λ-fold $K_{2,5}$-designs

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Abstract

In this note, we present necessary conditions for decomposing λK_n into copies of $K_{2,5}$, and show that these conditions are sufficient except for $\lambda = 5$ and $n = 8$, and possibly for the following cases: $\lambda = 1$ and $n = 40$; and $\lambda = 3$ and $n = 16$ or 20.

1 Introduction

Let λK_n denote the complete multigraph of order n in which exactly λ edges join each pair of vertices. A *G-decomposition* of a (multi)graph H is a partition of the edge set of H into isomorphic copies of G. In particular, a *G-design* of order n and index λ (or *λ-fold G-design* of order n) is a partition of the edge set of λK_n into copies of a graph G. Many have studied the *spectrum problem* for G-designs, which can be stated as follows: "For which n and λ does a λ-fold G-design of order n exist?" For example, the spectrum problem for stars [9], complete simple graphs on less than six vertices [6], and other simple connected graphs on less than five vertices as well as $K_{2,3}$ [3, 7] has been solved for all λ while the spectrum problem for paths [10], cycles of length at most 50 [1] and various other small graphs has been solved for $\lambda = 1$ [2, 3].

For a complete multigraph λK_n to be decomposed into isomorphic copies of a bipartite graph $K_{a,b}$ $(a \leq b)$ there are some obvious necessary conditions. First, the number of edges of λK_n must be divisible by the number of edges of $K_{a,b}$, so $2ab|\lambda n(n-1)$. Second, the degree of each vertex of λK_n must equal some non-negative, integer, linear combination of the degrees of the vertices of $K_{a,b}$, so $\lambda(n-1) = ax + by$ for some non-negative integers x and y. Therefore any common divisor of a and b must also be a divisor of $\lambda(n-1)$, so, in particular, $\gcd(a,b)\,|\lambda(n-1)$.

The case when $a = 1$ (these bipartite graphs are also known as stars) was completely solved by M. Tarsi [9]. When $a, b \geq 2$, a theorem of Graham and Pollak asserts that the complete graph K_n can be edge-partitioned into no fewer than $n-1$ complete bipartite graphs [5]. However, DeCaen and Hoffman have shown the impossibility of decomposing K_n into exactly $n-1$ complete bipartite graphs [4]. Thus, $\frac{n(n-1)}{2ab} > n-1$ or $n > 2ab$.

A related problem is to decompose a bipartite graph $K_{c,d}$ into isomorphic copies of the bipartite graph $K_{a,b}$. The following theorem is due to Hoffman and Liatti [8].

Theorem 1.1 *Let $a, b, c,$ and d be positive integers. Let $g = \gcd(a,b)$; let e, f be integers satisfying $ae - bf = g$; and let $h = ae + bf$. For each integer x, let $\alpha(x) = \left\lceil \frac{xf}{a} \right\rceil$, $\beta(x) = \left\lfloor \frac{xe}{b} \right\rfloor$ and $\gamma(x) = \frac{x}{ab}$.*

Then the edges of the complete bipartite graph $K_{c,d}$ can be partitioned into copies of the complete bipartite graph $K_{a,b}$ if and only if the following conditions are true:

(i) $ab|cd$;

(ii) $g|c$ *and* $\alpha(c) \leq \beta(c)$;

(iii) $g|d$ *and* $\alpha(d) \leq \beta(d)$; *and*

(iv) $c\alpha(d) + d\alpha(c) \leq h\gamma(cd) \leq c\beta(d) + d\beta(c)$.

2 Preliminary Results

We begin with a recursive construction for decomposing complete graphs into $K_{a,b}$, where $2 \leq a \leq b$. We denote the copy of $K_{a,b}$ in Figure 1 by $(1, 2, 3, ..., a) - (a+1, a+2, ..., a+b)$.

Lemma 2.1 *If K_n has a $K_{a,b}$-decomposition then so does K_{n+2ab}.*

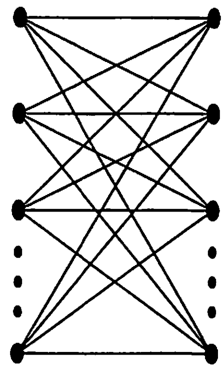

Figure 1: Block of $K_{a,b}$

Proof: Let $G = K_{a,b}$, and let K_{n+2ab} be defined on the vertex set $X \cup \mathbb{Z}_{2ab+1}$, where X is an $(n-1)$-set. By assumption, there is a $K_{a,b}$-decomposition of K_n defined on $X \cup \{0\}$. The following base block, when developed (mod $2ab+1$), gives a $K_{a,b}$-decompostion of K_{2ab+1} defined on $\mathbb{Z}_{2ab+1}$: $(b, 2b, 3b, ..., ab) - (0, 1, 2, ..., b-1)$. Theorem 1.1 guarantees a $K_{a,b}$-decomposition of $K_{n-1,2ab}$. □

Since we do not have an analogous result to Theorem 1.1 for $\lambda > 1$, we now restrict our attention to the case when $a = 2$ and $b = 5$. We will make use of the following proposition.

