigcup\{\infty_1, \infty_2\}, (\mathcal{B}\backslash C) \bigcup\{A, B\})$ with the repeat-arcs graph

$$RG = \langle\infty_2, c_k, x_1\rangle \bigcup \langle\infty_1, c_{\frac{k+1}{2}}\rangle \bigcup \langle c_1, x_{\frac{k-7}{2}}\rangle \bigcup (\bigcup_{i=1}^{\frac{k-9}{2}} \langle x_i, x_{i+1}\rangle).$$

Case $k = 5$: Without loss generality, let $C = \langle\infty_1, c_1, c_2, c_3, c_4\rangle \in \mathcal{B}$ and $c_1, c_2, c_3, c_4 \in X$. Taking x from $X\backslash\{c_1, c_2\}$ and replacing the C from $\mathcal{B}$ by $\langle\infty_1, c_1, c_2, x, \infty_2\rangle$ and $\langle c_2, c_3, c_4, \infty_1, \infty_2\rangle$, we can get a 5-$CM(v)$ with the repeat-arcs graph $RG = \langle\infty_2, c_2, x\rangle$.

Furthermore, for odd $k \geq 5$, since $\overline{\lambda} = k$ and

$$L_\lambda = \begin{cases} 2\lambda & (1 \leq \lambda \leq \frac{k+1}{2}) \\ 2\lambda - k & (\frac{k+3}{2} \leq \lambda < k) \end{cases},$$

$$R_\lambda = \begin{cases} k - 2\lambda & (1 \leq \lambda \leq \frac{k-3}{2}) \\ 2k - 2\lambda & (\frac{k-1}{2} \leq \lambda \leq k-1) \end{cases},$$

we have

$$L_\lambda = L_{\lambda-1} + L_1 \quad (2 \leq \lambda \leq k-1, \lambda \neq \frac{k+3}{2}), \; L_{\frac{k+3}{2}} = L_{\frac{k+1}{2}} - R_1,$$

$$R_\lambda = R_{\lambda-1} - L_1 \quad (2 \leq \lambda \leq k-1, \lambda \neq \frac{k-1}{2}), \; R_{\frac{k-1}{2}} = R_{\frac{k-3}{2}} + R_1.$$

By Theorem 4 and the structure of LG and RG, all optimal k-$PM_\lambda(v)$ and k-$CM_\lambda(v)$ can be obtained, where the structure of the leave-arcs graph of k-$PM_{\frac{k+1}{2}}(v)$ need to be taken as the form

$$\langle x_1, x_3\rangle \bigcup (\bigcup_{i=1}^{\frac{k-1}{2}} \langle x_i, x_{i+1}\rangle)$$

in order to get k-$PM_{\frac{k+3}{2}}(v)$. Of course, this can be done. □

Corollary 2. For $v(v-1) \equiv k-2 \pmod{k}$, if there exists an optimal k-$CM(v)$ with $RG = C_2$ and there exists an optimal k-$PM(v)$ with $LG = \bigcup_i \langle a_i, b_i\rangle$ if k even or $LG = \langle x, y, z\rangle \bigcup (\bigcup_i \langle a_i, b_i\rangle)$ if k odd, then there exist optimal k-$PM_\lambda(v)$ and k-$CM_\lambda(v)$ for any $\lambda \in [1, \overline{\lambda}]$ where $\overline{\lambda} = \frac{k}{gcd(2,k)}$.

Proof. By the discussion in the proof of Lemma 7 and the duality of L_λ and R_λ. □

Lemma 8. For $k \geq 5$, there exist optimal k-$PM(k+2)$ and k-$PM(2k-1)$. For odd $k \geq 5$, there exist optimal k-$PM(2k+2)$ and k-$PM(3k-1)$.

Proof.

(1) k-$PM(k+2)$, $k \geq 7$.

Points: $\{\infty_1, \infty_2\} \bigcup Z_k$.

k-circuits (k odd):

$$\begin{cases} [-1]_k, [\frac{k-1}{2}]_k, [-\frac{k-1}{2}]_k, \\ [\infty_1, A[2, \frac{k-3}{2}], \infty_2, A^{-1}[1, \frac{k-3}{2}]] \text{ with starters } (0, a), \end{cases}$$

where $a = \begin{cases} \frac{k+3}{4}, & k \equiv 1 \pmod 4 \\ -\frac{k+1}{4}, & k \equiv 3 \pmod 4 \end{cases}$.

k-circuits (k even): There is $d \in Z_k^* \backslash \{\pm 1\}$ such that $gcd(d, k) = 1$, since $\phi(k) > 2$.

$$\begin{cases} [1]_k, [-1]_k, [d]_k, \\ [\infty_1, A[2, \frac{k-2}{2}], \frac{k}{2}, -A^{-1}[2, \frac{k-2}{2}]]. \end{cases}$$

where the last sequence "..., $x, x+d$, ..." is replaced by "..., $x, \infty_2, x+d$, ...".

(2) k-$PM(2k-1)$, $k \geq 7$.

Points: $\{\infty_1, \infty_2\} \bigcup Z_{2k-3}$, k-circuits are listed below.

when $k \equiv 3 \pmod 4$:

$[A[1, k-2], \frac{k-3}{2}, -(k-2)]$,

$[\infty_1, -A[1, \frac{k-5}{2}], \infty_2, -A[\frac{k-1}{2}, k-3]]$ with starters $(0, \frac{k-3}{4})$;

when $k \equiv 1 \pmod 4$:

$[A[1, k-3], -(k-2), k-3, \frac{k-1}{2}]$,

$[\infty_1, -A[1, \frac{k-3}{2}], \infty_2, -A[\frac{k+1}{2}, k-4], k-2]$ with starters $(0, \frac{k-1}{4})$;

when $k \equiv 0 \pmod 4$:

$[A[1, k-2], -1, \frac{k}{2}]$,

$[\infty_1, A[2, \frac{k-2}{2}], \infty_2, -A[\frac{k+2}{2}, k-2]]$with starters $(0, \frac{k+4}{4})$;

when $k \equiv 2 \pmod 4$:

$[A[1, k-3], k-2, -(k-3), \frac{k}{2}]$,

$[\infty_1, -A[1, \frac{k-2}{2}], \infty_2, A[\frac{k+2}{2}, k-4], -(k-2)]$ with starters $(0, -\frac{k-2}{4})$.

(3) k-$PM(2k+2)$, k odd, $k \geq 5$.

Points: $\{\infty_1, \infty_2\} \bigcup Z_{2k}$, k-circuits are listed below.

$[-2]_k, [4]_k, [-4]_k$, $[\infty_1, A([1, k] \backslash \{2, 4\})]$, $[\infty_2, -A([1, k-1] \backslash \{3, 4\}), 3]$.

(4) k-$PM(3k-1)$, k odd, $k \geq 5$.

Points: $\{\infty_1, \infty_2\} \bigcup Z_{3k-3}$, k-circuits are listed below.

$$\left.\begin{array}{l}[A[1,k-1],\frac{k-1}{2}]\\[\infty_1,-A([1,\frac{3k-5}{2}]\backslash[\frac{k-1}{2},k-2])]\\[\infty_2,-A([\frac{k+1}{2},\frac{3k-3}{2}]\backslash\{k-1\})]\end{array}\right\} k\equiv 1 \pmod 4;$$

$$\left.\begin{array}{l}[A[2,k],\frac{k-1}{2}]\\[\infty_1,A([1,\frac{3k-5}{2}]\backslash[\frac{k-1}{2},k-1]),-1]\\[\infty_2,-A([\frac{k+1}{2},\frac{3k-3}{2}]\backslash\{k\})]\end{array}\right\} k\equiv 3 \pmod 4.$$

(5)

5-$PM(5+2)$: $\langle\infty_1,4,\infty_2,0,1\rangle$ develop 5, $[-1]_5,[2]_5,[-2]_5$.
5-$PM(2\times 5-1)$: $\langle\infty_1,0,1,6,2\rangle$ and $\langle\infty_2,0,6,1,5\rangle$ develop 7.
6-$PM(6+2)$: $\langle\infty_1,0,2,\infty_2,3,1\rangle$ develop 6, $\langle 0,1,2,5,4,3\rangle+2i$, $0\le i\le 2$.
6-$PM(2\times 6-1)$: $\langle\infty_1,0,1,8,2,7\rangle$ and $\langle\infty_2,0,8,1,7,2\rangle$ develop 9. □

Theorem 5. There exist all optimal k-$PM_\lambda(v)$ and k-$CM_\lambda(v)$ for $5\le k\le 14$, $v(v-1)\equiv 2 \pmod k$ and any λ.

Proof. By Theorem 2, we only need to consider the range $v\in[k+2,2k-1]$ if k is even or $v\in[k+2,3k-1]$ if k is odd. And, in the range, all possible (v,k) satisfying $v(v-1)\equiv 2 \pmod k$ are classified into three classes:

(1) $v\ge\frac{3k-3}{2}$ and $v\equiv 2,-1 \pmod k$, $k\ge 5$.

By Lemma 7 and Lemma 8, the conclusion holds. Note that the special k-circuit C for Lemma 7 can be found in the corresponding k-$PM(v)$ given by Lemma 8 except for the 5-$PM(7)$. But, replacing the blocks $\langle\infty_1,4,\infty_2,0,1\rangle$ and $\langle 1,4,2,0,3\rangle\in[-2]_5$ from the 5-$PM(5+2)$ by $\langle\infty_1,4,2,0,1\rangle$ and $\langle 1,4,\infty_2,0,3\rangle$, we can get a new 5-$PM(7)$ with the required special 5-circuit $C=\langle\infty_1,4,2,0,1\rangle$.

(2) $v\ge\frac{3k-3}{3}$ and $v\not\equiv 2,-1 \pmod k$.

For $5\le k\le 14$, these orders are $(k,v)=(9,14),(9,23),(10,14),(10,17),(14,20)$ and $(14,23)$. By Lemma 7, we only need to construct a k-$PM(v)$ with a special k-circuit C for these orders. In the following constructions, the points are $\{\infty_1,\infty_2\}\bigcup Z_{v-2}$ and the leave-arcs graph $LG=\langle\infty_1,\infty_2\rangle$.

9-$PM(14)$: $[\infty_1,4,-5,6,\infty_2,-4,5]$
with starters $(0,1)$, $[1,-2,-3]_3,[-1,2,3]_3$.

9-$PM(23)$: $[\infty_1,A([1,10]\backslash[4,6])]$, $[\infty_2,-A([1,10]\backslash[4,6])]$,

$[4,-5,-6]_3, [-4,5,6]_3$.

10-$PM(14)$: $[\infty_1,-1,2,4,\infty_2,-4,5,6]$
with starters $(0,2)$, $[1,-2,3,-5,-3]_2$.

10-$PM(17)$: $[\infty_1,3,-6,7,\infty_2,-3,6,-7]$
with starters $(0,-1)$, $[1,-4]_5,[-1,4]_5,[2,-5]_5,[-2,5]_5$.

14-$PM(20)$: $[\infty_1,-1,3,-6,7,8,\infty_2,9,-8,7,-6,-5]$
with starters $(0,7)$, $[1,-2,3,-4,5,2,4]_2$.

14-$PM(23)$: $[\infty_1,3,-6,-8,-9,10,\infty_2,-3,6,-7,9,-10]$ with starters $(0,-1),[1,-4]_7,[-1,4]_7,[2,-5]_7,[-2,5]_7,[7,8]_7$.

(3) $v < \frac{3k-3}{2}$.

These orders are $(k,v) = (k,k+2)$, $k \geq 8$. By Lemma 8, there exists a k-$PM(v)$ for these orders. But, since $v < \frac{3k-3}{2}$, it is impossible to obtain a k-$CM(v)$ immediately from Lemma 7. Now, let us give these k-$CM(k+2)$.

8-$CM(10)$:

$\langle\infty,7,1,8,5,6,2,4\rangle + i,\ i \in Z_9^*$,
$\langle 0,4,8,3,7,2,6,5\rangle, \langle\infty,7,1,0,5,6,2,4\rangle$,
$\langle\infty,4,3,2,1,8,5,0\rangle,\langle\infty,0,8,7,6,1,5,4\rangle$.
$RG = \langle 5,0\rangle \bigcup \langle 0,\infty\rangle \bigcup \langle\infty,4\rangle$.

9-$CM(11)$:

$\langle\infty,5,6,4,0,2,9,3,8\rangle + i,\ i \in Z_{10}^*$,
$\langle 0,3,6,9,2,5,8,1,7\rangle, \langle\infty,1,4,0,2,9,3,8,7\rangle$,
$\langle\infty,7,6,5,4,3,2,1,9\rangle, \langle\infty,5,6,4,7,1,0,9,8\rangle$.
$RG = \langle\infty,7\rangle \bigcup \langle 1,7\rangle \bigcup \langle\infty,1,9\rangle$.

10-$CM(12)$:

$\langle\infty,2,4,7,0,5,3,10,9,6\rangle + i,\ i \in Z_{11}^*$,
$\langle\infty,0,1,2,3,4,5,6,7,8\rangle, \langle\infty,2,4,8,9,10,0,6,1,7\rangle$,
$\langle\infty,7,2,8,3,9,4,10,5,0\rangle, \langle\infty,8,4,7,0,5,3,10,9,6\rangle$.
$RG = \langle\infty,0\rangle \bigcup \langle\infty,7\rangle \bigcup \langle\infty,8\rangle \bigcup \langle 4,8\rangle$.

11-$CM(13)$:

$\langle\infty_1,0,\infty_2,8,1,9,7,6,2,5,10\rangle + i,\ i \in Z_{11}^*$,
$\langle\infty_1,\infty_2,0,1,2,3,4,5,6,7,8\rangle, \langle\infty_2,\infty_1,8,9,10,0,6,1,7,2,3\rangle$,
$\langle\infty_1,0,\infty_2,8,3,9,7,6,2,5,10\rangle, \langle\infty_2,3,7,2,8,1,9,4,10,5,0\rangle$,$[2]_{11}$.
$RG = \langle\infty_1,8\rangle \bigcup \langle\infty_2,0\rangle \bigcup \langle\infty_2,3\rangle \bigcup \langle 2,3,7\rangle$.

12-$CM(14)$:

$\langle\infty,4,3,1,11,6,12,8,0,2,5,9\rangle+i,\ i\in Z_{13}^*\backslash\{6\}$,
$\langle\infty,0,1,2,3,4,12,5,6,8,9,10\rangle, \langle\infty,4,3,10,11,12,0,7,1,8,2,9\rangle$,
$\langle\infty,10,3,1,11,6,12,8,0,2,5,9\rangle, \langle\infty,9,3,10,4,11,5,12,6,7,8,0\rangle$,
$\langle\infty,10,9,7,4,5,1,6,0,8,11,2\rangle$.
$RG=\langle\infty,0\rangle\bigcup\langle\infty,9\rangle\bigcup\langle\infty,10\rangle\bigcup\langle 0,8\rangle\bigcup\langle 3,10\rangle$.

13-$CM(15)$:
$\langle\infty_1,6,\infty_2,9,2,10,1,11,7,5,8,0,12\rangle+i,\ i\in Z_{13}^*$,
$\langle\infty_1,\infty_2,0,1,2,3,4,5,6,7,8,9,10\rangle$,
$\langle\infty_2,\infty_1,10,11,12,0,7,1,8,2,9,3,4\rangle$,
$\langle\infty_1,6,\infty_2,9,3,10,4,11,7,5,8,0,12\rangle$,
$\langle\infty_2,4,3,\infty_1,9,2,10,1,11,5,12,6,0\rangle$,$[2]_{13}$.
$RG=\langle\infty_1,10\rangle\bigcup\langle\infty_2,0\rangle\bigcup\langle\infty_2,4\rangle\bigcup\langle 3,4\rangle\bigcup\langle\infty_1,9,3\rangle$.

14-$CM(16)$:
$\langle\infty,11,9,12,8,13,7,14,1,0,4,10,5,2\rangle+i,\ i\in Z_{15}^*\backslash\{1,-2\}$,
$\langle\infty,0,1,2,3,4,5,6,7,8,9,10,11,12\rangle$,
$\langle\infty,12,13,14,0,8,1,9,2,10,3,6,4,7\rangle$,
$\langle\infty,9,7,4,6,11,5,12,14,13,2,8,3,0\rangle$,
$\langle\infty,11,9,12,8,13,6,14,7,0,4,10,5,2\rangle$,
$\langle\infty,12,10,13,7,14,8,0,2,1,5,11,6,3\rangle$,
$\langle\infty,7,10,6,3,11,4,12,5,13,9,14,1,0\rangle$,
$RG=\langle\infty,0\rangle\bigcup\langle\infty,12\rangle\bigcup\langle\infty,7\rangle\bigcup\langle 7,4\rangle\bigcup\langle 4,6\rangle\bigcup\langle 6,3\rangle$. □

5 Constructions for $L_1 = 6$

Lemma 9. There exist optimal k-$PM(k+3)$ and k-$PM(2k-2)$ for $k \geq 5$. And, there exist optimal k-$PM(2k+3)$ and k-$PM(3k-2)$ for odd $k \geq 5$. The leave-arcs graph of each of these packing designs is DK_3.

Proof.

(1) k-$PM(k+3)$, k odd, $k \geq 7$.

Points: $\{\infty_1,\infty_2,\infty_3\}\bigcup Z_k$.

k-circuits: $[1]_k, [-1]_k, [-2]_k, [\frac{k-1}{2}]_k, [-\frac{k-1}{2}]_k$,
$[\infty_1,*,\infty_2,A[2,\frac{k-3}{2}],\infty_3,A^{-1}[3,\frac{k-3}{2}]]$ with starters $(0,-1,a)$,
where $a=\frac{k+3}{4}$ $(k\equiv 1 \bmod 4)$ or $-\frac{k+1}{4}$ $(k \equiv 3 \bmod 4)$.

(2) k-$PM(k+3)$, k even, $k \geq 14$.

Points: $\{\infty_1, \infty_2, \infty_3\} \bigcup Z_k$.

It is easy to see that $gcd(\pm 1, k) = 1$ and

$$gcd(\pm\tfrac{k-2}{2}, k) = 1 \text{ if } k \equiv 0 \pmod 4,$$
$$gcd(\pm\tfrac{k-4}{2}, k) = 1 \text{ if } k \equiv 2 \pmod 4.$$

Since $\phi(k) > 5$ when $k \geq 14$, there is $d \in Z_k^* \backslash \{\pm 1\}$ (and $d \neq \pm\frac{k-2}{2}$ or $\pm\frac{k-4}{2}$) such that $gcd(d,k) = 1$. Define the difference sequence

$$M = (M_1, d, M_2) = \begin{cases} (A[2, \frac{k}{2} - 2], \frac{k}{2}, -A^{-1}[2, \frac{k}{2} - 2]) & \text{if } k \equiv 0 \pmod 4 \\ (A[2, \frac{k}{2} - 3], -\frac{k-2}{2}, \frac{k}{2}, \frac{k-2}{2}, -A^{-1}[2, \frac{k}{2} - 3]) & \text{if } k \equiv 2 \pmod 4 \end{cases}$$

Then a k-$PM(k+3)$ consists of these k-circuits:

$$\begin{cases} [\infty_1, M_1, \infty_2, M_2, \infty_3, *] \text{ with starters } (0, a, b), \\ [1]_k, [-1]_k, [d]_k, [c]_k, [-c]_k, \end{cases}$$

where a depends on d (the starter of M_1 is 0, so the ordered pair corresponding to the difference d in $M = (M_1, d, M_2)$ is $(a - d, a)$) and

$$b = \begin{cases} \frac{k+4}{4} & k \equiv 0 \pmod 4 \\ \frac{k+6}{4} & k \equiv 2 \pmod 4 \end{cases}, c = \begin{cases} \frac{k-2}{2} & k \equiv 0 \pmod 4 \\ \frac{k-4}{2} & k \equiv 2 \pmod 4 \end{cases}.$$

For example,

14-$PM(17)$: Taking $d = 3$, $M = (2, -3, 4, -6, 7, 6, -4, 3, -2)$ corresponds to a number-tuple (0, 2, −1, 3, −3, 4, −4, 6, −5, 7). Thus, the 14-circuits are:

$\langle \infty_1, 0, 2, -1, 3, -3, 4, -4, 6, \infty_2, -5, 7, \infty_3, 5 \rangle$ develop 14,

$[1]_{14}, [-1]_{14}, [3]_{14}, [5]_{14}, [-5]_{14}$.

16-$PM(19)$: Taking $d = -5$, $M = (2, -3, 4, -5, 6, 8, -6, 5, -4, 3, -2)$ corresponds a number-tuple $(0, 2, -1, 3, -2, 4, -4, 6, -5, 7, -6, 8)$. Thus the 16-circuits are:

$\langle \infty_1, 0, 2, -1, 3, \infty_2, -2, 4, -4, 6, -5, 7, -6, 8, \infty_3, 5 \rangle$ develop 16,

$[1]_{16}, [-1]_{16}, [-5]_{16}, [7]_{16}, [-7]_{16}$.

(3) k-$PM(2k-2)$, $k \geq 5$.

Points: $\{\infty_1, \infty_2, \infty_3\} \bigcup Z_{2k-5}$

k-circuits: $[\infty_1, A[1, k-3], (-1)^k (k-4)]$,

$[\infty_2, -A[1, k-5], \infty_3, (-1)^{k-1}(k-3)]$ with starters $(0, a)$,

where $a = \frac{k-4}{2}$ (k even) or $-\frac{k-3}{2}$ (k odd).

(4) k-$PM(2k+3)$, k odd, $k \geq 9$.

Points: $\{\infty_1, \infty_2, \infty_3\} \bigcup Z_{2k}$

k-circuits: $[\infty_2, -A([1,7]\backslash\{2,4\}), \infty_3, -A[9, k-1]]$ with starters $(0,4)$,

$[\infty_1, A([1,k]\backslash\{2,4\})]$, $[2]_k, [-2]_k, [4]_k, [-4]_k, [8]_k$.

(5) k-$PM(3k-2)$, k odd, $k \geq 5$.

Points: $\{\infty_1, \infty_2, \infty_3\} \bigcup Z_{3k-5}$.

k-circuits: $[\infty_1, A[1, k-2]], [\infty_2, -A[1, k-2]]$,

$[\infty_3, A[k-1, \frac{3k-7}{2}], \frac{3k-5}{2}, -A^{-1}[k-1, \frac{3k-7}{2}]]$.

(6) 5-$PM(5+3)$: $[\infty_2, -2, \infty_3, *]$ with starters $(2,1)$, $[\infty_1, 1, 2, -1]$.

8-$PM(8+3)$: $[\infty_1, -3, 4, \infty_2, *, \infty_3, *]$ with starters $(0,2,3)$,

$[1, -2, 3, 2]_2, [-1]_8$.

10-$PM(10+3)$: $[\infty_1, -1, 2, \infty_2, 4, \infty_3, 5]$ with starters $(0,2,3)$,

$[1, -2, 3, -4, 3]_2$.

12-$PM(12+3)$: $[\infty_1, 2, -3, \infty_2, -2, 4, \infty_3, -4, 6]$ with starters

$(0, 3, -4), [1]_{12}, [5]_{12}, [-5]_{12}, [-1, 3]_6$.

5-$PM(10+3)$: $[\infty_1, 1, -2, 3]$, $[\infty_2, -1, 2, -3]$, $[\infty_3, 4, 5, -4]$.

7-$PM(14+3)$: $[\infty_2, -1, 3, -5, \infty_3, *]$ with starters $(0,1)$,

$[\infty_1, 1, -3, 5, -6, 7]$, $[2]_7, [-2]_7, [4]_7, [-4]_7, [6]_7$. □

Theorem 6. There exists all optimal k-$PM_\lambda(v)$ and k-$CM_\lambda(v)$ for $5 \leq k \leq 14, v(v-1) \equiv 6 \pmod{k}$ and any λ.

Proof. For each k we list a table giving L_λ and R_λ for $1 \leq \lambda < \overline{\lambda} = \frac{k}{gcd(k,6)}$, and give the necessary designs.

Note that, from Lemma 9, the leave-arcs graph of k-$PM(v)$ is $LG = DK_3$ for all v.

(1) $k = 5$ $(v = 8, 13), \overline{\lambda} = 5$.

λ	1	2	3	4
	6	2	3	4
L_λ		$L_1 - R_1$	$L_1 - R_2$	$L_2 + L_2$
	4	3	2	6
R_λ			$R_1 - L_2$	$R_2 + R_2$

It is easy to see that in order to get all designs listed in the table we need only a 5-$CM(v)$ with $RG = \langle x, y\rangle \bigcup \langle y, z\rangle$ and a 5-$CM_2(v)$ with $RG = \langle u, v, w\rangle$.

Replacing 5-circuits $A = \langle \infty_1, a_1, a_2, a_3, a_4\rangle$ and $B = \langle \infty_3, b_1, b_2, b_3, b_4\rangle$ from the 5-$PM(v)$ in Lemma 9 by four 5-circuits $(a_i, b_i \in Z_{v-5}, 1 \leq i \leq 4)$:

$\langle\infty_2,\infty_1,a_1,a_2,a_3\rangle,\langle\infty_1,\infty_3,\infty_2,a_3,a_4\rangle,$
$\langle\infty_2,\infty_3,b_1,b_2,b_3\rangle,\langle\infty_3,\infty_1,\infty_2,b_3,b_4\rangle,$
we can obtain an optimal 5-$CM(v)$ with $RG = \langle a_3,\infty_2\rangle\bigcup\langle\infty_2,b_3\rangle$. The existence of A and B is obvious for $v = 13$. As for $v = 8$, we can replace two 5-circuits $\langle\infty_1,0,1,3,2\rangle$ and $\langle\infty_2,2,0,\infty_3,1\rangle$ from the original 5-$PM(8)$ by $\langle\infty_1,0,1,\infty_2,2\rangle$ and $\langle\infty_3,1,3,2,0\rangle$, we can get a new 5-$PM(8)$ with A and B.

Let the 5-$PM(v)$ and 5-$CM(v)$ obtained above are $\mathcal{A}$ and $\mathcal{B}$. Transforming $\mathcal{B}$ to $\mathcal{B}'$ under the mapping $\infty_1\leftrightarrow a_3$, $\infty_3\leftrightarrow b_3$, and replacing a 5-circuit $\langle\infty_1,c_1,c_2,c_3,c_4\rangle$ from $\mathcal{A}\bigcup\mathcal{B}'$ by two 5-circuits $\langle\infty_1,\infty_3,d,c_3,c_4\rangle$ and $\langle\infty_3,\infty_1,c_1,c_2,c_3\rangle$ $(c_1,c_2,c_3,c_4\in Z_{v-5}$ and $d\in Z_{v-5}\backslash\{c_3,c_4\})$, we obtain an optimal 5-$CM_2(v)$ with $RG = \langle\infty_3,d,c_3\rangle$.

(2) $k = 7$ $(v = 10,12,17,19)$, $\overline{\lambda} = 7$.

λ	1	2	3	4	5	6
L_λ	6	5	4 L_1-R_2	3 L_1-R_3	2 L_3-R_2	8 L_1+L_5
R_λ	8	2 R_1-L_1	3 R_1-L_2	4 R_2+R_2	5 R_2+R_3	6 R_3+R_3

It is easy to see that we only need to construct a 7-$CM(v)$ with $RG = DK_3\bigcup C_2$ and a 7-$PM_2(v)$ with $LG = C_3\bigcup C_2$. Below, let us give these constructions.

$v = 10$, Points : $X = Z_7\bigcup\{\infty_1,\infty_2,\infty_3\}$.

7-$PM(10) = (X,\mathcal{A})$ from Lemma 9, where its $LG = DK_3$ is on the set $\{\infty_2,5,6\}$.

7-$CM(10) = (X,\mathcal{B}_1\bigcup\mathcal{B}_2)$, where

$\mathcal{B}_1$: $\langle\infty_2,\infty_1,\infty_3,3,4,6,1\rangle,\langle\infty_3,2,6,3,0,4,1\rangle,$
$\langle\infty_2,4,\infty_3,\infty_1,1,3,5\rangle,\langle\infty_1,\infty_2,6,5,0,2,4\rangle;$

$\mathcal{B}_2$: $\langle\infty_1,0,\infty_2,1,\infty_3,2,3\rangle+i, i\in Z_7^*\backslash\{1,3,4\},$
$\langle\infty_1,\infty_3,\infty_2,1,5,2,3\rangle,\langle\infty_1,3,\infty_2,5,\infty_3,6,0\rangle,$
$\langle\infty_2,2,\infty_3,\infty_1,0,5,6\rangle,\langle\infty_2,\infty_3,5,6,4,2,0\rangle,$
$\langle\infty_1,4,\infty_2,5,3,1,6\rangle,[-1]_7,[3]_7.$

$RG = \langle\infty_1,\infty_3\rangle\bigcup\langle\infty_2,5,6\rangle\bigcup\langle\infty_2,6,5\rangle.$

7-$PM_2(10) = (X,\mathcal{A}\bigcup\mathcal{B}_1'\bigcup\mathcal{B}_2)$, where

$\mathcal{B}_1'$: $\langle\infty_1,1,\infty_2,6,3,0,4\rangle,\langle\infty_2,4,1,\infty_3,2,6,5\rangle,\langle0,2,4,6,1,3,5\rangle.$

$LG_2 = \langle\infty_1,\infty_2\rangle\bigcup\langle\infty_3,3,4\rangle.$

$v = 12$, Points : $X = Z_9\bigcup\{\infty_1,\infty_2,\infty_3\}$.

7-$PM(12) = (X, \mathcal{A})$ from Lemma 9, where its $LG = DK_3$ is on the set $\{\infty_2, 1, 7\}$.

7-$CM(12) = (X, \mathcal{B}_1 \bigcup \mathcal{B}_2 \bigcup \mathcal{B}_3)$

$\mathcal{B}_1 = \{B + i; i \in Z_9^* \backslash \{\pm 1\}\}$, where $B = \langle \infty_3, 8, \infty_2, 7, 3, 6, 4 \rangle$;

$\mathcal{B}_2 : \langle \infty_2, \infty_1, 1, 2, 6, \infty_3, 7 \rangle, \langle \infty_2, \infty_3, 6, 2, 5, 7, 1 \rangle$;

$\mathcal{B}_3 : \langle \infty_1, 1, 2, 6, 3, 5, 4 \rangle + i, i \in Z_9^* \backslash \{-1, \pm 2\}$,

$\langle \infty_1, \infty_3, \infty_2, 6, 3, 2, 4 \rangle, \langle \infty_2, 0, \infty_2, 1, 5, 4, 3 \rangle$,

$\langle \infty_3, \infty_1, \infty_2, 7, 3, 6, 4 \rangle, \langle \infty_2, 8, 5, \infty_3, 0, 1, 7 \rangle$,

$\langle \infty_2, 7, 6, \infty_1, 3, 4, 8 \rangle, \langle \infty_1, 8, 4, 1, 3, 5, 2 \rangle, \langle \infty_3, 8, 0, 4, 7, 5, 3 \rangle$.

$RG = \langle \infty_2, 1, 7 \rangle \bigcup \langle \infty_2, 7, 1 \rangle \bigcup \langle \infty_3, 6 \rangle$.

7-$PM_2(12) = (X, \mathcal{A} \bigcup \mathcal{B}_1' \bigcup \mathcal{B}_2' \bigcup \mathcal{B}_3)$, where

$\mathcal{B}_1' = \{B + i; i \in Z_9^* \backslash \{\pm 1, -4\}\}$;

$\mathcal{B}_2' : \langle \infty_2, \infty_1, 1, 2, 0, \infty_3, 4 \rangle, \langle \infty_2, 3, 8, 2, 5, 7, 1 \rangle$.

$LG_2 = \langle \infty_2, \infty_3, 7 \rangle \bigcup \langle 2, 6 \rangle$.

<u>$v = 17$, Points : $X = Z_{14} \bigcup \{\infty_1, \infty_2, \infty_3\}$.</u>

7-$PM(17) = (X, \mathcal{A})$ from Lemma 9, where its $LG = DK_3$ is on the set $\{\infty_3, 0, 7\}$.

7-$CM(17) = (X, \mathcal{B}_1 \bigcup \mathcal{B}_2)$

$\mathcal{B}_1 :\ \langle \infty_2, \infty_1, \infty_3, 11, 0, 7, 8 \rangle, \langle \infty_3, \infty_1, 8, 13, 4, 9, 0 \rangle$,

$\langle \infty_3, 7, 0, 9, 4, 3, 1 \rangle, \langle 0, 7, 12, 3, 8, 5, 9 \rangle, \langle 0, 8, 2, 10, 4, 12, 6 \rangle$;

$\mathcal{B}_2 :\ \langle \infty_1, 8, 5, 9, 4, 3, 1 \rangle + i$ and $\langle \infty_2, 1, \infty_3, 11, 0, 7, 8 \rangle + i, i \in Z_{14}^*$,

$\langle \infty_1, \infty_2, \infty_3, 0, 5, 10, 1 \rangle, \langle \infty_3, \infty_2, 1, 6, 11, 2, 7 \rangle$,

$\langle 1, 9, 3, 11, 5, 13, 7 \rangle, [2]_7, [-4]_7, [6]_7$.

$RG = \langle \infty_3, 0, 7 \rangle \bigcup \langle \infty_3, 7, 0 \rangle \bigcup \langle 0, 9 \rangle$.

7-$PM_2(17) = (X, \mathcal{A} \bigcup \mathcal{B}_1' \bigcup \mathcal{B}_2)$, where

$\mathcal{B}_1' :\ \langle \infty_2, \infty_1, \infty_3, 7, 12, 3, 8 \rangle, \langle \infty_3, \infty_1, 8, 13, 4, 3, 1 \rangle$,

$\langle 0, 7, 8, 2, 10, 4, 9 \rangle, \langle 0, 8, 5, 9, 4, 12, 6 \rangle$.

$LG_2 = \langle \infty_3, 11, 0 \rangle \bigcup \langle 0, 7 \rangle$.

<u>$v = 19$, Points : $X = Z_{16} \bigcup \{\infty_1, \infty_2, \infty_3\}$.</u>

7-$PM(19) = (X, \mathcal{A})$ from Lemma 9, where its $LG = DK_3$ is on the set $\{\infty_3, 0, 8\}$.

7-$CM(19) = (X, \mathcal{B}_1 \bigcup \mathcal{B}_2 \bigcup \mathcal{B}_3)$

$\mathcal{B}_1 = \{B + i; i \in Z_{16}^* \backslash \{-1\}$, where $B = \langle \infty_1, 0, 11, 1, 10, 14, 13 \rangle$;

$\mathcal{B}_2 : \langle \infty_3, 8, 0, 11, 1, 10, 14 \rangle, \langle \infty_1, 0, 8, \infty_3, 14, 13, 12 \rangle$;

$\mathcal{B}_3 : \langle 0,14,1,13,2,12,3\rangle + i$ and $\langle \infty_2,13,\infty_3,14,6,8,9\rangle + i, i \in Z_{16}^*$,
$\langle \infty_1,\infty_2,\infty_3,0,14,1,13\rangle, \langle \infty_3,\infty_1,15,10,0,9,13\rangle$,
$\langle \infty_2,\infty_1,\infty_3,14,6,8,9\rangle, \langle \infty_3,\infty_2,13,2,12,3,0\rangle$.

$RG = \langle \infty_3,0,8\rangle \bigcup \langle \infty_3,8,0\rangle \bigcup \langle \infty_3,14\rangle$.

7-$PM_2(19) = (X, \mathcal{A} \bigcup \mathcal{B}_1' \bigcup \mathcal{B}_2' \bigcup \mathcal{B}_3)$, where

$\mathcal{B}_1' = \{B + i, i \in Z_{16}^* \backslash \{-1,-5\}\}$;

$\mathcal{B}_2' : \langle \infty_1,11,1,10,14,13,12\rangle, \langle 0,11,6,12,5,9,8\rangle$.

$LG_2 = \langle \infty_1,0,8\rangle \bigcup \langle \infty_3,8\rangle$.

(3) $k = 8$ $(v = 11,14)$, $\overline{\lambda} = 4$. By Corollary 2 and Lemma 9, we only need to construct an 8-$CM(v)$ with $RG = C_2$.

8-$CM(11)$: $\langle \infty,0,6,1,9,3,5,2\rangle + i,\ i \in Z_{10}$, and $\langle 0,1,2,3,4,5,6,7\rangle$, $\langle 7,8,9,0,3,6,5,4\rangle, \langle 4,3,2,1,0,9,8,7\rangle, \langle 6,9,2,5,8,1,4,7\rangle$.
$RG = \langle 4,7\rangle$.

8-$CM(14)$: $\langle \infty_1,9,\infty_2,0,1,6,2,5\rangle + i,\ i \in Z_{11}^* \backslash \{1\}$,
$\langle \infty_3,0,10,1,9,2,8,6\rangle + i,\ i \in Z_{11}^* \backslash \{2\}$,
$\langle \infty_1,\infty_2,\infty_3,0,1,6,2,5\rangle, \langle \infty_3,\infty_1,10,1,9,2,8,6\rangle$,
$\langle \infty_1,\infty_3,\infty_2,1,2,7,3,6\rangle, \langle \infty_2,\infty_1,9,1,3,0,4,10\rangle$,
$\langle \infty_2,0,10,8,\infty_3,2,1,9\rangle$. $RG = \langle 1,9\rangle$.

(4) $k = 9$ $(v = 12,16,21,25)$, $\overline{\lambda} = 3$.

Obviously, $L_1 = R_2 = 6, L_2 = R_1 = 3$ and $L_3 = L_2 - R_1 = 0, R_3 = R_1 - L_2 = 0$. Thus, we only need to construct a 9-$CM(v)$ with $RG = C_3$.

9-$CM(12) = (Z_{12}, \mathcal{B})$, $\mathcal{B}$: $\langle 0,11,1,10,2,9,3,6,4\rangle + i,\ i \in Z_{12}$, and
$\langle 0,1,6,11,4,9,2,3,8\rangle, \langle 0,4,5,6,7,8,9,10,11\rangle, \langle 0,5,10,3,4,8,1,2,7\rangle$.
$RG = \langle 0,4,8\rangle$.

9-$CM(16) = (Z_{16}, \mathcal{B})$, $\mathcal{B}$:
$\langle 0,15,1,14,2,13,3,12,4\rangle + i,\ i \in Z_{16}$,
$\langle 0,10,15,6,4,7,14,12,13\rangle + i,\ 0 \le i \le 6$,
$\langle 1,6,11,14,5,3,8,2,7\rangle, \langle 3,4,5,6,13,11,12,15,9\rangle$,
$\langle 4,9,14,8,13,7,12,5,10\rangle, \langle 5,13,12,6,7,8,9,10,11\rangle$. $RG = \langle 5,13,12\rangle$.

9-$CM(21) = (Z_{18} \bigcup \{\infty_1,\infty_2,\infty_3\}, \mathcal{B})$, $\mathcal{B}$:
$\langle \infty_1,5,\infty_2,0,15,16,3,10,4\rangle + i,\ i \in Z_{18}^* \backslash \{1\}$,
$\langle \infty_3,0,17,2,15,3,14,4,13\rangle + i,\ i \in Z_{18}^* \backslash \{7\}$,
$\langle \infty_1,\infty_2,\infty_3,0,15,16,3,10,4\rangle, \langle \infty_3,\infty_1,6,9,4,10,3,11,2\rangle$,
$\langle \infty_1,\infty_3,\infty_2,1,16,17,4,11,5\rangle, \langle \infty_2,\infty_1,3,14,4,13,\infty_3,7,6\rangle$,

$\langle\infty_3,\infty_1,5,\infty_2,0,17,2,15,3\rangle,[2]_9,[-2]_9,[4]_9,[-4]_9,[-8]_9$.

$RG=\langle\infty_3,\infty_1,3\rangle$.