Proposition 2.2 *Suppose $n \geq 5$. If λn is divisible by 2 or by 5, then there exists a $K_{2,5}$-decomposition of $\lambda K_{n,20}$.*

Proof: Define $V(\lambda K_{n,20})$ with bipartition (X, Y), where $X = \mathbb{Z}_n$ and $Y = \{\infty_0, \infty_1, \ldots, \infty_{19}\}$. Then if $2 \mid \lambda n$, then $B = \{(2j, 2j+1) - (\infty_{5i}, \infty_{5i+1}, \infty_{5i+2}, \infty_{5i+3}, \infty_{5i+4}) \mid 0 \leq j \leq (\lambda n - 2)/2, 0 \leq i \leq 3\}$, where all sums involving j are reduced modulo n, is the desired set of blocks for the decomposition.

If $5 \mid \lambda n$, then $B = \{(\infty_{2i}, \infty_{2i+1}) - (5j, 5j+1, 5j+2, 5j+3, 5j+4) \mid 0 \leq i \leq 9, 0 \leq j \leq (\lambda n - 5)/5\}$, where all sums involving j are reduced modulo n, is the desired set of blocks for the decomposition. □

This proposition is useful in the proof of the following.

Proposition 2.3 *Suppose $n \geq 5$, $\lambda \geq 2$, and $\lambda \neq 3$. If λn is divisible by 2 or 5, and if there exists a λ-fold $K_{2,5}$-design of order n, then there exists a λ-fold $K_{2,5}$-design of λK_{n+20}. Furthermore, if $\lambda = 3$, if $n \equiv 0, 1, 5,$ or 16 (mod 20), and if there exists a $K_{2,5}$-design of λK_n, then there is a λ-fold $K_{2,5}$-design of λK_{n+20}*

Proof: Let $G = K_{2,5}$, and let K_{n+20} be defined on the vertex set $Y \cup \mathbb{Z}_{20}$, where Y is an n-set. By assumption, there is a $K_{2,5}$-decomposition of λK_n defined on Y. Furthermore, $K_{2,5}$-decompositions of $2K_{20}$ and $5K_{20}$ can be found in the appendix. More generally, if $\lambda \geq 2$ and $\lambda \neq 3$, then one can obtain a $K_{2,5}$-decomposition of λK_{20} by taking $\lambda/2$ copies of a $K_{2,5}$-decomposition of $2K_{20}$ if λ is even and by taking one copy of a $K_{2,5}$-decomposition of $5K_{20}$ and $(\lambda - 5)/2$ copies of a $K_{2,5}$-decomposition of $2K_{20}$ if λ is odd. Proposition 2.2 guarantees a decomposition of $\lambda K_{n,20}$.

Now suppose $\lambda = 3$. Let K_{n+20} be defined on the vertex set $X \cup \mathbb{Z}_{21}$, where X is an $(n-1)$-set. By assumption, there is a $K_{2,5}$-decomposition of $3K_n$ defined on $X \cup \{0\}$. The following base block gives a $K_{2,5}$-decompostion of K_{21} defined on $\mathbb{Z}_{21}$ when developed (mod 21): $(5, 10) - (0, 1, 2, 3, 4)$. Putting together three copies of this decomposition gives a $K_{2,5}$-decomposition of $3K_{21}$. Theorem 1.1 guarantees a $K_{2,5}$-decomposition of $K_{n-1,20}$ and, thus, a $K_{2,5}$-decomposition of $3K_{n-1,20}$. □

3 Constructing λ-fold $K_{2,5}$-designs

In this section we construct λ-fold G-designs, where $G = K_{2,5}$.

The following lemma states necessary conditions for the existence of λ-fold $K_{2,5}$-designs of order n.

Lemma 3.1 *Let n and λ be positive integers. If there exists a λ-fold $K_{2,5}$-design of order n, and:*

(i) if $\lambda \equiv 1, 3, 7,$ or 9 (mod 10), then $n \equiv 0, 1, 5,$ or 16 (mod 20), $n \geq 21$ if $\lambda = 1$, and $n \geq 16$ if $\lambda > 1$;

(ii) if $\lambda \equiv 2, 4, 6,$ or 8 (mod 10), then $n \equiv 0$ or 1 (mod 5), $n \geq 10$;

(iii) if $\lambda \equiv 5$ (mod 10), then $n \equiv 0$ or 1 (mod 4), $n \geq 8$; and

(iv) if $\lambda \equiv 0$ (mod 10), then $n \geq 7$.

Proof: If there exists a $K_{2,5}$-decomposition of λK_n, then the number of edges in λK_n must be divisible by 10, and certainly $n \geq 7$. This implies that $\frac{\lambda n(n-1)}{20}$ is an integer. We consider each case in turn.

Case 1: $\lambda \equiv 1, 3, 7,$ or 9 (mod 10).

In this case, 20 and λ are relatively prime, so we need only consider when 20 divides $n(n-1)$. This is precisely when $n \equiv 0, 1, 5,$ or 16 (mod 20). Certainly, since $n \geq 7$ and $n \equiv 0, 1, 5,$ or 16 (mod 20), it follows that $n \geq 16$. However, by a theorem of DeCaen and Hoffman [4], $n \geq 21$ when $\lambda = 1$.

Case 2: $\lambda \equiv 2, 4, 6,$ or 8 (mod 10).

If $\lambda \equiv 2, 6, 14$, or 18 (mod 20), then $\gcd(20, \lambda) = 2$. Therefore, for these cases it suffices to consider when 10 divides $n(n-1)$. If $\lambda \equiv 4, 8, 12$, or 16 (mod 20), then $\gcd(20, \lambda) = 4$, so we need only consider when 5 divides $n(n-1)$. In all cases, this means that $n \equiv 0$ or 1 (mod 5). Since $n \geq 7$, and $n \equiv 0$ or 1 (mod 5), it follows that $n \geq 10$.

Case 3: $\lambda \equiv 5$ (mod 10).

If $\lambda \equiv 5$ (mod 10), then $\gcd(20, \lambda) = 5$. It follows that we need only consider when 4 divides $n(n-1)$. This means that $n \equiv 0$ or 1 (mod 4), and since $n \geq 7$, it follows that $n \geq 8$.

Case 4: $\lambda \equiv 0$ (mod 10).

If $\lambda \equiv 0$ (mod 10), then $\gcd(20, \lambda) = 10$ or 20, so it suffices to consider when 2 divides $n(n-1)$. Therefore, the only limitation is that $n \geq 7$. □

Interestingly enough, the obvious necessary conditions are not always sufficient, as we see from the following lemma.

Lemma 3.2 *There exists no $K_{2,5}$-design of order 8 and index 5.*

Proof: We assume that such a design $D = (S, B)$ exists, and work toward a contradiction. This design D has $|S| = 8$, and $|B| = 14$. Furthermore, since $\deg(v) = 35$ for every vertex v in $5K_8$, it follows that each vertex must appear as a vertex of degree 5 in an odd number of blocks of B. Let $d_2(v)$ and $d_5(v)$ denote the number of blocks of B in which v is a vertex of degree 2 and of degree 5, respectively. Suppose for some vertex x that $d_5(x) = 1$. Then $d_2(x) = 15$, which is impossible since $|B| = 14$. Therefore, $d_5(v) \geq 3$ for each $v \in S$. Since $|B| = 14$, it follows that there is exactly one vertex x for which $d_5(x) = 7$ or there are exactly two vertices x_1, x_2, for which $d_5(x_1) = 5 = d_5(x_2)$. For every other $v \in S$, $d_5(v) = 3$. In any event, there exist two vertices v_1, v_2 for which $d_5(v_1) = d_5(v_2) = 3$ which are contained as vertices of degree 5 in the same block. This leaves at most four other blocks of B which contain exactly one of v_1 or v_2 as a vertex of degree 5. Since the edge v_1v_2 occurs in a block $b \in B$ if and only if one of v_1 and v_2 has degree 5 in b and the other has degree 2 in b, it follows that v_1v_2 occurs in at most four blocks of B, thus giving a contradiction. □

This brings us to our main result.

Theorem 3.3 *Suppose λ and n are positive integers which satisfy the conditions of Lemma 3.1. Then there exists a λ-fold $K_{2,5}$-design of order n except when $\lambda = 5$ and $n = 8$, and except possibly in the following cases: $\lambda = 1$ and $n = 40$; and $\lambda = 3$ and $n = 16$ or 20.*

Proof: In light of Lemma 2.1, when decomposing K_n into $K_{a,b}$, we need only consider those special cases where n satisfies the obvious necessary conditions and $n < 2ab + m_{a,b}$, where $m_{a,b}$ denotes the smallest order m

for which there exists a $K_{a,b}$-design of order m. More specifically, in light of Proposition 2.3, when decomposing λK_n into $K_{2,5}$, it suffices to consider those special cases where n satisfies the obvious necessary conditions and $n < 20 + m_{2,5}(\lambda)$, where $m_{2,5}(\lambda)$ denotes the smallest order m for which there exists a λ-fold $K_{2,5}$-design of order m. Certainly, if there exist $K_{2,5}$-decompositions of $\lambda_1 K_n$ and $\lambda_2 K_n$, then there exists a $K_{2,5}$-decomposition of $(\lambda_1 + \lambda_2)K_n$. With these ideas in mind, we consider several cases based upon the value of λ.

Case 1: $\lambda = 1$.

By Lemma 3.1 and a theorem of DeCaen and Hoffman [4], we need only consider orders $n \equiv 0, 1, 5$, or 16 (mod 20), where $n \geq 21$. Therefore, it suffices to find $K_{2,5}$-designs of K_{21}, K_{25}, K_{36}, and K_{40}. The first three designs are found in the appendix; however, it is unknown whether a $K_{2,5}$-design of K_{40} exists. However, since a $K_{2,5}$-design of K_{60} exists (as seen in the appendix), there exist $K_{2,5}$-designs of K_n for all $n \equiv 0, 1, 5$, or 16 (mod 20), $n \geq 21$, except possibly for $n = 40$.

Case 2: $\lambda \equiv 2, 4, 6$, or 8 (mod 10).

In this case, $n \equiv 0$ or 1 (mod 5), $n \geq 10$. Certainly, it suffices to find all possible 2-fold $K_{2,5}$-designs for these values of n. It follows that we only need to find 2-fold $K_{2,5}$-designs of orders $n = 10, 11, 15, 16, 20, 21, 25$, and 26. However, when $n = 21$ or 25, we can simply take 2 copies of a $K_{2,5}$-design or order n and index 1. The remaining 2-fold $K_{2,5}$-designs are found in the appendix.

Case 3: $\lambda \equiv 1, 3, 7$, or 9 (mod 10), $\lambda > 1$.