9-$CM(25)=(Z_{22}\bigcup\{\infty_1,\infty_2,\infty_3\},\mathcal{B})$, $\mathcal{B}$:

$\langle 0,21,1,20,2,19,3,18,4\rangle+i,\ i\in Z_{22}$,

$\langle\infty_1,5,\infty_2,21,12,10,11,14,2\rangle+i, i\in Z_{22}^*\backslash\{1\}$

$\langle\infty_3,21,4,20,5,19,6,18,7\rangle+i, i\in Z_{22}^*\backslash\{1\}$,

$\langle\infty_1,\infty_2,\infty_3,21,12,10,11,14,2\rangle,\langle\infty_3,\infty_1,5,21,6,20,7,19,8\rangle$,

$\langle\infty_2,\infty_1,19,6,18,7,\infty_3,0,5\rangle$, $\langle\infty_1,\infty_3,\infty_2,0,13,11,12,15,3\rangle$,

$\langle\infty_3,\infty_1,6,\infty_2,21,4,20,5,19\rangle$. $RG=\langle\infty_3,\infty_1,19\rangle$.

(5) $k=10$ $(v=13,18)$, $\overline{\lambda}=5$.

λ	1	2	3	4
L_λ	6	2 L_1-R_1	8 L_1+L_2	4 L_2+L_2
R_λ	4	8 R_1+R_1	2 R_1-L_2	6 R_1+R_3

We only need to construct a 10-$CM(v)$ with $RG=2C_2=\langle x,y\rangle\bigcup\langle y,z\rangle$.

10-$CM(13)=(Z_{12}\bigcup\{\infty\},\mathcal{B})$, $\mathcal{B}$:

$\langle\infty,9,0,11,3,1,6,2,8,5\rangle+i,\ i\in Z_{12}^*$, and

$\langle 0,1,2,3,4,5,6,7,8,10\rangle$, $\langle 8,9,10,11,0,7,2,4,6,1\rangle$,

$\langle 8,3,10,5,0,2,9,11,1,4\rangle$,$\langle 1,3,5,7,9,0,11,6,8,4\rangle$,

$\langle\infty,9,4,11,3,1,6,2,8,5\rangle$. $RG=\langle 1,4\rangle\bigcup\langle 4,8\rangle$.

10-$CM(18)=(Z_{10}\bigcup\{a,b,c,d\},\mathcal{B})$, $\mathcal{B}$:

$\langle a,2,b,9,c,0,d,13,12,5\rangle+i,\ i\in Z_{14}^*$,

$\langle 0,4,6,2,7,1,13,8,5,11\rangle+i,\ i\in Z_{14}$,

$\langle a,b,d,c,0,1,2,3,4,5\rangle,\langle d,a,c,b,9,10,11,12,13,0\rangle$,

$\langle c,d,b,a,2,3,6,7,8,9\rangle$, $\langle b,c,a,d,13,12,5,6,3,2\rangle$.

$RG=\langle 2,3\rangle\bigcup\langle 3,6\rangle$.

(6) $k=11$ $(v=14,20,25,31)$, $\overline{\lambda}=11$.

λ	1	2	3	4	5
L_λ	6	12 L_1+L_1	7 L_2-R_1	2 L_1-R_3	8 L_1+L_4
R_λ	5	10 R_1+R_1	4 R_2-L_1	9 R_1+R_3	3 R_1-L_4

λ	6	7	8	9	10
	3	9	4	10	5
L_λ	$L_1 - R_5$	$L_1 + L_6$	$L_4 + L_4$	$L_1 + L_8$	$L_4 + L_6$
	8	2	7	12	6
R_λ	$R_1 + R_5$	$R_1 - L_6$	$R_1 + R_7$	$R_1 + R_8$	$R_5 + R_5$

It is not difficult to see that we only need to construct an 11-$CM(v)$ with $RG = C_3 \bigcup C_2$.

11-$CM(14) = (Z_{11} \bigcup \{\infty_1, \infty_2, \infty_3\}, \mathcal{B})$, $\mathcal{B}$:
$\langle \infty_1, 2, \infty_2, 9, \infty_3, 3, 10, 4, 8, 5, 7\rangle + i, i \in Z_{11}^*$, and
$\langle \infty_1, \infty_2, \infty_3, 3, 6, 9, 10, 0, 1, 4, 7\rangle$, $\langle \infty_1, \infty_3, 4, 8, 5, 7, 10, 2, \infty_2, 9, 1\rangle$,
$\langle \infty_3, \infty_2, \infty_1, 2, 5, 8, 0, 3, 10, 4, 1\rangle$,$\langle \infty_3, \infty_1, 1, 2, 3, 4, 5, 6, 7, 8, 9\rangle$,
$[-1]_{11}, [-2]_{11}, [-5]_{11}$. $RG = \langle \infty_3, 4, 1\rangle \bigcup \langle \infty_1, 1\rangle$.

11-$CM(20) = (Z_{17} \bigcup \{\infty_1, \infty_2, \infty_3\}, \mathcal{B})$, $\mathcal{B}$:
$\langle \infty_1, 10, 11, 2, 1, 3, 0, 4, 16, 5, 15\rangle + i$ and
$\langle \infty_2, 6, \infty_3, 2, 0, 3, 16, 4, 15, 5, 14\rangle + i, i \in Z_{17}^* \backslash \{1\}$,
$\langle \infty_1, \infty_2, \infty_3, 2, 1, 3, 0, 4, 16, 5, 15\rangle$, $\langle \infty_3, \infty_1, 11, 12, 3, 2, 4, 1, 5, 0, 6\rangle$,
$\langle \infty_2, \infty_1, \infty_3, 3, 1, 4, 0, 5, 16, 6, 15\rangle$, $\langle \infty_3, \infty_2, 6, 16, \infty_1, 10, 11, 2, 0, 1, 3\rangle$,
$\langle \infty_2, 7, \infty_3, 1, 0, 3, 16, 4, 15, 5, 14\rangle$. $RG = \langle \infty_3, 1, 3\rangle \bigcup \langle 0, 1\rangle$.

11-$CM(25) = (Z_{22} \bigcup \{\infty_1, \infty_2, \infty_3\}, \mathcal{B})$, $\mathcal{B}$:
$\langle \infty_1, 20, 5, 2, 16, 3, 15, 14, 17, 0, 1\rangle + i, i \in Z_{22}^* \backslash \{2\}$, and
$\langle \infty_2, 1, \infty_3, 2, 19, 3, 18, 4, 17, 5, 16\rangle + i, i \in Z_{22}^* \backslash \{1\}$.
$\langle \infty_1, \infty_2, \infty_3, 2, 16, 3, 15, 14, 17, 0, 1\rangle$, $\langle \infty_1, 0, 7, 4, 18, 5, 17, 16, \infty_2, 2, 3\rangle$,
$\langle \infty_2, \infty_1, \infty_3, 3, 20, 4, 19, 5, 18, 6, 17\rangle$, $\langle \infty_3, \infty_2, 0, 3, 18, 4, 17, 5, 16, 19, 2\rangle$,
$\langle \infty_2, 1, \infty_3, \infty_1, 20, 5, 2, 19, 3, 0, 4\rangle$, $[2]_{11}, [-2]_{11}, [4]_{11}, [-4]_{11}, [-6]_{11}$.
$RG = \langle \infty_2, 0, 4\rangle \bigcup \langle 0, 3\rangle$.

11-$CM(31) = (Z_{28} \bigcup \{\infty_1, \infty_2, \infty_3\}, \mathcal{B})$, $\mathcal{B}$:
$\langle 0, 26, 1, 25, 2, 24, 3, 23, 4, 22, 5\rangle + i, i \in Z_{28}$, and
$\langle \infty_1, , 11, 24, 2, 16, 17, 21, 23, 22, 10, 7\rangle + i, i \in Z_{28}^* \backslash \{3\}$.
$\langle \infty_2, 26, \infty_3, 2, 23, 3, 22, 4, 21, 5, 20\rangle + i, i \in Z_{28}^* \backslash \{1\}$,
$\langle \infty_1, \infty_2, \infty_3, 2, 16, 17, 21, 23, 22, 10, 7\rangle$,
$\langle \infty_3, \infty_2, 0, 3, 22, 4, 21, 5, 20, 24, 26\rangle$,
$\langle \infty_2, \infty_1, \infty_3, 3, 24, 4, 23, 5, 22, 6, 21\rangle$,
$\langle \infty_1, 14, 27, 5, 19, 20, \infty_2, 26, 25, 13, 10\rangle$,
$\langle \infty_2, 27, \infty_3, \infty_1, 11, 24, 2, 23, 3, 0, 4\rangle$. $RG = \langle \infty_2, 0, 4\rangle \bigcup \langle 0, 3\rangle$.

(7) $k = 12(v = 15, 22$ and 18, 19), $\overline{\lambda} = 2$.

Obviously, $L_1 = R_1 = 6$ and $L_2 = R_2 = 0$. We need to construct a 12-$CM(v)$ with $RG = DK_3$ for $v = 15, 22$. And, for $v = 18, 19$, we need to construct a 12-$PM(v)$ (with LG) and a 12-$CM(v)$ (with RG) such that the LG and the RG are isomorphic. From [8], an optimal 12-$PM(v)$ with $LG = C_6$ for $v = 18, 19$ can be found. And, from [7], an optimal 12-$CM(19)$ with $RG = C_6$ can be found. Here, we give an optimal 12-$CM(18) = (Z_{18}, \mathcal{B})$ with $RG = C_6$, where $\mathcal{B}$:

$\langle 0,15,3,17,6,1,9,10,2,13,7,16\rangle$ develop 18,

$\langle 0,17,3,1,6,5,9,7,12,11,15,13\rangle + i \mod 18,\ 1 \le i \le 5$,

$\langle 0,3,6,9,12,11,15,13,16,1,4,7\rangle$, $\langle 0,1,3,4,14,17,2,5,9,7,12,15\rangle$,

$\langle 0,17,3,1,6,5,8,11,14,7,10,13\rangle$. $RG = \langle 0,1,3,4,14,7\rangle$.

(8) $k = 13$ ($v = 16,\ 24,\ 29,\ 37$), $\overline{\lambda} = 13$.

λ	1	2	3	4	5	6
	6	12	5	11	4	10
L_λ		$L_1 + L_1$	$L_2 - R_1$	$L_1 + L_3$	$L_1 - R_4$	$L_3 + L_3$
	7	14	8	2	9	3
R_λ		$R_1 + R_1$	$R_2 - L_1$	$R_1 - L_3$	$R_1 + R_4$	$R_3 - L_3$

λ	7	8	9	10	11	12
	3	9	2	8	14	7
L_λ	$L_1 - R_6$	$L_1 + L_7$	$L_5 - R_4$	$L_5 + L_5$	$L_1 + L_{10}$	$L_5 + L_7$
	10	4	11	5	12	6
R_λ	$R_1 + R_6$	$L_4 + L_4$	$R_1 + R_8$	$R_4 + R_6$	$R_1 + R_{10}$	$R_6 + R_6$

It is easy to verify that, from Lemma 9 ($v = k+3, 2k-2, 2k+3, 3k-2$), we only need to construct a 13-$CM(v)$ with $RG = \langle x,y,z\rangle \bigcup \langle u,v\rangle \bigcup \langle v,w\rangle$.

13-$CM(16) = (\{\infty_1,\infty_2,\infty_3\} \bigcup Z_{13}, \mathcal{B})$, $\mathcal{B}$:

$\langle \infty_1,0,\infty_2,12,1,11,2,10,\infty_3,4,9,5,8\rangle + i,\ i \in Z_{13}^*$,

$\langle \infty_1,\infty_2,\infty_3,4,9,5,3,1,12,10,8,7,6\rangle$,

$\langle \infty_1,\infty_3,\infty_2,3,6,4,2,0,11,9,7,5,8\rangle$,

$\langle \infty_3,\infty_2,\infty_1,6,5,4,3,2,1,0,12,11,10\rangle$,

$\langle \infty_3,\infty_1,0,\infty_2,12,1,11,2,10,9,8,6,3\rangle$,

$[1]_{13}, [6]_{13}, [-6]_{13}$. $RG = \langle \infty_3,\infty_2,3\rangle \bigcup \langle 3,6\rangle \bigcup \langle 6,\infty_1\rangle$.

13-$CM(29) = (\{\infty_1,\infty_2,\infty_3\} \bigcup Z_{26}, \mathcal{B})$, $\mathcal{B}$:

$\langle \infty_1,0,1,24,3,23,4,22,5,21,6,20,7\rangle + i,\ i \in Z_{26}$,

$\langle \infty_2,0,25,22,17,23,16,\infty_3,4,21,5,20,6\rangle + i,\ i \in Z_{26}^*$,

$\langle\infty_1,\infty_2,\infty_3,4,21,5,20,18,16,14,12,10,8\rangle$,
$\langle\infty_2,\infty_1,9,7,5,3,1,25,10,24,22,20,6\rangle$,
$\langle\infty_1,\infty_3,\infty_2,0,25,23,21,19,17,15,13,11,9\rangle$,
$\langle\infty_3,\infty_1,8,6,4,2,0,24,25,22,17,23,16\rangle$.
$[2]_{13},[4]_{13},[-4]_{13},[8]_{13}$. $RG=\langle 10,24,25\rangle\bigcup\langle 9,\infty_1\rangle\bigcup\langle\infty_1,8\rangle$.

13-$CM(24)=(\{\infty_1,\infty_2,\infty_3\}\bigcup Z_{21},\mathcal{B})$, $\mathcal{B}$:
$\langle\infty_2,2,1,3,0,4,20,5,19,6,\infty_3,18,7\rangle+i,\ i\in Z_{21}^*$,
$\langle\infty_1,0,1,20,2,19,3,18,4,17,5,16,7\rangle+i,\ i\in Z_{21}^*\backslash\{10\}$,
$\langle\infty_2,\infty_3,\infty_1,10,11,9,12,8,13,7,14,6,1\rangle$,
$\langle\infty_2,\infty_1,0,1,20,5,19,3,4,6,\infty_3,18,7\rangle$,
$\langle\infty_1,\infty_3,\infty_2,2,1,6,3,18,4,17,5,16,7\rangle$,
$\langle\infty_1,\infty_2,1,3,0,4,20,2,19,6,15,5,17\rangle$.
$RG=\langle\infty_2,1\rangle\bigcup\langle 1,6\rangle\bigcup\langle 6,3,4\rangle$.

13-$CM(37)=(\{\infty_1,\infty_2,\infty_3\}\bigcup Z_{34},\mathcal{B})$, $\mathcal{B}$:
$\langle\infty_1,0,1,33,2,32,3,31,4,30,5,29,6\rangle+i,\ i\in Z_{34}^*$,
$\langle\infty_2,17,16,18,15,19,14,20,13,21,12,22,11\rangle+i,\ i\in Z_{34}^*$,
$\langle\infty_3,0,12,33,13,32,14,31,15,30,16,29,17\rangle+i,\ i\in Z_{34}^*$,
$\langle\infty_1,\infty_2,\infty_3,0,12,33,21,31,4,30,5,29,6\rangle$,
$\langle\infty_2,\infty_1,0,1,33,2,32,3,31,21,12,22,11\rangle$,
$\langle\infty_1,\infty_3,\infty_2,17,16,18,15,19,14,20,13,21,33\rangle$,
$\langle\infty_3,\infty_1,0,33,13,32,14,31,15,30,16,29,17\rangle$.
$RG=\langle\infty_1,0,33\rangle\bigcup\langle 21,33\rangle\bigcup\langle 21,31\rangle$.

(9) $k=14$ ($v=17,26$ and $19,24$), $\overline{\lambda}=7$.

λ	1	2	3	4	5	6
	6	12	4	10	2	8
L_λ		L_1+L_1	L_1-R_2	L_1+L_3	L_3-R_2	L_3+L_3
	8	2	10	4	12	6
R_λ		R_1-L_1	R_1+R_2	R_1-L_3	R_1+R_4	R_2+R_4

Case $v=17,26$ By Lemma 9 ($v=k+3$ and $2k-2$), it is not difficult to see that we only need to construct a 14-$CM(v)$ with $RG=DK_3\bigcup C_2$.

14-$CM(17)=(Z_{14}\bigcup\{\infty_1,\infty_2,\infty_3\},\mathcal{B})$, $\mathcal{B}$:
$\langle\infty_1,0,2,13,3,11,4,10,6,\infty_2,9,7,\infty_3,5\rangle+i,\ i\in Z_{14}^*\backslash\{3,8\}$,
$\langle\infty_3,\infty_1,\infty_2,9,12,1,4,7,10,13,2,5,8,11\rangle$,
$\langle\infty_2,\infty_3,5,\infty_1,9,7,8,10,11,0,1,2,3,6\rangle$,

$\langle\infty_3,\infty_2,\infty_1,8,9,0,2,13,3,11,4,10,6,7\rangle$,
$\langle\infty_1,\infty_3,11,12,13,0,\infty_2,3,4,5,2,6,9,8\rangle$,
$\langle\infty_1,0,5,10,1,6,11,2,7,12,3,8,13,9\rangle$,
$\langle\infty_1,3,5,6,0,7,13,4,9,\infty_2,12,10,\infty_3,8\rangle$,
$\langle\infty_1,8,9,10,7,11,5,12,4,0,3,1,\infty_3,13\rangle$, $[-1]_{14},[-5]_{14}$.
$RG=\langle\infty_3,11\rangle\bigcup\langle\infty_1,8,9\rangle\bigcup\langle\infty_1,9,8\rangle$.

14-$CM(26)=(Z_{23}\bigcup\{\infty_1,\infty_2,\infty_3\},\mathcal{B})$, $\mathcal{B}$:
$\langle\infty_1,1,2,0,3,22,4,21,5,20,6,19,7,17\rangle+i,i\in Z_{23}^*\backslash\{5\}$,
$\langle\infty_2,0,22,1,21,2,20,3,19,4,18,\infty_3,5,17\rangle+i,i\in Z_{23}^*\backslash\{-1,3,7\}$,
$\langle\infty_1,\infty_2,\infty_3,5,8,4,9,3,10,2,11,1,12,22\rangle$,
$\langle\infty_1,\infty_3,\infty_2,0,3,22,4,21,5,20,6,19,7,17\rangle$,
$\langle\infty_3,\infty_1,6,16,0,22,1,21,2,20,3,19,4,18\rangle$,
$\langle\infty_2,\infty_1,1,2,\infty_3,4,16,12,0,6,22,7,5,17\rangle$,
$\langle\infty_2,7,6,8,5,9,4,10,3,11,2,0,12,1\rangle$,
$\langle\infty_2,3,2,4,1,5,0,16,6,7,21,\infty_3,8,20\rangle$,
$\langle\infty_2,22,21,0,20,1,19,2,18,3,17,\infty_3,12,16\rangle$.
$RG=\langle 6,16\rangle\bigcup\langle 0,12,16\rangle\bigcup\langle 0,16,12\rangle$.

Case $v=19,24$ We can verify that, from the table above, it is enough to construct a 14-$PM(v)$ (with $LG=\langle a,b\rangle\bigcup\langle b,c\rangle\bigcup\langle c,d\rangle$) and 14-$CM(v)$ (with $RG=\langle x,y\rangle\bigcup\langle y,z\rangle\bigcup\langle z,u\rangle\bigcup\langle u,w\rangle$).

$v=19$, Points :$\{\infty\}\bigcup Z_{18}$.

$\mathcal{A}=\{\langle\infty,14,17,13,0,12,1,11,2,10,3,9,4,8\rangle+i,i\in A\}$,
$\mathcal{B}=\{\langle\infty,14,17,13,0,12,1,11,10,3,5,7,9,8\rangle$,
$\langle 0,2,4,6,8,9,10,11,12,13,14,15,16,17\rangle$,
$\langle 0,17,16,14,13,12,11,9,4,8,7,6,5,2\rangle$,
$\langle 0,16,13,10,.8,5,4,2,17,15,12,9,6,3\rangle$,
$\langle 0,15,13,11,2,10,9,7,4,3,1,17,14,16\rangle$,
$\langle 1,3,4,5,6,7,8,10,12,14,11,13,15,17\rangle$,
$\langle 1,16,15,14,12,10,7,5,3,9,11,8,6,4\rangle\}$.