In this case, we need only consider orders $n \equiv 0, 1, 5$, or 16 (mod 20), $n \geq 16$. There exist $K_{2,5}$-designs of order n and index 1 for all of these values of n except for $n = 16$ and 20, and possibly for $n = 40$. Since there exist 2-fold $K_{2,5}$-designs of order n for all $n \equiv 0$ or 1 (mod 5), $n \geq 10$, it suffices to find 3-fold $K_{2,5}$-designs of order $n = 16, 20$, and 40. Such a design of order 40 is found in the appendix, but it is unknown whether there exist 3-fold $K_{2,5}$-designs of orders 16 and 20. However, 5-fold $K_{2,5}$-designs of orders 16 and 20 exist and are found in the appendix. Therefore, for $\lambda = 3$, there exists a $K_{2,5}$-design of order n for all $n \equiv 0, 1, 5$, or 16 (mod 20), where $n \geq 21$. For $\lambda \equiv 1, 3, 7$, or 9 (mod 10), where $\lambda \geq 7$, and for $n \equiv 0, 1, 5$, or 16 (mod 20), where $n \geq 21$, we can obtain a λ-fold $K_{2,5}$-design of order n by taking one copy of a 3-fold $K_{2,5}$-design and $(\lambda - 3)/2$ copies of a 2-fold $K_{2,5}$ design of order n. For these same values of λ and for $n = 16$ or 20, we can construct a λ-fold $K_{2,5}$-design of order n by taking one copy of a 5-fold $K_{2,5}$-design of order n and a $(\lambda - 5)/2$ copies of a 2-fold $K_{2,5}$-design of order n.

Case 4: $\lambda \equiv 5$ (mod 10).

Here we need only consider orders $n \equiv 0$ or 1 (mod 4), where $n \geq 8$ if $\lambda > 5$ and $n \geq 9$ if $\lambda = 5$. Therefore, it suffices to find 5-fold $K_{2,5}$-designs of order $n = 9, 12, 13, 16, 17, 20, 24$, and 28, a 10-fold $K_{2,5}$-design of order $n = 8$, and a 15-fold $K_{2,5}$-design of order $n = 8$, all of which are found in the appendix. If $\lambda \equiv 5 \pmod{10}$ and $\lambda > 5$, then we can find λ-fold $K_{2,5}$-designs of order $n \equiv 0$ or $1 \pmod 4$, where $n \geq 9$ by simply taking $\lambda/5$ copies of the appropriate 5-fold design of order n. When $\lambda \equiv 5 \pmod{10}$, $\lambda \geq 15$, and $n = 8$, we need only take a 15-fold $K_{2,5}$-design of order 8 and $(\lambda - 15)/10$ copies of a 10-fold $K_{2,5}$-design of order 8 to obtain a λ-fold $K_{2,5}$-design of order 8.

Case 5: $\lambda \equiv 0 \pmod{10}$.

Here we need to consider all orders $n \geq 7$. For $\lambda \geq 10$, where $\lambda \equiv 0 \pmod{10}$, we can construct λ-fold $K_{2,5}$-designs of order n simply by taking $\lambda/10$ copies of a 10-fold $K_{2,5}$-design of order n. It suffices to find 10-fold $K_{2,5}$-designs for all orders n where $7 \leq n \leq 26$. There exist 1-fold $K_{2,5}$-designs of orders $n = 21$ and 25. In addition, there exist 2-fold $K_{2,5}$-designs of orders $n = 10, 11, 15, 16, 20$, and 26. Furthermore, there exist 5-fold $K_{2,5}$-designs of order $n = 9, 12, 13, 17$, and 24. Therefore, we need only construct 10-fold $K_{2,5}$-designs of order $n = 7, 8, 14, 18, 19, 22$, and 23. Each of these is found in the appendix.

References

[1] E. Bell, Decomposition of K_n into cycles of length at most 50. *Ars Combin.* 40 (1995) 49–58.

[2] J.-C. Bermond, and J. Schönheim, G-Decomposition of K_n Where G Has Four Vertices or Less. *Discrete Math.* 19 (1977) 113–120.

[3] J.-C. Bermond, C. Huang, A. Rosa and D. Sotteau, Decomposition of Complete Graphs into Isomorphic Subgraphs with Five Vertices. *Ars Combin.* 10 (1980) 211–254.

[4] D. De Caen and D. G. Hoffman, Impossibility of Decomposing the Complete Graph on n points into $n-1$ Isomorphic Complete Bipartite Graphs, *SIAM Journal of Discrete Mathematics* Vol. 2, No. 1 (1989) 48-50.

[5] R. L. Graham and H. O. Pollak, On embedding graphs in squashed cubes, *Springer Lecture Notes 303*, Springer-Berlag, New York (1973) 99-110.

[6] H. Hanani, The existence and construction of Balanced Incomplete Block Designs. *Annals of Mathematical Statistics* 32 (1961) 361–386.

[7] D. G. Hoffman and K. S. Kirkpatrick, G-designs of order n and index λ where G has 5 vertices or less. *Australas. J. Combin.* 18 (1998) 13–37.

[8] D. G. Hoffman and M. Liatti, Bipartite Designs. *J. Combin. Des.* 3 (1995) 499–454.

[9] M. Tarsi, Decomposition of Complete Multigraphs into Stars. *Discrete Math.* 26 (1979) 273–278.

[10] M. Tarsi, Decomposition of a complete multigraph into simple paths. *J. Combin. Th.* A 34 (1983), no. 1, 60–70.

Appendix

- K_{21} on the vertex set $\mathbb{Z}_{21}$. Here is a base block to be developed cyclically (mod 21): $(10,5)-(0,1,2,3,4)$

- K_{25} on the vertex set $\mathbb{Z}_5\times\mathbb{Z}_5$. Here are the base blocks to be developed cyclically (mod (5,–)): $((0,0)\,(0,1))-((1,0)\,(2,0)\,(1,1)\,(2,1)\,(0,2))$
$((1,2)\,(2,2))-((0,2)\,(0,0)\,(3,0)\,(0,1)\,(3,1))$
$((0,0)\,(0,4))-((0,1)\,(0,3)\,(1,3)\,(1,4)\,(2,4))$
$((0,2)\,(0,3))-((1,3)\,(2,3)\,(1,4)\,(2,4)\,(3,4))$
$((0,2)\,(1,1))-((0,3)\,(3,3)\,(4,3)\,(0,4)\,(4,4))$
$((1,3)\,(2,4))-((2,0)\,(3,0)\,(4,0)\,(0,1)\,(1,1))$

- K_{36} on the vertex set $\mathbb{Z}_9\times\mathbb{Z}_4$. Here are the base blocks to be developed cyclically (mod (9,–)): $((0,0)\,(0,2))-((0,3)\,(1,2)\,(2,2)\,(3,2)\,(4,2))$
$((0,1)\,(7,1))-((1,1)\,(2,1)\,(0,0)\,(6,3)\,(5,3))$
$((0,0)\,(7,0))-((1,0)\,(2,0)\,(1,3)\,(2,3)\,(2,1))$
$((7,3)\,(0,2))-((3,3)\,(4,3)\,(5,3)\,(6,3)\,(5,1))$
$((0,2)\,(7,1))-((4,0)\,(1,3)\,(2,3)\,(7,3)\,(8,3))$
$((0,0)\,(2,0))-((8,1)\,(0,2)\,(8,2)\,(7,3)\,(8,3))$
$((0,1)\,(4,1))-((8,0)\,(0,2)\,(1,2)\,(2,2)\,(3,2))$

- K_{60} on the vertex set $\mathbb{Z}_{24}\cup\{\infty_0,\infty_1,\dots,\infty_{34}\}\cup\{\infty\}$:

  First place a decomposition of K_{25} on the vertex set $\mathbb{Z}_{24}\cup\{\infty\}$, and then place a decomposition of K_{36} on the vertex set $\{\infty_0,\infty_1,\dots,\infty_{34}\}\cup\{\infty\}$. Finish the construction with the following set of blocks: $\{(2i,2i+1)-(\infty_{5j},\infty_{5j+1},\infty_{5j+2},\infty_{5j+3},\infty_{5j+4})\mid 0\le i\le 11, 0\le j\le 6\}$.

- $2K_{5,5}$ on the vertex set $\mathbb{Z}_5\times\mathbb{Z}_2$:

  Let $V_i=\{(0,i),(1,i),(2,i),(3,i),(4,i)\}$, for $0\le i\le 1$. Define $2K_{5,5}$ with bipartition (V_0,V_1). Then $(V_0\cup V_1,C)$ is a $K_{2,5}$-design of $2K_{5,5}$, where

$$C=\{\{((2i,0),(2i+1,0))-((0,1),(1,1),(2,1),(3,1),(4,1))\}\mid 0\le i\le 4\},$$

  where all sums are reduced modulo 5.

- $2K_{25}\setminus 2K_{20}$ on the vertex set $\mathbb{Z}_5\cup\{\infty_0,\infty_1,\dots,\infty_{19}\}$:

  Let $2K_{25}\setminus 2K_{20}$ denote the complete multigraph $2K_{25}$ with the edges of $2K_{20}$ removed (also called $2K_{25}$ with a hole of size 20). Furthermore, let $\{\infty_0,\infty_1,\dots,\infty_{19}\}$ be the vertices in the hole of size 20. The following sets of blocks give a $K_{2,5}$-decomposition of $2K_{25}\setminus 2K_{20}$:

(i) $\{\{(i, i+3) - (i+1, \infty_{4j}, \infty_{4j+1}, \infty_{4j+2}, \infty_{4j+3})\} \mid 0 \le i \le 4, 0 \le j \le 1\}$, where all sums including i are reduced modulo 5; and

(ii) two copies of each block in the following set: $\{\{(\infty_{2i}, \infty_{2i+1}) - (0,1,2,3,4)\} \mid 4 \le i \le 9\}$.

- $2K_{10}$ on the vertex set $\mathbb{Z}_{10}$:

  $(0,6) - (1,2,3,4,5)$
  $(1,7) - (2,3,4,5,6)$
  $(2,8) - (3,4,5,6,7)$
  $(3,9) - (4,5,6,7,8)$
  $(4,0) - (5,6,7,8,9)$
  $(1,6) - (0,5,7,8,9)$
  $(2,7) - (0,1,5,8,9)$
  $(3,8) - (0,1,2,5,9)$
  $(4,9) - (0,1,2,3,5)$.

- $2K_{11}$ on the vertex set $\mathbb{Z}_{11}$:

  Here is a base block to be developed cyclically (mod 11): $(0,1) - (2,4,6,8,10)$

- $2K_{15}$ on the vertex set $\mathbb{Z}_{10} \cup \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4\}$:

  First place a decomposition of $2K_{10}$ on the vertices in $\mathbb{Z}_{10}$. Then form the following sets of blocks:

  - $\{(\infty_i, \infty_{i+3}) - (7,8,9,\infty_{i+1}, \infty_{i+2}) \mid 0 \le i \le 4\}$, where all sums are reduced modulo 5;
  - $\{(2i, 2i+1) - (\infty_0, \infty_1, \infty_2, \infty_3, \infty_4) \mid 0 \le i \le 6\}$, where all sums are reduced modulo 7.