14-$PM(19)$: $\mathcal{A}(A=Z_{18}^*)$ and $\mathcal{B}$. $LG=\langle 0,1\rangle\bigcup\langle 1,2\rangle\bigcup\langle 2,3\rangle$.
14-$CM(19)$: $\mathcal{A}(A=Z_{18}^*\backslash\{-4,-6\}),\mathcal{B}$ and
$\langle\infty,8,11,9,12,6,13,5,14,4,15,3,16,2\rangle$,
$\langle\infty,10,13,9,14,8,11,7,12,3,2,1,0,4\rangle$,
$\langle 0,1,2,3,12,9,11,8,15,7,16,6,17,5\rangle$.

$RG = \langle 3,12\rangle \bigcup \langle 12,9\rangle \bigcup \langle 9,11\rangle \bigcup \langle 11,8\rangle$.

$v = 24$, Points :$\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \bigcup Z_{19}$.

$\mathcal{A} = \{\langle \infty_1, 0, \infty_2, 17, \infty_3, 3, \infty_5, 4, 11, 5, 10, 6, 9, 7\rangle + i; i \in A\}$,

$\mathcal{B} = \{\langle \infty_4, 10, 0, 18, 1, 17, 2, 16, 3, 15, 4, 14, 6, 7\rangle + i, i \in B\}$,

$\mathcal{C} = \{\langle \infty_1, \infty_4, \infty_2, \infty_5, \infty_3, 3, 17, 4, 11, 5, 10, 6, 9, 7\rangle$,

$\langle \infty_4, \infty_1, \infty_3, \infty_5, \infty_2, 17, 2, 18, 3, 15, 4, 14, 6, 7\rangle$,

$\langle \infty_3, \infty_1, \infty_5, 4, 16, 5, 15, 7, 8, \infty_4, 11, 1, \infty_2, 0\rangle$,

$\langle \infty_2, \infty_4, \infty_5, 5, 12, 6, 11, 7, 10, 8, \infty_1, 1, 3, 0\rangle$,

$\langle \infty_5, \infty_1, 2, 16, 8, 9, \infty_4, 10, 0, 18, 1, 17, \infty_3, 4\rangle$,

$\langle \infty_5, \infty_4, 12, 2, 1, 0, 4, 18, \infty_3, 5, 17, 6, 16, 3\rangle$,

$\langle \infty_1, 0, 2, \infty_2, 18, 5, \infty_5, 6, 13, 7, 12, 8, 11, 9\rangle\}$.

14-$PM(24)$: $\mathcal{A}(A = Z_{19}^*\backslash\{1,2\}), \mathcal{B}(B = Z_{19}^*\backslash\{1,2\})$ and $\mathcal{C}$.

$LG = \langle \infty_1, \infty_2\rangle \bigcup \langle \infty_2, \infty_3\rangle \bigcup \langle \infty_3, \infty_4\rangle$.

14-$CM(24)$: $\mathcal{A}(A = Z_{19}^*\backslash\{1,2\}), \mathcal{B}(B = Z_{19}^*\backslash\{1,2,3\}), \mathcal{C}$ and

$\langle \infty_1, \infty_2, \infty_3, \infty_4, 13, 3, 2, 4, 1, 5, 0, 14, 15, 16\rangle$,

$\langle \infty_4, \infty_3, \infty_2, \infty_1, 16, 15, 14, 0, 6, 18, 7, 17, 9, 10\rangle$.

$RG = \langle \infty_1, 16\rangle \bigcup \langle 16, 15\rangle \bigcup \langle 15, 14\rangle \bigcup \langle 14, 0\rangle$. □

6 Constructions for $L_1 = 12$

Lemma 10. There exist optimal k-$PM(k+4)$ and k-$PM(2k-3)$ for $k \geq 11$. And, there exists optimal k-$PM(2k+4)$ and k-$PM(3k-3)$ for odd $k \geq 11$. The leave-arcs graph of each of these packing designs is DK_4.

Proof.

(1) k-$PM(k+4)$, $k \geq 11$ and $k \neq 12$.

Let the points be $\{\infty_1, \infty_2, \infty_3, \infty_4\} \bigcup Z_k$. Since $\phi(k) \geq 8$ when $k \geq 11$ and $k \neq 12, 14, 18$, there are seven distinct numbers from Z_k^* as follows:

$$\pm d_1, \pm d_2, \pm d_3 \ \text{ and } \ \begin{cases} 1 & (k \ \text{even}) \\ (-1)^{\frac{k-1}{2}} \frac{k-1}{2} & (k \ \text{odd}) \end{cases},$$

such that $gcd(d_i, k) = 1$ for $1 \leq i \leq 3$. Then, the following k-circuits form an optimal k-$PM(k+4)$ with $LG = DK_4$:

when k is even:

$[\infty_1, -A([2, \frac{k}{2}]\backslash\{d_1, d_2, d_3\}), -A^{-1}([1, \frac{k}{2} - 1]\backslash\{d_1, d_2, d_3\}), \infty_2, *, \infty_3, *, \infty_4, *]$,
$[1]_k, [d_1]_k, [-d_1]_k, [d_2]_k, [-d_2]_k, [d_3]_k, [-d_3]_k$;

when k is odd:

$[\infty_1, A([1,\frac{k-1}{2}]\backslash\{d_1,d_2,d_3\}), -A^{-1}([1,\frac{k-3}{2}]\backslash\{d_1,d_2,d_3\}), \infty_2, *, \infty_3, *, \infty_4, *]$,
$[(-1)^{\frac{k-1}{2}}\frac{k-1}{2}]_k, [d_1]_k, [-d_1]_k, [d_2]_k, [-d_2]_k, [d_3]_k, [-d_3]_k$.

where the first starter (following ∞_1) is 0, and the other three starters (i.e. number $*$) may be easily chosen from the remaining numbers of Z_k.

The 12-$PM(12+4)$, in fact, is a 12-$MD(16)$. As for 14-$PM(14+4)$ and 18-$PM(18+4)$, these can be constructed as follows.

14-$PM(18)$: $[1,2,3,5,-6,4,-2]_2$,
$[\infty_1, -3, 6, 7, \infty_2, -1, \infty_3, -4, \infty_4, -5]$ with starters $(0,2,-6,-2)$.

18-$PM(22)$: $[5]_{18}, [1,-2,3,-4,-1,-3]_3$,
$[\infty_1, 2, -A[5,9], -A^{-1}[4,8], \infty_2, *, \infty_3, *, \infty_4, *]$ with starters $(0,1,-1,-2)$.

(2) k-$PM(2k-3)$, $k \geq 11$.

Points: $Z_{2k-7} \bigcup \{\infty_1, \infty_2, \infty_3, \infty_4\}$.

k-circuits: $[\infty_1, A[1,k-4], \infty_2, *]$ with starters $(0, k-4)$,
$[\infty_3, -A[1,k-4], \infty_4, *]$ with starters $(0, k-4)$.

(3) k-$PM(2k+4)$, k odd, $k \geq 11$.

Points: $Z_{2k} \bigcup \{\infty_1, \infty_2, \infty_3, \infty_4\}$.

k-circuits: $[\infty_1, A([1,k-2]\backslash\{2,4,8\}), k, \infty_2, *]$ with starters $(0,k)$,
$[\infty_3, -A([1,k-1]\backslash\{2,4,8\}), \infty_4, *]$ with starters $(0,k)$,
$[i]_k$ for $i = \pm 2, \pm 4, \pm 8, k-1$.

(4) k-$PM(3k-3)$, k odd, $k \geq 11$.

Points: $Z_{3k-7} \bigcup \{\infty_1, \infty_2, \infty_3, \infty_4\}$.

k-circuits: $[\infty_1, A([1,k-2]]$, $[\infty_2, -A([1,k-2]]$,
$[\infty_3, A[k-1, \frac{3k-7}{2}], -A^{-1}[k-1, \frac{3k-9}{2}], \infty_4, *]$ with starters $(0,1)$. □

Theorem 7. There exists all optimal k-$PM_\lambda(v)$ ad k-$CM_\lambda(v)$ for $11 \leq k \leq 14$, $v(v-1) \equiv 12 \pmod{k}$ and any λ.

Proof.

(1) $k = 11$ $(v = 15, 19, 26, 30)$, $\overline{\lambda} = 11$.

λ	1	2	3	4	5
	12	2	3	4	5
L_λ		$L_1 - R_1$	$L_1 - R_2$	$L_2 + L_2$	$L_2 + L_3$
	10	9	8	7	6
R_λ			$R_1 - L_2$	$R_1 - L_3$	$R_1 - L_4$

λ	6	7	8	9	10
	6	7	8	9	10
L_λ	L_3+L_3	L_2+L_5	L_4+L_4	L_3+L_6	L_5+L_5
	5	4	3	2	12
R_λ	R_1-L_5	R_1-L_6	R_1-L_7	R_1-L_8	R_2+R_8

Here the graph LG_1 is DK_4 by Lemma 10. Therefore, if we give an optimal 11-$CM(v)$ with $RG_1 = DK_4\backslash C_2$ and an optimal 11-$CM_2(v)$ with $RG_2 = DK_4\backslash C_3$, then, by Lemma 10 and the table, we can obtain all optimal 11-$PM_\lambda(v)$ and 11-$CM_\lambda(v)$ for given v and any λ.

$v = 15$, Points : $X = Z_{11}\bigcup\{\infty_1,\infty_2,\infty_3,\infty_4\}$.

11-$PM(15) = (X,\mathcal{A})$ from Lemma 10, where its $LG = DK_4$ is on the set $\{\infty_1,\infty_4,0,6\}$.

11-$CM(15) = (X,\mathcal{B}_1\bigcup\mathcal{B}_2\bigcup\mathcal{B}_3)$

$\mathcal{B}_1 = \{B+i; i\in Z_{11}^*\}$, where $B = \langle\infty_1,8,\infty_2,1,\infty_3,9,\infty_4,0,3,7,6\rangle$;

$\mathcal{B}_2$: $\langle\infty_1,\infty_2,\infty_3,\infty_4,0,1,2,3,4,5,6\rangle, \langle\infty_4,\infty_1,0,2,8,10,1,3,9,7,6\rangle,$
$\langle\infty_2,\infty_1,\infty_4,0,6,7,5,3,1,10,8\rangle, \langle\infty_1,\infty_4,\infty_3,9,0,3,5,7,2,4,6\rangle$;

$\mathcal{B}_3$: $\langle\infty_1,\infty_2,\infty_3,\infty_4,6,1,7,8,9,10,0\rangle, \langle\infty_4,\infty_1,6,8,3,7,9,4,10,5,0\rangle,$
$\langle\infty_4,\infty_2,1,\infty_3,\infty_1,8,6,4,2,0,9\rangle, [-3]_{11}, [-4]_{11}, [5]_{11}$.

$RG = \langle\infty_1,\infty_4\rangle\bigcup\langle\infty_1,0\rangle\bigcup\langle\infty_1,6\rangle\bigcup\langle\infty_4,0\rangle\bigcup\langle\infty_4,6\rangle$.

11-$CM_2(15) = (X,\mathcal{A}\bigcup\mathcal{B}_1'\bigcup\mathcal{B}_2'\bigcup\mathcal{B}_3)$, where

$\mathcal{B}_1' = \{B+i; i\in Z_{11}^*\backslash\{2,3,-5,-6\}\}$;

$\mathcal{B}_2'$: $\langle\infty_1,\infty_2,\infty_3,\infty_4,0,1,2,7,4,5,6\rangle, \langle\infty_1,3,9,7,2,4,\infty_4,5,8,1,0\rangle,$
$\langle\infty_1,\infty_4,2,4,1,7,5,3,\infty_3,0,6\rangle, \langle\infty_2,\infty_1,\infty_4,0,6,7,2,3,1,10,8\rangle,$
$\langle\infty_4,\infty_1,0,2,8,10,1,3,5,7,6\rangle, \langle\infty_4,\infty_3,9,8,\infty_1,10,\infty_2,3,4,6,0\rangle,$
$\langle\infty_2,6,\infty_3,4,7,1,2,5,9,0,3\rangle, \langle\infty_1,2,\infty_2,4,\infty_3,1,\infty_4,3,6,10,9\rangle,$
$\langle\infty_1,0,\infty_2,7,\infty_3,3,\infty_4,6,9,2,1\rangle$.

$RG_2 = \langle 1,2,4\rangle\bigcup\langle 1,7\rangle\bigcup\langle 2,7\rangle\bigcup\langle 4,7\rangle$.

$v = 19$, Points :$X = Z_{15}\bigcup\{\infty_1,\infty_2,\infty_3,\infty_4\}$.

11-$PM(19) = (X,\mathcal{A})$, from Lemma 10, where its $LG = DK_4$ is on the set $\{4,5,11,12\}$.

11-$CM(19) = (X,\mathcal{B}_1\bigcup\mathcal{B}_2\bigcup\mathcal{B}_3)$

$\mathcal{B}_1 = \{B+i; i\in Z_{15}^*\backslash\{1,2,6,9\}\}\bigcup\{C+i; i\in Z_{15}^*\backslash\{1,5,6\}\}$, where

$B = \langle\infty_1,11,\infty_2,0,1,14,2,13,3,12,4\rangle$,

$C = \langle\infty_3,9,\infty_4,1,0,2,14,3,13,4,12\rangle$;

$\mathcal{B}_2 : \langle \infty_3, 0, \infty_4, 7, 6, 8, 5, 11, 4, 10, 3 \rangle,$
$\langle \infty_1, 11, \infty_2, 6, 7, 5, 12, 4, 9, 3, 10 \rangle,$
$\langle \infty_1, \infty_2, \infty_3, \infty_4, 1, 0, 2, 13, 5, 8, 4 \rangle;$
$\mathcal{B}_3 : \langle \infty_3, \infty_1, \infty_4, 2, 1, 3, 0, 4, 12, 5, 13 \rangle,$
$\langle \infty_1, \infty_3, \infty_2, 1, 2, 14, 3, 13, 4, 0, 5 \rangle,$
$\langle \infty_1, 5, \infty_2, 9, 4, 8, 11, 7, 12, 6, 13 \rangle,$
$\langle \infty_3, 14, \infty_4, 6, 5, 7, 4, 13, 3, 9, 2 \rangle,$
$\langle \infty_2, \infty_4, \infty_3, 9, 10, 8, 3, 12, 4, 14, 2 \rangle,$
$\langle \infty_3, 10, \infty_4, \infty_2, \infty_1, 2, 3, 14, 5, 4, 12 \rangle,$
$\langle \infty_4, \infty_1, 12, \infty_2, 0, 1, 14, 4, 11, 5, 9 \rangle,$
$\langle \infty_1, 13, \infty_2, 2, 0, 3, 1, 4, 5, 14, 6 \rangle.$
$RG = \langle 4, 5 \rangle \bigcup \langle 4, 11 \rangle \bigcup \langle 5, 11 \rangle \bigcup \langle 4, 12 \rangle \bigcup \langle 5, 12 \rangle.$

11-$CM_2(19) = (X, \mathcal{A} \bigcup \mathcal{B}_1' \bigcup \mathcal{B}_2' \bigcup \mathcal{B}_3)$ where
$\mathcal{B}_1' = \{B + i; i \in Z_{15}^* \setminus \{1, 2, 6, 7, 9\}\} \bigcup \{C + i; i \in Z_{15}^* \setminus \{1, 5, 6, 8\}\};$
$\mathcal{B}_2' : \langle \infty_1, \infty_2, \infty_3, \infty_4, 1, 0, 2, 3, 12, 8, 4 \rangle,$
$\langle \infty_1, 11, \infty_2, 6, 9, 5, 12, 2, 8, 3, 10 \rangle,$
$\langle \infty_3, 0, \infty_4, 9, 8, 2, 12, 11, 6, 7, 5 \rangle,$
$\langle \infty_1, 3, \infty_2, 7, 8, 6, 12, 5, 10, 4, 11 \rangle,$
$\langle \infty_3, 2, \infty_4, 7, 6, 8, 5, 11, 4, 10, 3 \rangle,$
$\langle 2, 13, 5, 8, 10, 7, 11, 12, 4, 9, 3 \rangle.$
$RG_2 = \langle 3, 12, 8 \rangle \bigcup \langle 2, 3 \rangle \bigcup \langle 2, 8 \rangle \bigcup \langle 2, 12 \rangle.$

$v = 26$, Points : $X = Z_{22} \bigcup \{\infty_1, \infty_2, \infty_3, \infty_4\}.$

11-$PM(26) = (X, \mathcal{A})$ from Lemma 10, where its $LG = DK_4$ is on the set $\{\infty_1, \infty_2, 0, 16\}$.

11-$CM(26) = (X, \mathcal{B}_1 \bigcup \mathcal{B}_2 \bigcup \mathcal{B}_3)$
$\mathcal{B}_1 = \{B + i; i \in Z_{22}^* \setminus \{1\}\} \bigcup \{C + i; i \in Z_{22}^*\}$, where
$B = \langle \infty_1, 6, \infty_2, 2, 3, 0, 11, 16, 9, 17, 7 \rangle,$
$C = \langle \infty_3, 21, \infty_4, 3, 2, 5, 15, 10, 17, 8, 0 \rangle;$
$\mathcal{B}_2 : \langle \infty_2, \infty_4, \infty_3, 21, 19, 17, 15, 13, 11, 9, 7 \rangle;$
$\mathcal{B}_3 : \langle \infty_1, \infty_3, \infty_2, 2, 3, 0, 11, 16, 9, 17, 7 \rangle,$
$\langle \infty_3, \infty_1, \infty_4, 3, 2, 5, 15, 10, 17, 8, 0 \rangle,$
$\langle \infty_2, \infty_1, 0, 9, 18, 5, 14, 1, 10, 19, 6 \rangle,$
$\langle \infty_2, \infty_3, \infty_4, \infty_1, 6, 15, 2, 11, 20, 7, 16 \rangle,$
$\langle \infty_1, \infty_2, 16, 3, 12, 21, 8, 17, 4, 13, 0 \rangle,$

$\langle\infty_1, 16, 0, \infty_2, 3, 1, 12, 17, 10, 18, 8\rangle$,
$\langle\infty_4, \infty_2, 0, 16, \infty_1, 7, 5, 3, 4, 1, 21\rangle$,
$\langle 0, 20, 18, 16, 14, 12, 10, 8, 6, 4, 2\rangle$,
$[2]_{11}, [4]_{11}, [-4]_{11}, [6]_{11}, [-6]_{11}$.

$RG = \langle\infty_1, 0\rangle \bigcup \langle\infty_1, 16\rangle \bigcup \langle\infty_2, 0\rangle \bigcup \infty_2, 16\rangle \bigcup \langle 0, 16\rangle$.

11-$CM_2(26) = (X, \mathcal{A} \bigcup \mathcal{B}_1' \bigcup \mathcal{B}_2' \bigcup \mathcal{B}_3)$, where

$\mathcal{B}_1' = \{B + i; i \in Z_{22}^* \backslash \{1, 2\}\} \bigcup \{C + i; i \in Z_{22}^* \backslash \{-1, -10\}\}$;

$\mathcal{B}_2$: $\langle\infty_1, \infty_2, \infty_4, \infty_3, 21, 19, 17, 15, 5, 8, 9\rangle$,
$\langle\infty_2, \infty_1, 8, 5, 15, 13, 18, 11, 19, 9, 7\rangle$,
$\langle\infty_3, 20, \infty_4, 2, 1, 4, 5, 9, 16, 7, 21\rangle$,
$\langle\infty_3, 11, \infty_4, 15, 14, 9, 5, 0, 7, 20, 12\rangle$,
$\langle\infty_2, 4, 14, 17, 5, 2, 13, 11, 9, 15, 8\rangle$.

$RG_2 = \langle 5, 8\rangle \bigcup \langle 5, 9\rangle \bigcup \langle 5, 15\rangle \bigcup \langle 8, 9, 15\rangle$.

$\underline{v = 30, \text{Points} : X = Z_{26} \bigcup \{\infty_1, \infty_2, \infty_3, \infty_4\}.}$

11-$PM(30) = (X, \mathcal{A})$ from Lemma 10, where its $LG = DK_4$ is on the set $\{0, 3, 17, 18\}$.