- $2K_{16}$ on the vertex set $\mathbb{Z}_{11} \cup \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4\}$:

  First place a decomposition of $2K_{11}$ on the vertices in $\mathbb{Z}_{11}$. Then form the following sets of blocks:

  - $\{(\infty_i, \infty_{i+3}) - (8,9,10,\infty_{i+1}, \infty_{i+2}) \mid 0 \le i \le 4\}$, where all sums are reduced modulo 5;
  - $\{(2i, 2i+1) - (\infty_0, \infty_1, \infty_2, \infty_3, \infty_4) \mid 0 \le i \le 7\}$, where all sums are reduced modulo 8.

- $2K_{20}$ on the vertex set $\mathbb{Z}_{10} \times \mathbb{Z}_2$:

  Place a decomposition of $2K_{10}$ on each of the vertex sets $V_0 = \{(i,0) \mid 0 \le i \le 9\}$ and $V_1 = \{(i,1) \mid 0 \le i \le 9\}$. Then, by Theorem 1.1, there exists a $K_{2,5}$ decomposition of $K_{10,10}$ with bipartition (V_0, V_1). Take two copies of this decomposition to complete the construction.

- $2K_{26}$ on the vertex set $\mathbb{Z}_{16} \cup \{\infty_0, \infty_1, \ldots, \infty_9\}$:

  First place a decomposition of $2K_{16}$ on the vertices in $\mathbb{Z}_{16}$. Then place a decomposition of $2K_{10}$ on the vertices in $\{\infty_0, \infty_1, \ldots, \infty_9\}$.

By Theorem 1.1, there exists a $K_{2,5}$ decomposition of $K_{10,16}$ with bipartition $(\mathbb{Z}_{16}, \{\infty_0, \infty_1, \ldots, \infty_9\})$. Take two copies of this decomposition to complete the construction.

- $2K_{40}$ on the vertex set $\mathbb{Z}_{19} \cup \{\infty_0, \infty_1, \ldots, \infty_{19}\} \cup \{\infty\}$:

  First place a decomposition of $2K_{20}$ on the vertex set $\mathbb{Z}_{19} \cup \{\infty\}$, and then place a $2K_{21}$ design on the vertex set $\{\infty_0, \infty_1, \ldots, \infty_{19}\} \cup \{\infty\}$. Finish the construction with the following set of blocks:

  $$\{(2i, 2i+1) - (\infty_{5j}, \infty_{5j+1}, \infty_{5j+2}, \infty_{5j+3}, \infty_{5j+4}) \mid 0 \le i \le 18, 0 \le j \le 3\},$$

  where all sums involving i are reduced modulo 19.

- $3K_{25} \setminus 3K_{20}$ on the vertex set $\mathbb{Z}_5 \cup \{\infty_0, \infty_1, \ldots, \infty_{19}\}$:

  Let $3K_{25} \setminus 3K_{20}$ denote the complete multigraph $3K_{25}$ with the edges of $3K_{20}$ removed (also called $3K_{25}$ with a hole of size 20). Furthermore, let $\{\infty_0, \infty_1, \ldots, \infty_{19}\}$ be the vertices in the hole of size 20. The following sets of blocks give a $K_{2,5}$-decomposition of $3K_{25} \setminus 3K_{20}$:

  (i) $\{\{(i, i+3) - (i+1, \infty_{4j}, \infty_{4j+1}, \infty_{4j+2}, \infty_{4j+3})\} \mid 0 \le i \le 4, 0 \le j \le 2\}$, where all sums including i are reduced modulo 5;

  (ii) $\{(\infty_{2i}, \infty_{2i+1}) - (0,1,2,3,4)\} \mid 0 \le i \le 5\}$; and

  (iii) three copies of each block in the following set: $\{\{(\infty_{2i}, \infty_{2i+1}) - (0,1,2,3,4)\} \mid 6 \le i \le 9\}$.

- $3K_{40}$ on the vertex set $(\mathbb{Z}_5 \times \mathbb{Z}_4) \cup \{\infty_0, \infty_1, \ldots, \infty_{19}\}$:

  For $0 \le i \le 3$, let $V_i = \{(0,i), (1,i), (2,i), (3,i), (4,i)\}$, and let $V_4 = \{\infty_0, \infty_1, \ldots, \infty_{19}\}$. We form the blocks in the following manner.

  (i) Place a $K_{2,5}$-design of K_{25} on each set $V_i \cup V_4$, for $0 \le i \le 2$.

  (ii) Place a $K_{2,5}$-design of $2K_{5,5}$ with bipartition (V_i, V_j), for $0 \le i < j \le 3$.

  (iii) Place a $K_{2,5}$-design of $K_{5,10}$ with bipartition $(V_1, V_2 \cup V_3)$, $(V_2, V_0 \cup V_3)$, and $(V_0, V_1 \cup V_3)$.

  (iv) Place a $K_{2,5}$-design of $2K_{25} \setminus 2K_{20}$ on each set from $V_i \cup V_4$, for $1 \le i \le 3$, where V_4 contains all vertices in the hole of size 20.