11-$CM(30) = (X, \mathcal{B}_1 \bigcup \mathcal{B}_2 \bigcup \mathcal{B}_3)$

$\mathcal{B}_1 = \{A + i; i \in Z_{26}^* \backslash \{-3\}\} \bigcup \{B + i, C + i; i \in Z_{26}^* \backslash \{\pm 1\}\}$, where

$A = \langle 0, 24, 1, 23, 2, 22, 3, 21, 4, 20, 5\rangle$,
$B = \langle\infty_1, 17, \infty_2, 22, 2, 21, 3, 20, 4, 19, 5\rangle$,
$C = \langle\infty_3, 22, \infty_4, 2, 4, 17, 16, 13, 1, 5, 6\rangle$;

$\mathcal{B}_2$: $\langle\infty_2, \infty_4, \infty_1, 18, 3, 21, 4, 20, 5, 0, 17\rangle$,
$\langle\infty_3, 21, \infty_4, 1, 3, 18, 17, 0, 4, 5, 6\rangle$,
$\langle\infty_4, \infty_3, 22, 3, 17, 16, 13, 1, 20, 2, 23\rangle$;

$\mathcal{B}_3$: $\langle\infty_1, \infty_2, \infty_3, \infty_4, 2, 21, 3, 20, 4, 19, 5\rangle$,
$\langle\infty_1, \infty_3, \infty_2, 23, 3, 22, 4, 21, 5, 20, 6\rangle$,
$\langle\infty_3, \infty_1, \infty_4, 3, 5, 18, 17, 14, 2, 6, 7\rangle$,
$\langle\infty_4, \infty_2, \infty_1, 17, 18, 0, 24, 1, 23, 2, 22\rangle$,
$\langle\infty_2, 22, 2, 4, 17, 3, 16, 15, 12, 0, 18\rangle$,
$\langle\infty_3, 23, 21, 24, 20, 25, 19, 0, 18, 1, 5\rangle$,
$\langle\infty_1, 16, \infty_2, 21, 1, 17, 2, 19, 3, 18, 4\rangle$.

$RG = \langle 0, 17, 18\rangle \bigcup \langle 0, 18, 17\rangle \bigcup \langle 3, 17\rangle \bigcup \langle 3, 18\rangle$.

11-$CM_2(30) = (X, \mathcal{A} \bigcup \mathcal{B}_1' \bigcup \mathcal{B}_2' \bigcup \mathcal{B}_3)$, where

$\mathcal{B}_1' = \{A + i; i \in Z_{26}^* \backslash \{-3, 13\}\} \bigcup \{B + i; i \in Z_{26}^* \backslash \{\pm 1\}\} \bigcup \{C + i; i \in$

$Z_{26}^{*}\backslash\{\pm 1, -9\}\}$;

$\mathcal{B}_2 : \langle\infty_4, \infty_3, 22, 3, 17, 16, 13, 1, 18, 4, 23\rangle$,

$\langle\infty_2, \infty_4, \infty_1, 18, 23, 1, 4, 20, 5, 0, 17\rangle$,

$\langle\infty_4, 1, 20, 2, 23, 4, 18, 17, 0, 3, 21\rangle$,

$\langle\infty_3, 13, \infty_4, 19, 21, 4, 1, 3, 18, 22, 23\rangle$,

$\langle 13, 11, 14, 10, 15, 9, 16, 8, 7, 4, 18\rangle$,

$\langle\infty_3, 21, 8, 17, 7, 18, 3, 0, 4, 5, 6\rangle$.

$RG_2 = \langle 1, 4\rangle \bigcup \langle 4, 18\rangle \bigcup \langle 4, 23\rangle \bigcup \langle 1, 18, 23\rangle$.

(2) $k = 13$ ($v = 17, 23, 30, 36$), $\overline{\lambda} = 13$.

λ	1	2	3	4	5	6
L_λ	12	11	10 $L_1 - R_2$	9 $L_1 - R_3$	8 $L_1 - R_4$	7 $L_1 - R_5$
R_λ	14	2 $R_1 - L_1$	3 $R_1 - L_2$	4 $R_2 + R_2$	5 $R_2 + R_3$	6 $R_3 + R_3$

λ	7	8	9	10	11	12
L_λ	6 $L_1 - R_6$	5 $L_1 - R_7$	4 $L_1 - R_8$	3 $L_7 - R_3$	2 $L_9 - R_2$	14 $L_6 + L_6$
R_λ	7 $R_2 + R_5$	8 $R_4 + R_4$	9 $R_4 + R_5$	10 $R_5 + R_5$	11 $R_4 + R_7$	12 $R_6 + R_6$

It is not difficult to see that all optimal 13-$PM_\lambda(v)$ and 13-$CM_\lambda(v)$ for given v and any λ can be obtained by the table and Theorem 4, if we give

13-$PM(v)$ with $LG = \langle x, y, z\rangle \bigcup \langle x, z, y\rangle \bigcup \langle x, u, v\rangle \bigcup \langle x, v, u\rangle$;

13-$CM(v)$ with $LG = \langle x, y, z\rangle \bigcup \langle x, z, y\rangle \bigcup \langle x, u, v\rangle \bigcup \langle x, v, u\rangle \bigcup \langle v, w\rangle$;

13-$PM_2(v)$ with $LG_2 = \langle x, y, z\rangle \bigcup \langle x, u, v\rangle \bigcup \langle x, v, u\rangle \bigcup \langle v, w\rangle$,

where x, y, z, u, v, w are distinct.

$v = 17$, Points :$X = Z_{13} \bigcup \{a, b, c, d\}$.

$\mathcal{A} = \{\langle a, 7, b, 6, c, 8, d, 3, 4, 12, 5, 10, 9\rangle + i, i \in A\}$,

$\mathcal{B} = \{\langle a, 3, b, 2, c, 11, 4, d, 12, 0, 1, 6, 5\rangle, [3]_{13}, [-3]_{13}\}$,

$\mathcal{C} = \{\langle b, d, a, 4, 6, 8, 10, 12, 1, 3, 5, 7, 9\rangle, \langle a, 9, c, 4, b, 5, 6, 7, d, 2, 10, 3, 11\rangle$,
$\langle a, 6, b, 12, c, 0, d, 8, 9, 4, 10, 2, 1\rangle, \langle a, 0, b, 8, c, 1, d, 9, 10, 5, 11, 3, 2\rangle$,
$[-2]_{13}, [4]_{13}, [-4]_{13}\}$,

$\mathcal{D} = \{\langle a, b, c, 5, 3, 10, 4, 1, 7, 2, 8, 0, 12\rangle, \langle a, d, b, 11, c, 7, 12, 6, 0, 2, 4, 3, 8\rangle$,
$\langle a, c, d, 5, 12, 11, 0, 3, 1, 4, 9, 8, 7\rangle, \langle c, b, a, 8, 2, 9, 11, d, 6, 1, 3, 0, 5\rangle\}$,

13-$PM(17) = (X, \Omega)$, where Ω consists of $\mathcal{A}(A = \{0, 1, 4, 7, 8\})$, $\mathcal{B}$, $\mathcal{C}$ and

$\langle a,12,b,11,c,10,d,0,8,1,2,7,6\rangle, \langle a,d,b,3,c,7,12,11,0,2,4,9,8\rangle,$
$\langle a,5,b,4,c,6,d,1,9,2,3,8,7\rangle, \langle a,10,b,9,11,d,6,1,7,2,8,0,12\rangle,[-6]_{13}.$
$LG = \langle a,b,c\rangle \bigcup \langle a,c,b\rangle \bigcup \langle c,d,5\rangle \bigcup \langle d,c,5\rangle.$

13-$CM(17)$: $\mathcal{A}(A = \{0,4,7,8\}), \mathcal{B}, \mathcal{C}, \mathcal{D}$ and
$\langle d,c,a,0,7,1,8,b,9,3,4,11,5\rangle, \langle a,5,b,3,c,10,d,0,8,1,2,7,6\rangle,$
$\langle a,10,b,4,c,6,d,1,9,2,3,5,0\rangle, \langle a,12,b,7,c,9,d,4,5,0,6,11,10\rangle.$
$RG = \langle 1,4,3\rangle \bigcup \langle 1,3,4\rangle \bigcup \langle 3,0,5\rangle \bigcup \langle 0,3,5\rangle \bigcup \langle 0,a\rangle.$

13-$PM_2(17) = (X, \Omega' \bigcup \mathcal{B} \bigcup \mathcal{D} \bigcup \mathcal{E})$, where Ω' is an isomorphic image of Ω under the mapping $a \to 1, b \to 4, c \to 3, d \to 0$ and $5 \to 5$.

$\mathcal{E}$ consists of the following 14 blocks:
$\langle a,9,c,4,b,5,0,7,d,2,10,3,11\rangle, \langle a,0,b,6,c,8,d,3,4,12,5,7,9\rangle,$
$\langle a,1,b,8,c,12,d,9,10,5,11,3,2\rangle, \langle a,6,b,7,c,0,d,8,9,4,10,2,1\rangle,$
$\langle a,7,3,12,c,2,d,4,5,0,6,11,10\rangle, \langle a,2,b,1,9,3,d,11,12,7,0,5,4\rangle,$
$\langle d,1,5,9,0,4,8,12,3,7,11,2,6\rangle, \langle b,4,c,6,2,11,9,7,5,3,1,12,10\rangle,$
$\langle d,c,a,10,1,8,b,9,2,3,4,11,5\rangle, \langle a,11,b,3,c,10,d,0,8,1,2,7,6\rangle,$
$\langle a,12,b,10,c,9,d,7,8,3,5,1,0\rangle, \langle b,0,c,1,d,10,11,6,12,4,3,9,5\rangle,$
$\langle c,3,a,5,6,7,b,12,8,4,0,9,1\rangle, \langle b,d,a,4,2,0,11,7,1,3,5,10,9\rangle.$
$RG_2 = \langle 6,8,10\rangle \bigcup \langle 6,10,8\rangle \bigcup \langle 1,10,12\rangle \bigcup \langle 4,6\rangle.$

$v = 23$, Points :$X = Z_{19} \bigcup \{a,b,c,d\}.$

$\mathcal{A} = \{\langle a,11,b,0,1,18,2,17,3,16,4,15,5\rangle + i; i \in A\},$
$\mathcal{B} = \{\langle c,1,d,0,10,18,11,17,12,16,13,15,14\rangle + i; i \in B\},$
$\mathcal{C} = \{\langle c,2,d,0,10,18,11,17,12,16,13,15,14\rangle,$
$\langle a,d,b,0,1,18,2,17,3,16,4,15,5\rangle,$
$\langle c,7,d,a,11,0,12,18,13,17,14,16,15\rangle\},$
$\mathcal{D} = \{\langle b,d,6,16,5,17,4,18,3,0,2,1,11\rangle\},$
$\mathcal{E} = \{\langle d,1,c,a,b,4,5,3,6,2,7,11,10\rangle,$
$\langle a,15,17,16,c,d,10,11,7,1,8,0,9\rangle\}.$

13-$PM(23) = (X, \Omega)$, where Ω consists of $\mathcal{A}(A = Z_{19}^*)$, $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{B}(B = Z_{19}^* \backslash \{1,6\})$. $LG = \langle a,b,c\rangle \bigcup \langle a,c,b\rangle \bigcup \langle c,d,1\rangle \bigcup \langle d,c,1\rangle.$

13-$CM(23)$: $\mathcal{A}(A = Z_{19}^* \backslash \{1,4\}), \mathcal{B}(B = Z_{19}^* \backslash \{1,2,6\}), \mathcal{C}, \mathcal{D}, \mathcal{E}$ and
$\langle a,12,b,c,1,d,8,10,7,17,5,16,6\rangle, \langle d,c,b,1,2,0,3,18,4,17,7,10,8\rangle,$
$\langle b,a,c,3,d,2,12,1,13,0,14,18,15\rangle.$
$RG = \langle 7,11,10\rangle \bigcup \langle 7,10,11\rangle \bigcup \langle 10,d,8\rangle \bigcup \langle d,10,8\rangle \bigcup \langle 7,17\rangle.$

13-$PM_2(23) = (X, \Omega' \bigcup \mathcal{A} \bigcup \mathcal{B} \bigcup \mathcal{C} \bigcup \mathcal{E} \bigcup \mathcal{F})$, where Ω' is an isomorphic image of Ω under the mapping $a \to 7, b \to 11, c \to 10, d \to d$ and $1 \to 8$. $\mathcal{A}(A = Z_{19}^{*} \backslash \{1,2,4\}), \mathcal{B}(B = Z_{19}^{*} \backslash \{1,2,3,6,7,8,18\})$ and $\mathcal{F}$ consists of the following 8 blocks:

$\langle b,c,8,d,6,16,5,17,4,18,3,1,11\rangle, \langle a,13,b,d,c,1,4,0,5,18,6,17,7\rangle,$
$\langle c,9,d,7,10,16,6,18,5,0,4,1,3\rangle, \langle a,12,14,13,c,b,1,d,8,10,7,17,6\rangle,$
$\langle c,3,d,8,18,7,0,6,1,5,2,4,17\rangle, \langle b,a,c,4,d,2,12,1,13,0,14,18,15\rangle,$
$\langle b,2,c,0,d,18,9,17,5,16,11,15,12\rangle,$
$\langle d,3,13,2,14,1,15,0,16,18,17,10,8\rangle.$

$RG_2 = \langle 0,2,3\rangle \bigcup \langle 0,3,2\rangle \bigcup \langle 3,18,4\rangle \bigcup \langle 1,2\rangle.$

$v = 30$, Points $: X = Z_{26} \bigcup \{a,b,c,d\}.$

$\mathcal{A} = \{\langle a,6,b,0,1,24,3,23,4,21,5,20,7\rangle + i, i \in A\},$
$\mathcal{B} = \{\langle c,7,d,0,25,2,23,3,22,5,21,6,20\rangle + i; i \in B\},$
$\mathcal{C} = \{\langle c,20,d,a,6,9,4,10,3,12,2,13,1\rangle, [8]_{13}, [-8]_{13}, [-12]_{13}, [2]_{13}^{1},$
$[-2]_{13}^{1}, [4]_{13}^{1}, [-4]_{13}^{1}\},$
$\mathcal{D} = \{\langle c,14,d,0,25,2,23,3,22,5,21,6,20\rangle,$
$\langle a,d,b,0,1,24,3,23,4,21,5,20,7\rangle,$
$\langle b,d,13,12,15,10,16,9,18,8,19,7,6\rangle, [2]_{13}^{0}, [-2]_{13}^{0}, [4]_{13}^{0}, [-4]_{13}^{0}\},$
$\mathcal{E} = \{\langle d,c,b,3,4,1,6,0,7,24,20,25,12\rangle,$
$\langle b,a,c,8,d,1,0,3,24,4,23,6,22\rangle,$
$\langle d,7,c,a,b,16,17,14,19,13,20,8,25\rangle,$
$\langle a,22,7,21,c,d,25,8,20,11,16,10,23\rangle,$
$\langle a,19,b,13,14,11,21,10,17,8,18,7,20\rangle\}.$

13-$PM(30) = (X, \Omega)$, where Ω consists of $\mathcal{A}(A = Z_{26}^{*}), \mathcal{C}, \mathcal{D}$ and $\mathcal{B}(B = Z_{26}^{*} \backslash \{7,13\})$. $LG = \langle a,b,c\rangle \bigcup \langle a,c,b\rangle \bigcup \langle c,d,7\rangle \bigcup \langle d,c,7\rangle$.

13-$CM(30)$: $\mathcal{A}(A = Z_{26}^{*} \backslash \{3,13,16\}), \mathcal{B}(B = Z_{26}^{*} \backslash \{1,7,13\}), \mathcal{C}, \mathcal{D}, \mathcal{E}$ and $\langle a,9,b,c,7,d,12,25,20,24,8,23,10\rangle$.

$RG = \langle 8,20,25\rangle \bigcup \langle 8,25,20\rangle \bigcup \langle 25,d,12\rangle \bigcup \langle d,25,12\rangle \bigcup \langle 20,24\rangle.$

13-$PM_2(30) = (X, \Omega' \bigcup \mathcal{A} \bigcup \mathcal{B} \bigcup \mathcal{C} \bigcup \mathcal{E} \bigcup \mathcal{F})$, where Ω' is an isomorphic image of Ω under the mapping $a \to 8, b \to 20, c \to 25, d \to d$ and $7 \to 12$. $\mathcal{A}(A = Z_{26}^{*} \backslash \{3,7,13,16\}), \mathcal{B}(B = Z_{26}^{*} \backslash \{1,4,7,13,21\})$ and $\mathcal{F}$ consists of the following 10 blocks:

$\langle 0,25,2,24,22,20,16,12,10,8,6,1,7\rangle,$
$\langle 0,4,8,19,7,6,10,14,16,18,20,22,24\rangle,$

$\langle c, 11, d, 13, 12, 8, 4, 0, 22, 18, 14, 10, 24\rangle$,
$\langle c, 14, d, 0, 24, 2, 23, 3, 22, 5, 21, 6, 20\rangle$,
$\langle a, d, b, 0, 9, 25, 10, 12, 14, 18, 16, 20, 7\rangle$,
$\langle c, 2, d, 4, 3, 23, 18, 22, 0, 16, 14, 12, 15\rangle$,
$\langle 0, 1, 15, 10, 16, 9, 18, 8, 12, 25, 20, 24, 17\rangle$,
$\langle b, c, 7, d, 12, 16, 1, 24, 8, 5, 10, a, 9\rangle$,
$\langle a, 13, b, d, 21, 20, 23, 4, 11, 2, 12, 1, 14\rangle$,
$\langle b, 7, 8, 23, 10, 4, 21, 5, 20, 18, 24, 3, 6\rangle$.
$RG_2 = \langle 2,4,6\rangle \bigcup \langle 2,6,4\rangle \bigcup \langle 6,8,10\rangle \bigcup \langle 0,2\rangle$.

$v = 36$, Points $: X = Z_{32} \bigcup \{a, b, c, d\}$.

$\mathcal{A} = \{\langle a, 25, b, 0, 1, 31, 2, 30, 3, 10, 20, 9, 21\rangle + i; i \in A\}$,
$\mathcal{B} = \{\langle c, 26, d, 0, 31, 1, 30, 2, 29, 3, 28, 4, 27\rangle + i; i \in B\}$,
$\mathcal{C} = \{\langle 0, 24, 1, 23, 2, 22, 3, 21, 4, 20, 5, 19, 6\rangle + i; i \in C\}$,
$\mathcal{D} = \{\langle c, 20, d, a, 25, 27, 24, 28, 23, 29, 22, 30, 21\rangle$,
$\langle c, 25, d, 0, 31, 1, 30, 2, 29, 3, 28, 4, 27\rangle\}$,
$\mathcal{E} = \{\langle b, d, 31, 30, 0, 29, 1, 28, 2, 27, 3, 26, 25\rangle$,
$\langle a, d, b, 0, 1, 31, 2, 30, 3, 10, 20, 9, 21\rangle\}$,
$\mathcal{F} = \{\langle d, 26, c, a, b, 4, 5, 3, 6, 2, 7, 1, 10\rangle$,
$\langle b, a, c, 27, d, 1, 0, 2, 31, 3, 30, 4, 29\rangle\}$.

13-$PM(36) = (X, \Omega)$, where Ω consists of $\mathcal{A}(A = Z_{32}^*), \mathcal{C}(C = Z_{32})$, $\mathcal{D}$, $\mathcal{B}(B = Z_{32}^* \backslash \{26, 31\})$ and $\mathcal{E}$. $LG = \langle a,b,c\rangle \bigcup \langle a,c,b\rangle \bigcup \langle c,d,26\rangle \bigcup \langle d,c,26\rangle$.

13-$CM(36)$: $\mathcal{A}(A = Z_{32}^* \backslash \{2,4\}), \mathcal{B}(B = Z_{32}^* \backslash \{1,26,31\}), \mathcal{C}(C = Z_{32}), \mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ and

$\langle a, 27, b, c, 26, d, 6, 10, 7, 12, 22, 11, 23\rangle, \langle d, c, b, 2, 3, 1, 4, 0, 5, 12, 7, 10, 6\rangle$,
$\langle a, 29, 5, 28, c, d, 10, 1, 7, 14, 24, 13, 25\rangle$.
$RG = \langle 1,7,10\rangle \bigcup \langle 1,10,7\rangle \bigcup \langle 10,d,6\rangle \bigcup \langle d,10,6\rangle \bigcup \langle 7,12\rangle$.