  (v) Place a $K_{2,5}$-design of $3K_{25} \setminus 3K_{20}$ on the set $V_0 \cup V_4$, where V_4 contains all vertices in the hole of size 20.

- $5K_9$ on the vertex set $\mathbb{Z}_9$:

  Here are the base blocks to be generated cyclically modulo 9: $(0,4) - (1,2,3,5,6)$ and $(0,1) - (2,4,5,6,8)$.

- $5K_{12}$ on the vertex set $\mathbb{Z}_{11} \cup \{\infty\}$:

  Here are the base blocks to be generated cyclically modulo 11: $(\infty, 0)-(1,2,3,4,5)$, $(0,5)-(6,7,8,9,10)$, and $(0,5)-(6,7,8,9,10)$.

- $5K_{13}$ on the vertex set $\mathbb{Z}_{13}$:

  Here are the base blocks to be generated cyclically modulo 13: $(0,1)-(2,3,4,5,6)$, $(0,3)-(1,4,5,6,9)$, and $(0,4)$-$(1,5,6,9,10)$.

- $5K_{16}$ on the vertex set $\mathbb{Z}_{15} \cup \{\infty\}$:

  Here are the base blocks to be generated cyclically (mod 15): $(\infty,0)-(1,2,3,4,5)$, $(0,4)-(1,6,8,9,11)$, $(0,6)-(3,7,8,10,11)$, and $(0,12)-(6,7,9,13,14)$.

- $5K_{17}$ on the vertex set $\mathbb{Z}_8 \cup \{\infty_0, \infty_1, \ldots, \infty_7\} \cup \{\infty\}$:

  On each set of vertices $\mathbb{Z}_8 \cup \{\infty\}$ and $\{\infty_0, \infty_1, \ldots, \infty_7\} \cup \{\infty\}$, place a decomposition of $5K_9$. Then form the following set of blocks:

  $$\{(2i, 2i+1) - (\infty_{5j}, \infty_{5j+1}, \infty_{5j+2}, \infty_{5j+3}, \infty_{5j+4}) \mid 0 \le i \le 3, 0 \le j \le 7\},$$

  where all sums are reduced modulo 8.

- $5K_{20}$ on the vertex set $\mathbb{Z}_{19} \cup \{\infty\}$:

  Here are the base blocks to be generated cyclically (mod 19): $(\infty,0)-(1,2,3,4,5)$, $(0,1)-(6,7,8,9,10)$
  $(0,2)-(1,4,5,8,9)$, $(0,4)-(6,7,8,12,13)$, and
  $(0,6)-(1,2,3,4,5)$.

- $5K_{24}$ on the vertex set $\mathbb{Z}_{12} \times \mathbb{Z}_2$:

  Place a decomposition of $5K_{12}$ on each of the vertex sets $V_0 = \{(i,0) \mid 0 \le i \le 11\}$ and $V_1 = \{(i,1) \mid 0 \le i \le 11\}$. Then form the following set of blocks: $\{((2i,0),(2i+1,0))-((5j,1),(5j+1,1),(5j+2,1),(5j+3,1),(5j+4,1)) \mid 0 \le i \le 5, 0 \le j \le 11\}$, where all sums are reduced modulo 12.

- $5K_{28}$ on the vertex set $\mathbb{Z}_{12} \cup \{\infty_0, \infty_1, \ldots, \infty_{14}\} \cup \{\infty\}$.

  First place a decomposition of $5K_{13}$ on the vertex set $\mathbb{Z}_{12} \cup \{\infty\}$, and then place a $5K_{16}$ decomposition on the vertex set $\{\infty_0, \infty_1, \ldots, \infty_{14}\} \cup \{\infty\}$. Finish the construction by taking five copies of each block in the following set:

  $$\{(2i, 2i+1) - (\infty_{5j}, \infty_{5j+1}, \infty_{5j+2}, \infty_{5j+3}, \infty_{5j+4}) \mid 0 \le i \le 5, 0 \le j \le 2\}.$$

- $5K_{40}$ on the vertex set $\mathbb{Z}_{39} \cup \{\infty\}$:

  Here are the base blocks to be generated cyclically modulo 39:

$(\infty, 0) - (1, 2, 3, 4, 5)$	$(0, 1) - (2, 4, 6, 8, 20)$
$(0, 9) - (15, 16, 17, 18, 19)$	$(0, 15) - (1, 2, 3, 4, 5)$
$(0, 16) - (1, 2, 3, 4, 5)$	$(0, 19) - (5, 6, 7, 8, 9)$
$(0, 25) - (6, 7, 8, 9, 10)$	$(0, 25) - (6, 7, 8, 9, 11)$
$(0, 25) - (1, 2, 3, 4, 12)$	$(0, 27) - (9, 10, 11, 12, 13)$

- $10K_7$ on the vertex set $\mathbb{Z}_7$:

  For each pair $i, j \in \mathbb{Z}_7$, $i < j$, produce exactly one copy of $K_{2,5}$ which contains both i and j as vertices of degree 5.

- $10K_8$ on the vertex set $\mathbb{Z}_7 \cup \{\infty\}$:

  Here are the base blocks to be generated cyclically modulo 7: $(\infty, 0) - (1, 2, 3, 4, 6)$, $(\infty, 0) - (1, 3, 4, 5, 6)$, $(0, 1) - (2, 3, 4, 5, 6)$, and $(0, 3) - (1, 2, 4, 5, 6)$.