13-$PM_2(36) = (X, \Omega' \bigcup \mathcal{A} \bigcup \mathcal{B} \bigcup \mathcal{C} \bigcup \mathcal{D} \bigcup \mathcal{F} \bigcup \mathcal{G})$, where Ω' is an isomorphic image of Ω under the mapping $a \to 1, b \to 7, c \to 10, d \to d$ and $26 \to 6$. $\mathcal{A}(A = Z_{32}^* \backslash \{3,4\}), \mathcal{B}(B = Z_{32}^* \backslash \{1,3,4,5,26,31\}), \mathcal{C}(C = Z_{32}^* \backslash \{16\})$ and $\mathcal{G}$ consists of the following 9 blocks:

$\langle a, 28, b, c, 26, d, 10, 1, 7, 18, 6, 0, 24\rangle, \langle c, 30, d, 4, 2, 5, 1, 6, 10, 7, 0, 8, 31\rangle$,
$\langle c, 31, d, 5, 19, 6, 3, 7, 2, 4, 1, 9, 0\rangle, \langle a, 29, 5, 28, c, d, 6, 1, 7, 14, 24, 13, 25\rangle$,
$\langle d, b, c, 3, 5, 2, 22, 16, 8, 17, 7, 10, 6\rangle, \langle a, d, b, 0, 6, 31, 2, 8, 1, 5, 20, 9, 21\rangle$,
$\langle b, d, 31, 30, 0, 29, 1, 23, 2, 27, 3, 26, 25\rangle, \langle c, 29, d, 3, 2, 6, 19, 5, 0, 1, 31, 7, 30\rangle$,

$\langle 2,30,3,10,20,4,6,13,23,12,24,1,28\rangle$.

$RG_2 = \langle 3,4,21\rangle \bigcup \langle 4,3,21\rangle \bigcup \langle 4,20,5\rangle \bigcup \langle 3,22\rangle$.

(3) $k = 14$ ($v = 18, 25$), $\overline{\lambda} = 7$.

An optimal $14\text{-}CM(v)$, $v = 18$ and 25, can be found from Lemma 3 and Theorem 1 in [7]. For $14\text{-}PM(v)$, by Lemma 10 and Corollary 2. □

7 Construction for other L_1

1° Case $L_1 = 3$

(1) $k = 9$ ($v = 13, 15, 22, 24$)

Since $\overline{\lambda} = 3$ and $R_1 = 6$, we need to construct a $9\text{-}PM(v)$ with leave-arcs graph C_3 and a $9\text{-}CM(v)$ with repeat-arcs graph DK_3.

$v = 13$, Points : Z_{13}.

$\mathcal{A} = \{\langle 0,12,1,11,2,10,3,6,4\rangle + i; i \in A\}$

$9\text{-}PM(13)$: $\mathcal{A}(A = Z_{13})$ and

$\langle 0,7,1,8,2,9,10,11,12\rangle, \langle 0,1,2,3,4,11,5,12,6\rangle,$

$\langle 0,5,10,4,9,1,6,7,8\rangle, \langle 2,7,12,4,5,6,11,3,10\rangle$. $LG = \langle 3,8,9\rangle$.

$9\text{-}CM(13)$: $\mathcal{A}(A = Z^*_{13})$ and

$\langle 0,1,2,3,4,5,6,7,8\rangle, \langle 0,5,10,2,7,12,4,9,1\rangle,$

$\langle 0,7,1,8,9,10,11,5,12\rangle, \langle 0,1,9,3,10,4,11,12,6\rangle$.

$RG = \langle 0,1,9\rangle \bigcup \langle 0,9,1\rangle$.

$v = 15$, Points : Z_{15}.

$\mathcal{A} = \{\langle 0,4,14,5,8,6,2,1,3\rangle + i; i \in A\}$

$\mathcal{B} = \{\langle 1,10,4,11,3,8,9,14,7\rangle + i; 1 \le i \le 3\}$.

$9\text{-}PM(15)$: $\mathcal{A}(A = Z_{15}), \mathcal{B}$ and

$\langle 1,10,3,4,12,5,14,7,8\rangle$, $\langle 0,1,8,13,3,10,2,9,14\rangle$,

$\langle 0,5,13,6,14,8,9,2,7\rangle$, $\langle 0,7,12,13,14,4,11,3,8\rangle$,

$\langle 0,9,1,2,3,11,4,5,6\rangle$. $LG = \langle 1,6,7\rangle$.

$9\text{-}CM(15)$: $\mathcal{A}$ ($A = Z^*_{15}$), $\mathcal{B}$ and

$\langle 0,5,10,3,4,12,13,6,14\rangle$, $\langle 0,1,10,5,14,4,11,3,8\rangle$,

$\langle 0,4,5,13,14,8,1,2,7\rangle$, $\langle 0,7,8,13,3,11,4,5,6\rangle$,

$\langle 0,9,14,5,8,6,2,1,3\rangle, \langle 1,6,7,12,5,4,10,2,9\rangle,$

$\langle 1,8,9,2,3,10,4,14,7\rangle$. $RG = \langle 4,5,10\rangle \bigcup \langle 4,10,5\rangle$.

$v = 22$, Points : $Z_{16} \bigcup \{\infty_1, \infty_2, ..., \infty_6\}$.

$\mathcal{A} = \{\langle\infty_1, 3, 13, 4, 12, \infty_2, 1, 14, 2\rangle + i; i \in A\}$
$\mathcal{B} = \{\langle\infty_3, 3, 4, 6, 9, \infty_4, 1, 12, 2\rangle + i; i \in B\}$
$\mathcal{C} = \{\langle\infty_5, 3, 12, 11, 9, \infty_6, 1, 13, 2\rangle + i; i \in Z_{16}^*\}$
$\mathcal{D} = \{\langle\infty_1, \infty_5, \infty_6, \infty_3, \infty_4, \infty_2, 1, 14, 2\rangle,$
$\langle\infty_5\infty_2, \infty_1, \infty_4, \infty_3, \infty_6, 1, 13, 2\rangle,$
$\langle\infty_3, \infty_5, \infty_1, \infty_2, \infty_6, \infty_4, 1, 12, 2\rangle\}.$

9-$PM(22)$: $\mathcal{A}(A = Z_{16}^*\backslash\{4\})$, $\mathcal{B}(B = Z_{16}^*\backslash\{-5\})$, $\mathcal{C}, \mathcal{D}$ and
$\langle\infty_1, 7, 13, 4, 12, \infty_2, 5, 2, 6\rangle$,$\langle\infty_4, \infty_1, \infty_6, \infty_5, \infty_3, 3, 4, 6, 9\rangle$,
$\langle\infty_2, \infty_3, \infty_1, 3, 12, 7, 1, 8, 0\rangle$,$\langle\infty_4, \infty_5, 3, 13, \infty_3, 14, 15, 1, 4\rangle$,
$\langle\infty_6, \infty_1, \infty_3, \infty_2, \infty_5, \infty_4, 12, 11, 9\rangle$. $LG = \langle\infty_2, \infty_4, \infty_6\rangle$.

9-$CM(22)$: $\mathcal{A}(A = Z_{16}^*)$, $\mathcal{B}(B = Z_{16}^*\backslash\{4\})$, $\mathcal{C}, \mathcal{D}$ and
$\langle\infty_2, \infty_4, 5, 0, 13, \infty_5, 3, 4, 12\rangle$, $\langle\infty_4, \infty_1, \infty_6, \infty_5, \infty_3, 3, 12, 11, 9\rangle$,
$\langle\infty_6, \infty_1, \infty_3, \infty_2, \infty_5, 13, 4, 6, 9\rangle$, $\langle\infty_4, \infty_5, 0, 6, \infty_3, 7, 8, 10, 13\rangle$,
$\langle\infty_5, \infty_4, \infty_6, \infty_2, \infty_3, \infty_1, 3, 13, 0\rangle$.
$RG = \langle\infty_5, 0, 13\rangle \bigcup \langle\infty_5, 13, 0\rangle$.

$v = 24$, Points : $Z_{20} \bigcup \{\infty_1, \infty_2, \infty_3, \infty_4\}$.

$\mathcal{A} = \{\langle\infty_1, 0, 19, 1, 18, 2, 17, 3, 16\rangle + i; i \in Z_{20}^*\}$,
$\mathcal{B} = \{\langle\infty_2, 16, 3, 17, 2, 18, 1, 19, 0\rangle + i; i \in B\}$,
$\mathcal{C} = \{\langle\infty_3, 1, \infty_4, 16, 4, 15, 5, 14, 6\rangle + i; i \in C\}$.

9-$PM(24)$: $\mathcal{A}$, $\mathcal{B}(B = Z_{20}\backslash\{3\})$, $\mathcal{C}(C = Z_{20}^*\backslash\{1\})$ and
$\langle\infty_3, \infty_1, \infty_4, 16, 4, 15, 5, 4, 6\rangle$,$\langle\infty_3, \infty_2, \infty_4, 17, 5, 16, 6, 15, 7\rangle$,
$\langle\infty_2, \infty_1, 0, 19, 1, \infty_4, \infty_3, 2, 3\rangle$, $\langle\infty_4, \infty_2, 19, 6, 0, 5, 1, 4, 2\rangle$,
$\langle\infty_1, \infty_2, \infty_3, 1, 18, 2, 17, 3, 16\rangle$. $LG = \langle\infty_1, \infty_3, \infty_4\rangle$.

9-$CM(24)$: $\mathcal{A}$, $\mathcal{B}(B = Z_{20}^*)$, $\mathcal{C}(C = Z_{20}^*\backslash\{1, 3\})$ and
$\langle\infty_2, \infty_1, \infty_3, \infty_4, 17, 5, 1, 19, 0\rangle$, $\langle\infty_3, \infty_1, \infty_4, 16, 4, 15, 5, 14, 6\rangle$,
$\langle\infty_4, \infty_2, \infty_3, 2, 17, 3, 16, \infty_1, 1\rangle$, $\langle\infty_1, \infty_2, 16, 3, 17, 2, 18, 1, 5\rangle$,
$\langle\infty_4, \infty_3, 1, \infty_1, 0, 19, 7, 18, 2\rangle$,$\langle\infty_3, 4, \infty_4, 19, 1, 18, 8, 17, 9\rangle$,
$\langle\infty_3, \infty_2, \infty_4, \infty_1, 5, 16, 6, 15, 7\rangle$. $RG = \langle\infty_1, 1, 5\rangle \bigcup \langle\infty_1, 5, 1\rangle$.

(2) $k = 13$ ($v = 20, 33$).

λ	1	2	3	4	5	6
	3	6	9	12	2	5
L_λ		$L_1 + L_1$	$L_1 + L_2$	$L_2 + L_2$	$L_2 - R_3$	$L_1 + L_5$
	10	7	4	14	11	8
R_λ		$R_1 - L_1$	$R_2 - L_1$	$R_1 + R_3$	$R_2 + R_3$	$R_3 + R_3$

λ	7	8	9	10	11	12
	8	11	14	4	7	10
L_λ	L_1+L_6	L_1+L_7	L_1+L_8	L_5+L_5	L_1+L_{10}	L_1+L_{11}
	5	2	12	9	6	3
R_λ	R_6-L_1	R_7-L_1	R_1+R_8	R_2+R_8	R_3+R_8	$R_{11}-L_1$

It is not difficult to see that we only need to construct a 13-$PM(v)$ with $LG = C_3$ and a 13-$CM(v)$ with $RG = DK_3 \bigcup \langle x,y\rangle \bigcup \langle y,z\rangle$.

$v = 20$, Points : Z_{20}.

$\mathcal{A} = \{\langle 0,19,1,18,2,15,10,16,4,7,3,8,6\rangle + i; i \in A\}$,

$\mathcal{B} = \{\langle 0,10,11,3,4,16,7,17,6,15,5,12,19\rangle + i; i \in B\}$,

$\mathcal{C} = \{\langle 0,11,1,10,2,13,4,7,3,14,5,16,8\rangle$,
$\langle 0,7,18,5,15,4,14,3,12,1,8,19,11\rangle$,
$\langle 0,12,3,10,1,13,2,14,4,15,16,17,9\rangle$,
$\langle 1,12,2,11,18,19,6,16,3,15,7,8,9\rangle$,
$\langle 1,16,2,10,13,9,14,15,6,5,7,4,8\rangle$,
$\langle 2,15,10,19,9,16,5,14,12,6,17,4,11\rangle\}$.

13-$PM(20)$: $\mathcal{A}(A = Z_{20}^*\backslash\{6\})$, $\mathcal{B}(B = \{0,1,2,3\})$, $\mathcal{C}$ and
$\langle 0,19,1,18,2,9,10,16,4,13,3,8,6\rangle$. $LG = \langle 10,17,18\rangle$.

13-$CM(20)$: $\mathcal{A}(A = Z_{20}^*\backslash\{1,2,6,-4\})$, $\mathcal{B}(B = \{0,1,3\})$, $\mathcal{C}$ and
$\langle 0,2,19,3,16,11,17,12,18,6,13,14,1\rangle$,
$\langle 0,4,17,14,18,9,19,8,2,16,15,1,3\rangle$,
$\langle 0,19,1,2,9,5,10,16,4,13,3,8,6\rangle$,
$\langle 1,15,14,13,6,18,10,17,5,8,4,9,7\rangle$,
$\langle 1,18,2,12,13,5,6,9,10,8,17,7,14\rangle$,
$\langle 0,3,19,4,2,1,14,15,17,18,11,6,12\rangle$.
$RG = \langle 6,13\rangle \bigcup \langle 13,14\rangle \bigcup \langle 1,14,15\rangle \bigcup \langle 1,15,14\rangle$.

$v = 33$, Points : $Z_{32} \bigcup \{\infty\}$.

$\mathcal{A} = \{\langle 10,9,11,8,12,7,13,6,14,5,15,4,16\rangle + i; i \in A\}$,

$\mathcal{B} = \{\langle \infty,5,10,17,9,18,6,19,4,20,2,21,7\rangle + i; i \in B\}$,

$\mathcal{C} = \{\langle 0,1,31,2,17,18,16,19,9,20,3,25,4\rangle + i; 0 \le i \le 13\}$,

$\mathcal{D} = \{\langle \infty,5,10,9,18,6,19,4,20,16,14,17,7\rangle$,
$\langle \infty,30,26,22,18,1,23,19,2,24,3,31,0\rangle$,
$\langle 0,28,24,20,2,21,7,18,8,19,15,16,17\rangle$,
$\langle 0,30,3,10,17,9,11,31,12,29,13,27,14\rangle$,
$\langle 1,29,25,21,17,15,13,16,31,27,23,2,30\rangle\}$.

13-$PM(33)$: $\mathcal{A}(A = Z_{32}^*), \mathcal{B}(B = Z_{32}^*\backslash\{-7\}), \mathcal{C}, \mathcal{D}$ and

$\langle 2, 11, 8, 12, 7, 13, 6, 14, 5, 15, 4, 16, 10\rangle$. $LG = \langle 14, 15, 18\rangle$.

13-$CM(33)$: $\mathcal{A}(A = Z_{32}^*\backslash\{-10\}), \mathcal{B}(B = Z_{32}^*\backslash\{-7, 10\}), \mathcal{C}, \mathcal{D}$ and

$\langle\infty, 15, 20, 27, 19, 28, 16, 29, 14, 30, 18, 31, 17\rangle$,

$\langle 0, 18, 14, 5, 15, 4, 16, 10, 2, 11, 30, 12, 31\rangle$,

$\langle 0, 31, 1, 30, 11, 8, 12, 7, 13, 6, 14, 15, 18\rangle$,

$\langle 0, 31, 18, 30, 2, 29, 3, 28, 4, 27, 5, 26, 6\rangle$.

$RG = \langle 0, 18, 31\rangle \bigcup \langle 0, 31, 18\rangle \bigcup \langle 18, 30\rangle \bigcup \langle 11, 30\rangle$.

2° Case $L_1 = 4$

(1) $k = 8$ $(v = 12, 13)$

Since $\overline{\lambda} = 2$ and $R_1 = 4$, we only need to construct an 8-$PM(v)$ (with leave-arcs graph LG) and an 8-$CM(v)$ (with repeat-arcs graph RG) such that LG and RG are isomorphic.

$v = 12$, Points : $Z_{11} \bigcup \{\infty\}$.

$\mathcal{A} = \{\langle\infty, 1, 6, 2, 5, 3, 4, 8\rangle + i; i \in A\}$;

$\mathcal{B} = \{\langle 0, 10, 9, 8, 7, 6, 3, 5\rangle, \langle 0, 2, 4, 6, 8, 10, 1, 3\rangle, \langle 0, 8, 5, 2, 10, 7, 4, 1\rangle\}$.

8-$PM(12)$: $\mathcal{A}(A = Z_{11}^*), \mathcal{B}$ and $\langle 0, 6, 1, 7, 2, 8, 3, 9\rangle, \langle\infty, 1, 9, 6, 2, 5, 4, 8\rangle$,

$\langle 4, 3, 2, 1, 6, 5, 7, 9\rangle . LG = \langle 3, 4, 10, 5\rangle$.

8-$CM(12)$: $\mathcal{A}(A = Z_{11}), \mathcal{B}$ and $\langle 6, 1, 7, 2, 8, 3, 9, 4\rangle, \langle 5, 4, 3, 2, 1, 9, 0, 6\rangle$,

$\langle 4, 10, 5, 7, 9, 6, 0, 1\rangle . RG = \langle 0, 1, 4, 6\rangle$.

$v = 13$, Points : $Z_9 \bigcup \{\infty_1, \infty_2, \infty_3, \infty_4\}$.

$\mathcal{A} = \{\langle\infty_1, 0, \infty_2, 1, 5, 2, 4, 3\rangle + i; i \in Z_9\backslash\{1, 4\}\}$,

$\mathcal{B} = \{\langle\infty_3, 0, \infty_4, 3, 4, 2, 5, 1\rangle + i; i \in Z_9^*\}$,

$\mathcal{C} = \{\langle\infty_2, \infty_4, \infty_3, \infty_1, 4, 2, 5, 1\rangle, \langle\infty_1, \infty_4, \infty_2, 2, 6, 3, 5, 4\rangle$,

$\langle\infty_2, \infty_1, 1, \infty_3, 0, \infty_4, 3, 4\rangle\}$.

8-$PM(13)$: $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\langle\infty_1, \infty_3, \infty_2, 5, 0, 6, 8, 7\rangle$.

$LG = \langle\infty_1, \infty_2, \infty_3, \infty_4\rangle$.

8-$CM(13)$: $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\langle\infty_4, \infty_1, \infty_3, \infty_2, 5, 0, 6, 8\rangle$,

$\langle\infty_1, \infty_2, \infty_3, \infty_4, 2, 4, 8, 7\rangle . RG = \langle\infty_4, 2, 4, 8\rangle$.

(2) $k = 13$ $(v = 19, 21, 32, 34)$, $\overline{\lambda} = 13$.

λ	1	2	3	4	5	6
L_λ	4	8 L_1+L_1	12 L_1+L_2	3 L_2-R_2	7 L_1+L_4	11 L_1+L_5
R_λ	9	5 R_1-L_1	14 R_1+R_2	10 R_2+R_2	6 R_1-L_4	2 R_5-L_1

λ	7	8	9	10	11	12
L_λ	2 L_1-R_6	6 L_1+L_7	10 L_1+L_8	14 L_1+L_9	5 L_4+L_7	9 L_1+L_{11}
R_λ	11 R_1+R_6	7 R_2+R_6	3 R_2-L_7	12 R_5+R_5	8 R_5+R_6	4 R_6+R_6

It is not difficult to see that we only need to construct a 13-$PM(v)$ with $LG=\langle x,y\rangle\bigcup\langle x,u\rangle$ and a 13-$CM(v)$ with $RG=\langle x,y,z\rangle\bigcup\langle z,y,x\rangle\bigcup\langle x,u,v\rangle$, where $\{y,z\}\bigcap\{u,v\}=\phi$.

$v=19$, Points :Z_{19}.

$\mathcal{A}=\{\langle 0,5,1,4,2,15,14,16,13,17,12,18,11\rangle+i; i\in A\}$,

$\mathcal{B}=\{\langle 0,7,16,17,8,18,10,11,3,4,15,5,12\rangle+i; 1\le i\le 5\}$.

13-$PM(19)$: $\mathcal{A}(A=Z_{19}^*\backslash\{8,11,12\}),\mathcal{B}$ and

$\langle 0,7,16,17,8,18,10,11,3,4,13,9,12\rangle$,
$\langle 0,5,1,4,2,15,6,16,13,3,12,18,11\rangle$,
$\langle 0,8,13,17,12,10,4,3,5,2,6,1,7\rangle$,
$\langle 2,11,18,6,8,5,15,14,16,7,17,9,10\rangle$,
$\langle 3,11,16,12,17,13,7,6,15,5,9,4,10\rangle$,
$\langle 4,12,15,13,16,14,8,7,9,6,10,5,11\rangle$. $LG=\langle 4,14\rangle\bigcup\langle 5,14\rangle$.

13-$CM(19)$: $\mathcal{A}(A=Z_{19}^*),\mathcal{B}$ and

$\langle 0,7,16,17,8,18,6,15,5,14,4,13,1\rangle$,
$\langle 0,5,1,7,17,9,10,2,11,18,13,3,12\rangle$,
$\langle 0,1,18,10,11,3,4,14,5,15,6,16,7\rangle$,
$\langle 0,7,1,4,2,15,14,16,13,17,12,18,11\rangle$.
$RG=\langle 0,1,7\rangle\bigcup\langle 7,1,0\rangle\bigcup\langle 1,18,13\rangle$.

$v=21$, Points :$Z_{15}\bigcup\{a,b,c,d,e,f\}$.

$\mathcal{A}=\{\langle a,7,b,13,c,2,0,3,14,4,11,5,6\rangle+i; i\in A\}$,

$\mathcal{B}=\{\langle d,5,e,10,f,0,2,1,13,6,12,7,11\rangle+i; i\in B\}$.

$\mathcal{C}=\{\langle d,a,e,b,f,c,2,1,13,6,12,7,11\rangle,\langle a,14,d,b,e,c,f,0,3,4,11,5,6\rangle$,
$\langle c,b,d,e,f,a,7,10,6,11,3,12,13\rangle,\langle b,13,f,3,5,e,10,c,9,2,8,11,7\rangle$,
$\langle b,c,e,a,f,d,5,4,1,9,0,10,14\rangle,\langle a,5,b,11,c,0,13,12,2,9,3,14,4\rangle$,

$\langle a,d,f,b,5,c,10,8,3,7,12,4,13\rangle, \langle a,4,b,10,f,e,d,c,14,12,0,2,3\rangle,$
$\langle a,0,11,f,1,12,9,7,d,8,e,13,14\rangle, \langle b,6,c,d,7,e,12,f,11,1,8,2,0\rangle,$
$\langle a,12,b,3,c,7,13,8,4,9,1,10,11\rangle, \langle d,6,e,11,13,1,3,2,14,7,5,8,12\rangle\}.$

13-$PM(21)$: $\mathcal{A}(A = Z_{15}^*\backslash\{5,7,8,12,13\}), \mathcal{B}(B = Z_{15}^*\backslash\{1,2,3,11\}), \mathcal{C}$ and $\langle d,1,e,6,f,2,4,3,0,8,14,9,13\rangle$. $LG = \langle a,b\rangle \bigcup \langle a,c\rangle$.

13-$CM(21)$: $\mathcal{A}(A = Z_{15}^*\backslash\{5,7,8,12,13,14\}), \mathcal{B}(B = Z_{15}^*\backslash\{1,2,3,4,11,14\}), \mathcal{C}$ and

$\langle b,d,9,e,14,f,4,5,11,6,10,1,7\rangle, \langle b,a,c,1,14,2,13,3,10,d,7,4,6\rangle,$
$\langle c,a,b,7,d,4,e,9,f,14,1,0,12\rangle, \langle a,6,f,2,10,4,1,11,0,d,b,12,5\rangle,$
$\langle d,1,e,6,5,2,4,3,0,8,14,9,13\rangle$. $LG = \langle b,d,7\rangle \bigcup \langle d,b,7\rangle \bigcup \langle 1,7,4\rangle$.

$v = 32$, Points $: Z_{26} \bigcup \{a,b,c,d,e,f\}$.

$\mathcal{A} = \{\langle a,2,b,9,c,1,0,3,24,5,22,7,20\rangle + i; i \in A\}$,
$\mathcal{B} = \{\langle d,7,e,25,f,0,1,24,3,22,5,20,8\rangle + i; 1 \in Z_{26}^*\backslash\{1,23\}\}$.
$\mathcal{C} = \{\langle a,25,b,e,d,f,0,3,24,5,22,7,20\rangle,$
$\langle c,b,d,e,f,a,2,25,4,23,6,21,9\rangle,$
$\langle d,a,e,b,f,c,1,24,3,22,5,20,8\rangle,$
$\langle b,c,e,a,f,d,8,11,6,13,4,15,2\rangle,$
$\langle c,d,4,e,22,f,23,24,21,0,19,2,17\rangle,$
$\langle a,d,b,6,c,f,e,0,21,2,19,4,17\rangle,$
$\langle f,b,17,5,d,c,9,8,e,25,24,1,0\rangle,$
$\langle a,10,b,9,d,7,e,c,24,23,0,1,2\rangle,$
$[2]_{13},[-2]_{13},[4]_{13},[-4]_{13},[6]_{13},[-6]_{13},[8]_{13},[-8]_{13},[10]_{13},[-10]_{13}\}$.

13-$PM(32)$: $\mathcal{A}(A = Z_{26}^*\backslash\{8,23,24\}), \mathcal{B}, \mathcal{C}$ and
$\langle a,0,b,7,c,25,f,1,22,3,20,5,18\rangle, [12]_{13}$. $LG = \langle a,b\rangle \bigcup \langle a,c\rangle$.

13-$CM(32)$: $\mathcal{A}(A = Z_{26}^*\backslash\{1,8,23,24\}), \mathcal{B}, \mathcal{C}$ and
$\langle c,a,b,10,22,3,20,6,18,4,16,2,15\rangle,$
$\langle b,a,c,2,14,0,12,24,15,19,5,17,3\rangle,$
$\langle c,15,1,13,25,11,23,9,21,7,19,24,10\rangle,$
$\langle a,0,b,7,c,2,1,4,25,6,23,8,21\rangle,$
$\langle a,3,15,2,c,25,f,1,22,8,20,5,18\rangle.$
$RG = \langle c,2,15\rangle \bigcup \langle c,15,2\rangle \bigcup \langle 15,19,24\rangle$.

$v = 34$, Points $: Z_{28} \bigcup \{a,b,c,d,e,f\}$.

$\mathcal{A} = \{\langle a,1,b,26,c,0,13,14,12,15,11,16,2\rangle + i; i \in A\}$,
$\mathcal{B} = \{\langle d,12,e,4,f,13,0,7,27,8,26,9,25\rangle + i; i \in B\}\}$.

$\mathcal{C} = \{\langle 0,27,1,26,2,25,3,24,4,23,5,22,6\rangle + i, i \in Z_{28}\}$,

$\mathcal{D} = \{\langle c,b,d,e,f,a,1,8,0,9,27,10,26\rangle, \langle d,a,e,b,f,c,0,7,27,8,26,9,25\rangle,$

$\langle a,d,b,e,c,f,13,14,12,15,11,16,2\rangle,$

$\langle b,c,e,5,f,d,12,13,11,14,10,15,1\rangle,$

$\langle a,0,11,27,d,14,e,6,f,15,2,9,1\rangle,$

$\langle e,a,2,b,27,c,d,f,14,1,10,0,13\rangle,$

$\langle a,f,e,d,c,1,14,15,13,16,12,17,3\rangle,$

$\langle f,b,26,5,25,6,24,7,23,d,10,e,2\rangle\}$.

13-$PM(34)$: $\mathcal{A}(A = Z_{28}^*\backslash\{1,27\}), \mathcal{B}(B = Z_{28}^*\backslash\{1,2,26\}), \mathcal{C}, \mathcal{D}$ and

$\langle b,25,c,27,12,e,4,f,11,26,d,13,0\rangle$. $LG = \langle a,b\rangle \bigcup \langle a,c\rangle$.

13-$CM(34)$: $\mathcal{A}(A = Z_{28}^*\backslash\{1,26,27\}), \mathcal{B}(B = Z_{28}^*\backslash\{1,2,3,26\}), \mathcal{C}, \mathcal{D}$ and

$\langle b,a,c,2,15,16,14,17,13,18,4,26,0\rangle,$

$\langle a,3,b,0,d,15,e,7,f,11,26,16,4\rangle,$

$\langle c,a,b,26,4,16,3,10,2,11,1,12,0\rangle,$

$\langle b,25,c,27,12,e,4,f,16,26,d,13,0\rangle.$

$RG = \langle 4,16,26\rangle \bigcup \langle 4,26,16\rangle \bigcup \langle b,26,0\rangle$.

3° Case $L_1 = 5$

$k = 7$ ($v = 11, 18$), $\overline{\lambda} = 7$. By Corollary 2, we only need to construct an optimal 7-$PM(v)$ with $LG = C_3 \bigcup C_2$ and an optimal 7-$CM(v)$ with $RG = C_2$.

<u>$v = 11$, Points : $Z_7 \bigcup \{\infty_1, \infty_2, \infty_3, \infty_4\}$.</u>

$\mathcal{A} = \{\langle \infty_1, 1, 2, \infty_2, 3, 5, 4\rangle + i; i \in A\}$,

$\mathcal{B} = \{\langle \infty_3, 0, 4, \infty_4, 1, 6, 2\rangle + i; i \in B\}$,

$\mathcal{C} = \{\langle \infty_1, \infty_4, \infty_3, \infty_2, 3, 5, 4\rangle\}$.

7-$PM(11)$: $\mathcal{A}(A = Z_7^*), \mathcal{B}(B = Z_7^*), \mathcal{C}$ and

$\langle \infty_3, \infty_4, \infty_2, \infty_1, 1, 6, 2\rangle, \langle \infty_2, \infty_3, 0, 4, \infty_4, 1, 2\rangle$.

$LG = \langle \infty_1, \infty_2, \infty_4\rangle \bigcup \langle \infty_1, \infty_3\rangle$.

7-$CM(11)$: $\mathcal{A}(A = Z_7^*\backslash\{-1\}), \mathcal{B}(B = Z_7\backslash\{3\}), \mathcal{C}$ and

$\langle \infty_2, \infty_1, \infty_3, \infty_4, 0, 1, 2\rangle, \langle \infty_3, \infty_1, \infty_2, \infty_4, 4, 2, 5\rangle,$

$\langle \infty_4, \infty_1, 1, \infty_2, \infty_3, 3, 0\rangle, \langle \infty_4, \infty_2, 2, 4, 3, \infty_1, 0\rangle$.

$RG = \langle \infty_4, 0\rangle$.

<u>$v = 18$, Points : $Z_{14} \bigcup \{\infty_1, \infty_2, \infty_3, \infty_4\}$.</u>

$\mathcal{A} = \{\langle \infty_3, 0, 13, 1, 12, 2, 11\rangle + i; i \in A\}$.

$\mathcal{B} = \{\langle \infty_4, 1, 11, 2, 10, 3, 9\rangle + i; i \in B\}$,

$\mathcal{C} = \{\langle\infty_1, 10, 8, \infty_2, 12, 1, 2\rangle + i; i \in Z_{14}^*\}$,

$\mathcal{D} = \{\langle\infty_1, \infty_2, \infty_3, \infty_4, 1, 11, 2\rangle, \langle\infty_4, \infty_3, \infty_2, \infty_1, 10, 3, 9\rangle,$
$\langle\infty_3, \infty_1, \infty_4, \infty_2, 12, 2, 11\rangle\}$.

7-PM(18): $\mathcal{A}(A = Z_{14}^*\backslash\{1, 2\})$, $\mathcal{B}(B = Z_{14}^*\backslash\{1\})$, $\mathcal{C}, \mathcal{D}$ and
$\langle\infty_2, \infty_4, \infty_1, \infty_3, 2, 10, 8\rangle, \langle\infty_3, 0, 4, 13, 1, 3, 12\rangle,$
$\langle\infty_3, 1, 2, 12, 3, 0, 13\rangle, \langle\infty_4, 2, 13, 3, 11, 4, 10\rangle.$
$LG = \langle 0, 2, 1\rangle \bigcup \langle 1, 12\rangle.$

7-CM(18): $\mathcal{A}(A = Z_{14}^*), \mathcal{B}(B = Z_{14}^*), \mathcal{C}, \mathcal{D}$ and $\langle\infty_4, \infty_1, \infty_3, 0, 13, 1, 12\rangle,$
$\langle\infty_2, \infty_4, 12, 1, 2, 10, 8\rangle$. $RG = \langle\infty_4, 12\rangle$.

4° Case $L_1 = 7$

$k = 13$ $(v = 18, 22, 31, 35)$, $\overline{\lambda} = 13$.

λ	1	2	3	4	5	6
L_λ	7	14 $L_1 + L_1$	8 $L_2 - R_1$	2 $L_3 - R_1$	9 $L_1 + L_4$	3 $L_1 - R_5$
R_λ	6	12 $R_1 + R_1$	5 $R_2 - L_1$	11 $R_1 + R_3$	4 $R_1 - L_4$	10 $R_1 + R_5$

λ	7	8	9	10	11	12
L_λ	10 $L_1 + L_6$	4 $L_4 + L_4$	11 $L_4 + L_5$	5 $L_4 + L_6$	12 $L_5 + L_6$	6 $L_6 + L_6$
R_λ	3 $R_3 - L_4$	9 $R_3 + R_5$	2 $R_1 - L_8$	8 $R_1 + R_9$	14 $R_1 + R_{10}$	7 $R_3 + R_9$

It is not difficult to see that we only need to construct a 13-$PM(v)$ with $LG = \langle x, y\rangle \bigcup \langle y, z\rangle \bigcup \langle z, u, v\rangle$ and a 13-$CM(v)$ with $RG = DK_3$.

<u>$v = 18$, Points : Z_{18}.</u>

$\mathcal{A} = \{\langle 0, 4, 17, 5, 16, 6, 15, 7, 12, 8, 11, 9, 3\rangle + i; i \in A\}$,

$\mathcal{B} = \{\langle 0, 2, 4, 6, 8, 9, 10, 12, 13, 14, 7, 15, 17\rangle,$
$\langle 0, 17, 6, 13, 2, 9, 16, 5, 12, 1, 8, 14, 3\rangle,$
$\langle 0, 7, 9, 11, 13, 12, 8, 15, 6, 16, 3, 17, 2\rangle,$
$\langle 0, 6, 7, 14, 15, 4, 11, 9, 3, 10, 17, 1, 5\rangle,$
$\langle 0, 4, 17, 5, 16, 6, 15, 7, 12, 9, 13, 8, 11\rangle,$
$\langle 1, 3, 5, 6, 17, 7, 16, 8, 13, 9, 12, 10, 4\rangle\}$.

13-PM(18) : $\mathcal{A}(A = Z_{18}^*\backslash\{1, 9\}), \mathcal{B}$ and
$\langle 0, 1, 2, 3, 4, 5, 7, 8, 10, 11, 12, 14, 16\rangle,$
$\langle 0, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1\rangle.$
$LG = \langle 13, 15, 14\rangle \bigcup \langle 15, 16\rangle \bigcup \langle 16, 17\rangle.$

13-$CM(18)$: $\mathcal{A}(A = Z_{18}^*\backslash\{1,8,9,14\}), \mathcal{B}$ and

$\langle 0,12,11,10,9,8,7,6,17,4,3,2,1\rangle$,

$\langle 0,1,12,2,11,3,4,6,5,17,16,15,14\rangle$,

$\langle 0,13,1,2,3,8,12,7,5,4,17,14,16\rangle$,

$\langle 1,17,11,8,4,7,13,6,14,5,15,2,16\rangle$,

$\langle 4,5,7,8,10,11,12,14,13,15,16,17,6\rangle$.

$RG = \langle 4,6,17\rangle \bigcup \langle 4,17,6\rangle$.

$\underline{v = 22, \text{Points} : Z_{17} \bigcup \{a,b,c,d,e\}.}$

$\mathcal{A} = \{\langle a,8,b,2,1,3,0,4,16,5,14,7,15\rangle + i; i \in A\}$,

$\mathcal{B} = \{\langle c,5,d,11,e,0,1,16,2,15,3,14,4\rangle + i, i \in B\}$,

$\mathcal{C} = \{\langle e,b,d,a,c,5,7,4,8,3,9,1,11\rangle$,

$\langle d,c,e,a,8,b,2,1,3,0,4,16,5\rangle$,

$\langle a,12,d,1,e,7,8,6,5,10,4,11,2\rangle$,

$\langle a,e,d,11,c,12,b,6,9,5,14,7,15\rangle$.

13-$PM(22)$: $\mathcal{A}(A = Z_{17}^*\backslash\{4\}), \mathcal{B}(B = Z_{17}^*), \mathcal{C}$ and

$\langle c,a,d,b,e,0,1,16,2,15,3,14,4\rangle . LG = \langle a,b\rangle \bigcup \langle b,c\rangle \bigcup \langle c,d,e\rangle$.

13-$CM(22)$: $\mathcal{A}(A = Z_{17}^*\backslash\{4,10,15\}), \mathcal{B}(B = Z_{17}^*\backslash\{4,7,16\}), \mathcal{C}$ and

$\langle c,b,a,d,e,0,1,16,2,15,3,14,4\rangle$,

$\langle e,c,a,b,0,16,1,15,2,14,3,12,9\rangle$,

$\langle a,6,b,c,d,10,14,9,12,e,4,5,13\rangle$,

$\langle a,1,b,12,11,13,10,e,9,15,7,0,8\rangle$,

$\langle c,4,d,b,e,16,0,15,1,14,2,13,3\rangle$,

$\langle c,9,d,15,e,12,5,3,6,2,7,1,8\rangle$.

$RG = \langle e,9,12\rangle \bigcup \langle e,12,9\rangle$.

$\underline{v = 31, \text{Points} : Z_{26} \bigcup \{a,b,c,d,e\}.}$

$\mathcal{A} = \{\langle c,24,d,7,e,0,25,2,23,4,21,6,22\rangle + i; i \in A\}$,

$\mathcal{B} = \{\langle a,9,b,0,1,24,3,22,5,20,6,19,7\rangle + i, i \in Z_{26}^*\backslash\{4\}\}$,

$\mathcal{C} = \{\langle c,a,d,b,e,0,25,2,23,4,21,6,22\rangle$,

$\langle e,b,d,a,c,24,3,22,5,20,6,19,7\rangle$,

$\langle d,c,e,a,13,b,4,5,2,7,0,9,24\rangle$,

$\langle a,e,d,7,6,9,b,0,1,24,10,23,11\rangle$,

$\langle c,5,d,14,e,7,a,9,4,11,2,13,3\rangle$,

$[2]_{13}, [-2]_{13}, [4]_{13}, [-4]_{13}, [6]_{13}, [8]_{13}, [-6]_{13}^1, [-8]_{13}, [10]_{13}\}$.

13-$PM(31)$: $\mathcal{A}(A = Z_{26}^*\backslash\{7\}), \mathcal{B}, \mathcal{C}$ and

$\langle 0,20,14,8,2,22,16,10,4,24,18,12,6\rangle$. $LG = \langle a,b\rangle \bigcup \langle b,c\rangle \bigcup \langle c,d,e\rangle$.

13-$CM(31)$: $\mathcal{A}(A = Z_{26}^{*} \backslash \{1,3,7\}), \mathcal{B}, \mathcal{C}$ and

$\langle c,25,d,8,2,22,16,10,4,24,9,7,23\rangle, \langle a,9,25,c,1,d,10,e,3,2,5,0,7\rangle,$

$\langle a,b,c,d,e,1,0,3,24,5,22,7,9\rangle, \langle e,c,b,a,7,24,18,12,6,0,20,14,8\rangle.$

$RG = \langle a,7,9\rangle \bigcup \langle a,9,7\rangle$.