- $10K_{14}$ on the vertex set $\mathbb{Z}_7 \times \mathbb{Z}_2$:

  Place a decomposition of $10K_7$ on each of the vertex sets $\{(i, 0) \mid 0 \leq i \leq 6\}$ and $\{(i, 1) \mid 0 \leq i \leq 6\}$. Then form the following set of blocks:

  $$\{((2i, 0), (2i + 1, 0)) - ((5j, 1), (5j + 1, 1), (5j + 2, 1), (5j + 3, 1), (5j + 4, 1)) \mid 0 \leq i \leq 6, 0 \leq j \leq 6\},$$

  where all sums are reduced modulo 7.

- $10K_{18}$ on the vertex set $\mathbb{Z}_9 \times \mathbb{Z}_2$:

  Place a decomposition of $10K_9$ on each of the vertex sets $\{(i, 0) \mid 0 \leq i \leq 8\}$ and $\{(i, 1) \mid 0 \leq i \leq 8\}$. Then form the following set of blocks:

  $$\{((2i, 0), (2i + 1, 0)) - ((5j, 1), (5j + 1, 1), (5j + 2, 1), (5j + 3, 1), (5j + 4, 1)) \mid 0 \leq i, j \leq 8\},$$

  where all sums are reduced modulo 9.

- $10K_{19}$ on the vertex set $(\mathbb{Z}_6 \times \mathbb{Z}_3) \cup \{\infty\}$:

  Place a decomposition of $10K_7$ on each of the vertex sets $V_0 = \{(i, 0) \mid 0 \leq i \leq 5\} \cup \{\infty\}$, $V_1 = \{(i, 1) \mid 0 \leq i \leq 5\} \cup \{\infty\}$, and $V_2 = \{(i, 2) \mid 0 \leq i \leq 5\} \cup \{\infty\}$. Furthermore, for $0 \leq i \leq 2$, where all sums are reduced modulo 6, and for $0 \leq j \leq 2$, where all sums are reduced modulo 3, form two copies of each of the following blocks:
  $\{((2i, j), (2i+1, j)) - ((0, j+1), (1, j+1), (2, j+1), (3, j+1), (4, j+1))\}$
  $\{((2i, j), (2i+1, j)) - ((0, j+1), (1, j+1), (2, j+1), (3, j+1), (5, j+1))\}$

$\{((2i,j),(2i+1,j))-((0,j+1),(1,j+1),(2,j+1),(4,j+1),(5,j+1))\}$
$\{((2i,j),(2i+1,j))-((0,j+1),(1,j+1),(3,j+1),(4,j+1),(5,j+1))\}$
$\{((2i,j),(2i+1,j))-((0,j+1),(2,j+1),(3,j+1),(4,j+1),(5,j+1))\}$
$\{((2i,j),(2i+1,j))-((1,j+1),(2,j+1),(3,j+1),(4,j+1),(5,j+1))\}$.

- $10K_{22}$ on the vertex set $\mathbb{Z}_{11} \times \mathbb{Z}_2$:

  Place a decomposition of $10K_{11}$ on each of the vertex sets $\{(i,0) \mid 0 \le i \le 10\}$ and $\{(i,1) \mid 0 \le i \le 10\}$. Then form the following set of blocks:

  $$\{((2i,0),(2i+1,0)) - ((5j,1),(5j+1,1),(5j+2,1),(5j+3,1),(5j+4,1)) \mid 0 \le i,j \le 10\},$$

  where all sums are reduced modulo 11.

- $10K_{23}$ on the vertex set $\mathbb{Z}_{16} \cup \{\infty_0, \infty_1, \ldots, \infty_5\} \cup \{\infty\}$:

  First place a decomposition of $10K_{17}$ on the vertex set $\mathbb{Z}_{16} \cup \{\infty\}$. Then place a decomposition of $10K_7$ on the vertex set $\{\infty_0, \infty_1, \ldots, \infty_5\} \cup \{\infty\}$. Finally, for $0 \le i \le 7$, where all sums are reduced modulo 8, form two copies of each of the following blocks:
  $\{(2i,2i+1) - (\infty_0,\infty_1,\infty_2,\infty_3,\infty_4)\}$
  $\{(2i,2i+1) - (\infty_0,\infty_1,\infty_2,\infty_3,\infty_5)\}$
  $\{(2i,2i+1) - (\infty_0,\infty_1,\infty_2,\infty_4,\infty_5)\}$
  $\{(2i,2i+1) - (\infty_0,\infty_1,\infty_3,\infty_4,\infty_5)\}$
  $\{(2i,2i+1) - (\infty_0,\infty_2,\infty_3,\infty_4,\infty_5)\}$
  $\{(2i,2i+1) - (\infty_1,\infty_2,\infty_3,\infty_4,\infty_5)\}$.

- $15K_8$ on the vertex set $\mathbb{Z}_7 \cup \{\infty\}$:

  Here are the base blocks to be generated cyclically modulo 7:

$(\infty,0) - (1,2,3,4,5)$	$(\infty,0) - (1,2,3,4,6)$
$(\infty,0) - (1,2,3,5,6)$	$(0,1) - (2,3,4,5,6)$
$(0,2) - (1,3,4,5,6)$	$(0,3) - (1,2,4,5,6)$