$v = 35$, Points : $Z_{30} \bigcup \{a,b,c,d,e\}$.

$\mathcal{A} = \{\langle a,6,5,7,4,8,3,9,2,10,1,11,0\rangle + i; i \in A\},$

$\mathcal{B} = \{\langle c,5,d,4,e,0,12,29,13,28,14,27,15\rangle + i, i \in B\},$

$\mathcal{C} = \{\langle b,1,2,0,3,29,4,28,5,27,6,26,7\rangle + i, i \in Z_{30}^{*}\},$

$\mathcal{D} = \{\langle c,a,d,b,e,0,12,29,13,28,14,27,15\rangle,$

$\langle d,a,c,e,b,1,2,0,3,29,4,28,5\rangle,$

$\langle a,e,d,c,5,27,22,28,21,29,20,0,19\rangle,$

$\langle b,d,4,e,a,25,24,26,23,27,6,5,7\rangle,$

$\langle a,6,26,7,4,8,3,9,2,10,1,11,0\rangle\}.$

13-$PM(35)$: $\mathcal{A}(A = Z_{30}^{*} \backslash \{19\}), \mathcal{B}(B = Z_{30}^{*}), \mathcal{C}$ and $\mathcal{D}$.

$LG = \langle a,b\rangle \bigcup \langle b,c\rangle \bigcup \langle c,d,e\rangle$.

13-$CM(35)$: $\mathcal{A}(A = Z_{30}^{*} \backslash \{3,19,24\}), \mathcal{B}(B = Z_{30}^{*} \backslash \{1\}), \mathcal{C}, \mathcal{D}$ and

$\langle a,b,c,d,e,13,27,3,26,4,25,5,24\rangle,$

$\langle c,6,d,5,e,27,13,0,14,29,1,28,16\rangle,$

$\langle e,c,b,a,9,8,10,7,11,6,12,5,13\rangle,$

$\langle a,0,29,15,28,2,27,e,1,13,4,14,3\rangle.$

$RG = \langle e,13,27\rangle \bigcup \langle e,27,13\rangle$.

5° **Case** $L_1 = 8$

(1) $k = 11$ $(v = 17,28), \overline{\lambda} = 11$.

λ	1	2	3	4	5
L_λ	8	5 $L_1 - R_1$	2 $L_2 - R_1$	10 $L_1 + L_3$	7 $L_2 + L_3$
R_λ	3	6 $R_1 + R_1$	9 $R_2 + R_1$	12 $R_2 + R_2$	4 $R_2 - L_3$

λ	6	7	8	9	10
L_λ	4 $L_3 + L_3$	12 $L_1 + L_6$	9 $L_2 + L_6$	6 $L_3 + L_6$	3 $L_9 - R_1$
R_λ	7 $R_1 + R_5$	10 $R_2 + R_5$	2 $R_2 - L_6$	5 $R_1 + R_8$	8 $R_1 + R_9$

Obviously, if we construct an optimal 11-$PM(v)$ with $LG = DK_3 \bigcup C_2$ and an optimal 11-$CM(v)$ with $RG = C_3$, then all optimal 11-$PM_\lambda(v)$ and 11-$CM_\lambda(v)$ can be obtained by Theorem 4.

$\underline{v = 17, \text{Points}: Z_{11} \bigcup \{\infty_1, ..., \infty_6\}.}$

$\mathcal{A} = \{\langle\infty_1, 0, \infty_2, 1, \infty_3, 7, 8, 10, 5, 3, 2\rangle + i; i \in A\}$,

$\mathcal{B} = \{\langle\infty_4, 6, \infty_5, 7, \infty_6, 8, 1, 9, 3, 10, 2\rangle + i; i \in B\}$,

$\mathcal{C} = \{\langle\infty_1, \infty_4, \infty_2, \infty_5, \infty_3, \infty_6, 8, 10, 5, 3, 2\rangle$,
$\langle\infty_1, \infty_5, \infty_2, \infty_6, \infty_3, 8, 9, 0, 6, 4, 3\rangle$,
$\langle\infty_3, \infty_5, \infty_4, \infty_1, \infty_6, \infty_2, 1, 9, 3, 10, 2\rangle$,
$\langle\infty_4, \infty_6, \infty_5, \infty_1, 3, \infty_2, 4, \infty_3, 10, 0, 2\rangle\}$.

11-$PM(17)$: $\mathcal{A}(A = Z_{11}^* \backslash \{1, 3, 8\}), \mathcal{B}(B = Z_{11}^* \backslash \{2\}), \mathcal{C}$ and
$\langle\infty_6, \infty_4, \infty_5, 7, 2, 0, 10, \infty_1, 8, \infty_2, 9\rangle$,
$\langle\infty_2, \infty_4, 6, \infty_5, 9, \infty_3, 4, 5, 7, 8, 1\rangle$,
$\langle\infty_6, \infty_1, 0, \infty_2, 2, 8, 6, 5, 1, \infty_3, 7\rangle$,
$\langle\infty_5, \infty_6, 10, 3, 0, 5, \infty_1, 1, 4, \infty_4, 8\rangle$.
$LG = \langle\infty_1, \infty_2, \infty_3\rangle \bigcup \langle\infty_1, \infty_3, \infty_2\rangle \bigcup \langle\infty_3, \infty_4\rangle$.

11-$CM(17)$: $\mathcal{A}(A = Z_{11}^* \backslash \{1, 3, 5\}), \mathcal{B}(B = Z_{11}^* \backslash \{-1, 3\}), \mathcal{C}$ and
$\langle\infty_2, \infty_4, \infty_3, \infty_1, \infty_5, 7, 8, 1, 6, 2, 5\rangle$,
$\langle\infty_6, \infty_1, \infty_3, \infty_2, 2, 9, 1, \infty_4, 5, \infty_5, 6\rangle$,
$\langle\infty_1, \infty_2, \infty_3, 7, \infty_6, \infty_4, \infty_5, 2, 8, 6, 5\rangle$,
$\langle\infty_1, 1, \infty_3, \infty_4, 6, \infty_5, \infty_6, 7, 0, 8, 2\rangle$,
$\langle\infty_2, \infty_1, 5, \infty_4, 9, \infty_5, 10, \infty_6, 0, 4, 1\rangle$,
$\langle\infty_1, 0, \infty_2, 6, \infty_3, 1, 2, 4, 10, 8, 7\rangle$.
$RG = \langle\infty_1, \infty_5, 2\rangle$.

$\underline{v = 28, \text{Points}: Z_{22} \bigcup \{\infty_1, ..., \infty_6\}.}$

$\mathcal{A} = \{\langle 0, 20, 1, 19, 2, 18, 3, 17, 4, 16, 5\rangle + i; i \in A\}$,

$\mathcal{B} = \{\langle\infty_1, 0, \infty_2, 1, \infty_3, 16, 4, 17, 19, 3, 7\rangle + i; i \in B\}$,

$\mathcal{C} = \{\langle\infty_4, 5, \infty_5, 10, \infty_6, 4, 1, 0, 8, 9, 2\rangle + i; i \in C\}$,

$\mathcal{D} = \{\langle\infty_1, \infty_4, \infty_2, \infty_5, \infty_3, \infty_6, 4, 17, 19, 3, 7\rangle$,
$\langle\infty_3, \infty_5, \infty_4, \infty_1, \infty_6, \infty_2, 1, 0, 8, 9, 2\rangle$,
$\langle\infty_1, \infty_5, \infty_2, \infty_6, \infty_3, 17, 5, 18, 20, 4, 8\rangle\}$.

11-$PM(28)$: $\mathcal{A}(A = Z_{22}^*), \mathcal{B}(B = Z_{22}^* \backslash \{1, 2, -7\}), \mathcal{C}(C = Z_{22}^* \backslash \{-1\}), \mathcal{D}$ and

$\langle\infty_2, \infty_4, \infty_6, \infty_5, \infty_1, 2, 18, 3, 17, 4, 1\rangle$,

$\langle\infty_4,\infty_5,\infty_6,\infty_1,1,\infty_3,16,5,0,\infty_2,2\rangle$,
$\langle\infty_6,\infty_4,4,16,\infty_3,18,6,19,21,5,9\rangle$,
$\langle\infty_1,15,\infty_2,3,\infty_3,9,19,10,12,18,0\rangle$,
$\langle\infty_1,0,20,1,19,2,\infty_2,16,4,\infty_5,9\rangle$,
$\langle\infty_4,5,\infty_5,10,\infty_6,3,0,21,7,8,1\rangle$.
$LG=\langle\infty_1,\infty_2,\infty_3\rangle\bigcup\langle\infty_1,\infty_3,\infty_2\rangle\bigcup\langle\infty_3,\infty_4\rangle$.

11-$CM(28)$: $\mathcal{A}(A=Z_{22}^*\backslash\{-2,10\}),\mathcal{B}(B=Z_{22}^*\backslash\{1\}),\mathcal{C}(C=Z_{22}^*),\mathcal{D}$ and

$\langle\infty_2,\infty_4,\infty_6,\infty_5,\infty_1,\infty_3,16,5,0,20,1\rangle$,
$\langle\infty_6,\infty_4,\infty_3,\infty_2,\infty_1,0,5,14,4,15,10\rangle$,
$\langle\infty_2,\infty_3,\infty_4,5,\infty_5,\infty_6,\infty_1,1,19,2,0\rangle$,
$\langle\infty_3,\infty_1,\infty_2,2,18,3,17,0,16,4,1\rangle$,
$\langle\infty_4,\infty_5,10,8,11,7,12,6,13,5,2\rangle$,
$\langle 1,15,2,14,3,20,18,21,17,4,16\rangle$. $RG=\langle 0,5,2\rangle$.

(2) $k=12$ ($v=17,20$)

An optimal 12-$CM(17)$ with $RG=C_4$ has been obtained from Theorem 3 in [7]. And , an optimal 12-$PM(17)$ with $LG=2C_4$ has been obtained in [8]. Since $\overline{\lambda}$, $(L_1,R_1)=(8,4)$ and $(L_2,R_2)=(4,8)$, all optimal 12-$PM_\lambda(17)$ and 12-$CM_\lambda(17)$ for any λ can be obtained.

An optimal 12-$CM(20)$ with $RG=2C_2$ can be found from Theorem 3 in [7]. To obtain an optimal 12-$PM_2(20)$ by Theorem 4, we need construct an optimal 12-$PM(20)$ with $LG=4C_2$:

$\langle 0,19,2,18,3,17,4,16,5,15,6,14\rangle$ develop 20,
$\langle 1,5,7,4,17,10,8,12,13,14,11,6\rangle+2i \pmod{20}$, $0\le i\le 5$,
$\langle 1,19,16,9,10,7,2,17,15,13,6,3\rangle$, $\langle 0,13,17,1,18,11,4,8,9,7,5,3\rangle$,
$\langle 0,4,5,2,15,8,6,10,12,14,16,18\rangle$, $\langle 0,15,19,3,18,13,11,9,2,6,8,5\rangle$,
$\langle 1,3,5,6,7,8,10,11,12,9,4,19\rangle$. $LG=\langle 0,2\rangle\bigcup\langle 2,4\rangle\bigcup\langle 4,6\rangle\bigcup\langle 17,19\rangle$.

6° **Case** $L_1=9$

$k=11$ ($v=16,18,27,29$), $\overline{\lambda}=11$. By Corollary 2, we only need to construct an optimal 11-$PM(v)$ with $LG=DK_3\bigcup C_3$ and an optimal 11-$CM(v)$ with $RG=C_2$.

$v=16$, Points : $Z_{15}\bigcup\{\infty\}$.

$\mathcal{A}=\{\langle\infty_1,13,0,12,1,11,2,10,4,9,5\rangle+i;i\in A\}$,
$\mathcal{B}=\{\langle 1,8,7,5,6,9,10,13,12,4,3\rangle+i;0\le i\le 2\}$,

11-$PM(16)$: $\mathcal{A}(A = Z_{15}^*\backslash\{2\}), \mathcal{B}$ and

$\langle\infty, 0, 2, 14, 3, 13, 1, 4, 12, 11, 7\rangle, \langle\infty, 13, 0, 12, 6, 11, 2, 10, 4, 9, 5\rangle,$
$\langle 0, 3, 4, 7, 6, 13, 11, 9, 12, 10, 2\rangle, \langle 1, 14, 2, 5, 12, 13, 4, 11, 10, 8, 9\rangle,$
$\langle 0, 13, 14, 12, 1, 11, 3, 6, 4, 5, 8\rangle.$

$LG = \langle 0, 7, 14\rangle \bigcup \langle 0, 1\rangle \bigcup \langle 1, 2\rangle \bigcup \langle 2, 3\rangle.$

11-$CM(16)$: $\mathcal{A}(A = Z_{15}\backslash\{5\}), \mathcal{B}$ and

$\langle\infty, 3, 5, 2, 6, 1, 7, 0, 9, 12, 10\rangle, \langle 0, 1, 2, 3, 6, 4, 11, 10, 8, 7, 14\rangle,$
$\langle 0, 3, 4, 7, 6, 13, 11, 9, 1, 14, 2\rangle, \langle 0, 7, 8, 9, 14, 10, 2, 5, 12, 13, 1\rangle,$
$\langle 0, 13, 14, 12, 11, 3, 2, 1, 4, 5, 8\rangle.$ $\quad RG = \langle 7, 8\rangle.$

$v = 18$, Points : $Z_{13} \bigcup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}.$

$\mathcal{A} = \{\langle\infty_1, 7, \infty_2, 5, \infty_3, 2, 6, 11, 8, 3, 10\rangle + i; i \in A\},$
$\mathcal{B} = \{\langle\infty_4, 0, \infty_5, 1, 12, 2, 11, 10, 3, 4, 6\rangle + i; i \in B\},$
$\mathcal{C} = \{\langle\infty_1, \infty_3, \infty_5, \infty_2, \infty_4, 3, 7, 12, 9, 4, 11\rangle,$
$\langle\infty_1, \infty_4, \infty_2, \infty_5, \infty_3, 2, 6, 11, 8, 3, 10\rangle,$
$\langle\infty_4, \infty_5, \infty_1, 7, \infty_2, 5, \infty_3, 3, 9, 10, 12\rangle,$
$\langle\infty_4, \infty_1, \infty_5, 1, 12, 2, 11, 10, 3, 4, 6\rangle\}.$

11-$PM(18)$: $\mathcal{A}(A = Z_{13}^*\backslash\{1\}), \mathcal{B}(B = Z_{13}^*\backslash\{3, 6\}), \mathcal{C}$ and

$\langle\infty_3, \infty_1, 8, 4, 3, \infty_5, 7, 5, 1, 0, 6\rangle, \langle\infty_2, 6, 7, 9, \infty_4, 0, \infty_5, 4, 2, 5, 8\rangle.$

$LG = \langle\infty_5, \infty_4, 6\rangle \bigcup \langle\infty_1, \infty_2\rangle \bigcup \langle\infty_2, \infty_3\rangle \bigcup \langle\infty_3, \infty_4\rangle.$

11-$CM(18)$: $\mathcal{A}(A = Z_{13}^*\backslash\{1, 2\}), \mathcal{B}(B = Z_{13}^*\backslash\{1, 3, 6\}), \mathcal{C}$ and

$\langle\infty_5, \infty_4, \infty_3, \infty_2, \infty_1, 8, 4, 2, 5, 1, 0\rangle,$
$\langle\infty_2, \infty_3, \infty_1, 6, 7, 9, \infty_4, 0, 10, 5, 8\rangle,$
$\langle\infty_1, \infty_2, 7, \infty_3, \infty_4, 1, \infty_5, 4, 8, 0, 6\rangle,$
$\langle\infty_1, 9, \infty_2, 6, \infty_3, 4, 3, \infty_5, 7, 5, 12\rangle,$
$\langle\infty_4, 6, \infty_5, 2, 0, 3, 12, 11, 4, 5, 7\rangle.$ $\quad RG = \langle\infty_1, 6\rangle.$

$v = 27$, Points : $Z_{20} \bigcup \{a, b, c, d, e, f, g\}.$

$\mathcal{A} = \{\langle 0, 18, 1, 17, 2, 16, 3, 15, 4, 14, 5\rangle + i; i \in A\},$
$\mathcal{B} = \{\langle a, 18, b, 12, c, 2, 10, 11, 13, 17, 3\rangle + i; i \in B\},$
$\mathcal{C} = \{\langle d, 14, e, 13, f, 12, g, 11, 10, 7, 0\rangle + i; i \in C\},$
$\mathcal{D} = \{\langle a, c, e, d, f, g, b, 12, 14, 18, 4\rangle, \langle d, a, e, b, f, c, g, 11, 10, 7, 0\rangle,$
$\langle d, b, e, c, f, a, g, 12, 11, 8, 1\rangle, \langle a, f, b, g, c, 2, 10, 11, 13, 17, 3\rangle,$
$\langle g, e, a, d, 14, f, 13, c, 3, 11, 12\rangle, \langle c, a, 19, b, d, 15, e, 13, g, f, 12\rangle\}.$

11-$PM(27)$: $\mathcal{A}(A = Z_{20}^*), \mathcal{B}(B = Z_{20}^*\backslash\{1\}), \mathcal{C}(C = Z_{20}^*\backslash\{1, 4\}), \mathcal{D}$ and

$\langle d, g, a, 18, b, 13, f, e, 14, 11, 4\rangle, \langle e, g, d, 18, 1, 17, f, 16, 3, 15, 14\rangle,$

$\langle e,17,2,16,g,15,4,14,5,0,18\rangle$,
$LG = \langle a,b\rangle \bigcup \langle b,c\rangle \bigcup \langle c,d\rangle \bigcup \langle d,e,f\rangle$.

11-$CM(27)$: $\mathcal{A}(A = Z_{20}\backslash\{-3\}), \mathcal{B}(B = Z_{20}^*\backslash\{1,3\}), \mathcal{C}(C = Z_{20}^*\backslash\{1\})$, $\mathcal{D}$ and

$\langle f,e,g,d,c,b,a,18,14,19,13\rangle, \langle e,f,d,g,a,18,b,c,5,13,14\rangle$,
$\langle c,d,e,14,16,0,6,a,1,b,15\rangle, \langle a,b,13,0,12,1,11,2,17,15,18\rangle$.
$RG = \langle a,18\rangle$.

$v = 29$, Points :$Z_{24} \bigcup \{a,b,c,d,e\}$.

$\mathcal{A} = \{\langle a,2,b,17,5,18,4,19,3,20,0\rangle + i; i \in A\}$,
$\mathcal{B} = \{\langle 0,22,1,21,2,20,3,19,4,18,5\rangle + i; i \in Z_{24}^*\}$,
$\mathcal{C} = \{\langle c,0,d,18,e,17,16,13,19,20,22\rangle + i; i \in Z_{24}^*\backslash\{1\}\}$,
$\mathcal{D} = \{\langle c,a,d,b,e,17,16,13,19,20,22\rangle, \langle a,c,e,b,d,18,4,19,3,20,0\rangle$,
$\langle d,a,e,18,17,14,20,21,23,c,1\rangle, \langle e,c,0,22,1,21,2,b,17,5,18\rangle\}$,

11-$PM(29)$: $\mathcal{A}(A = Z_{20}^*), \mathcal{B}, \mathcal{C}, \mathcal{D}$ and
$\langle d,e,a,2,20,3,19,4,18,5,0\rangle$, $LG = \langle a,b\rangle \bigcup \langle b,c\rangle \bigcup \langle c,d\rangle \bigcup \langle e,d,19\rangle$.

11-$CM(29)$: $\mathcal{A}(A = Z_{24}^*\backslash\{4\}), \mathcal{B}, \mathcal{C}, \mathcal{D}$ and
$\langle e,d,c,b,a,2,20,3,19,4,18\rangle, \langle d,e,a,b,21,9,22,8,23,7,0\rangle$,
$\langle b,c,d,19,e,18,5,0,4,a,6\rangle$. $RG = \langle e,18\rangle$. □

8 Conclusion

Theorem 8. For $5 \le k \le 14$, $v \ge k$ and any positive integer λ, there exists an optimal k-$PM_\lambda(v)$ and an optimal k-$CM_\lambda(v)$, except for $(v,k,\lambda) = (6,6,1)$.

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