Special Super Edge Magic Labelings of Bipartite Graphs

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Dedicated to Rikio Ichishima and Eduardo Canale

ABSTRACT. Let G be a bipartite graph with bipartite sets V_1 and V_2. If f is a bijective function from the vertices and edges of G into the first $p+q$ positive intergers, where p and q denotes the order and size of G, respectively, meeting the properties that f is a super edge magic labeling and if the cardinal of V_i is p_i for $i = 1, 2$, then the image of the set V_1 is the set of the first p_i positive intergers and the image of the set V_2 is the set of intergers from $p_1 + 1$ up to p. If a bipartite graph G admits an special super edge magic labeling, we say that G is special super edge magic. Some properties of special super edge magic graphs are presented. However, this work is mainly devoted to the study of the relations existing between super edge magic and special super edge magic labelings.

1 Introduction

For most of the graph theory terminology used here, we follow [1]. In 1996, Enomoto, Lladó, Nakamigawa, and Ringel [2] defined a super edge magic labeling of a graph G to be a bijective function $f\colon V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$ and $f(u) + f(uv) + f(v) = K$ for every edge uv of $E(G)$.

If a graph G admits a super edge magic labeling, then it is said that G is a super edge magic graph For short, we will abbreviate Super Edge Magic to SEM, and the abbreviation will be used through the rest of the paper.

It is not hard to transform SEM labelings into equivalent vertex labelings. This fact was observed first by Figueroa *et al.* in [3] and it is presented next as Lemma 1.

Lemma 1. *A graph G is SEM if and only if there exists a bijective function $g\colon V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that the set $\{f(u) + f(v)\colon uv \in E(G)\}$ is a set of $|E(G)|$ consecutive intergers.*

In our study of SEM labelings the concept of special super edge magic labelings of bipartite graphs has emerged naturally. We define a special super edge magic labeling of a bipartive graph G with bipartite sets V_1 and V_2 to be a bijective function $f: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that f is a super edge magic labeling of G, with the extra property that $f(V_1) = \{1, 2, \ldots, |V_1|\}$ and $f(V_2) = \{|V_1| + 1, |V_1| + 2, \ldots, |V(G)|\}$.

If a graph G admits a special super edge magic labeling, then we say that G is special super edge magic. For short, we will abbreviate special super edge magic as SSEM through the rest of this paper.

Notice that it is possible to redefine SSEM labelings of bipartite graphs in such a way that only the vertices of the graph are labeled, and we do this in Lemma 2.

Lemma 2. *A bipartite graph G with bipartite sets V_1 and V_2 is SSEM if and only if there exits a bijective function $g: V(G) \rightarrow \{1, 2, \ldots, |V(G)|\}$ with the properties that $g(V_1) = \{1, 2, \ldots, |V_1|\}$, $g(V_2) = \{|V_1| + 1, |V_1| + 2, \ldots, |V(G)|\}$ and the set $\{g(u) + g(v) : uv \in E(G)\}$ is a set of $|E(G)|$ consecutive intergers.*

Proof: For the necessity, assume that a bipaffite graph G with bipartite sets V_1 and V_2 is SSEM. Let f be an SSEM labeling of G. Then, the function $f\ |_{V(G)}$ defined to be the restriction of the function f to the set $V(G)$, meets the properties of the function g as described in the statement of the Lemma. Therefore, take $g = f\ |_{V(G)}$. For the sufficiency, assume that g is a bijective function defined on the vertex set of a bipartite graph G, meeting the requirements of the statements of the Lemma. Define the function $f: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$ as $g(x)$ if $x \in V(G)$ and $\min\{g(u) + g(v) : uv \in E(G)\} + |V(G)| + |E(G)| - g(a) - g(b)$ if $x = ab \in E(G)$. Then, f is, in fact, a SSEM labeling of G. Therefore G is an SSEM graph. □

Enomoto *et al.* (see [2]), also observed the following interesting result.

Theorem 1. *If G is a SEM graph, then $|E(G)| \leq 2|V(G)| - 3$.*

With the aid of Lemma 2 and motivated by the previous theorem, we have been able to prove the next result concerning SSEM bipartite graphs.

Theorem 2. *If G is an SSEM bipartite graph with bipartite sets V_1 and V_2, then $|E(G)| \leq |V(G)| - 1$.*

Proof: Let G be an SSEM bipartite graph with bipartite sets V_1 and V_2. Then, by Lemma 2, there exists a bijective function $g: V(G) \rightarrow \{1, 2, \ldots, V(G)\}$ such that $g(V_1) = \{1, 2, \ldots, |V_1|\}$, $g(V_2) = \{|V_1| + 1, |V_2| + 2, \ldots, |V(G)|\}$ and $\{g(u) + g(v) : uv \in E(G)|\}$ is a set of $|E(G)|$ consecutive integers.

Now, $\min\{g(u)+g(v) \colon uv \in E(G)\} \geq 2+|V_1|$ and $\max\{g(u)+g(v) \colon uv \in E(G)\} \leq 2|V_1|+|V_2|$ hence, $|\{g(u)+g(v) \colon uv \in E(G)\}| \leq 2|V_1|+|V_2|-|V_1|-2+1 = |V(G)|-1$, but $|\{g(u)+g(v) \colon uv \in E(G)\}| = |E(G)|$ therefore, $|E(G)| \leq |V(G)|-1$. □

2 Relations among SSEM labelings, SEM labelings, chessboards 1-regular, and 2-regular graphs

In the abstract of this paper, it has been mentioned that the main goal of this work is to establish relations between SEM graphs and SSEM bipartite graphs. However, in order to do this, we need to define what in the near future will prove to be the "link" between the two concepts. This "link" is what we call an $n \times n$ chessboard. An $n \times n$ chessboard is defined to be a square that contains inside of it n rows and columns shaping n^2 new little squares inside of the original one. Figure 1 shows a 3×3 and 5×5 chessboard.

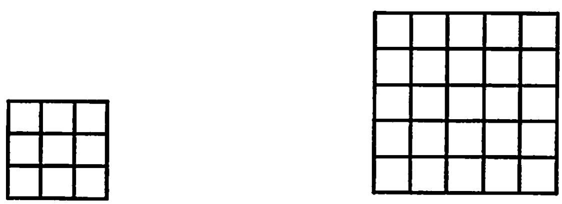

Figure 1

Now, notice that assigning different numbers to the columns and different numbers to the rows of an $n \times n$ chessboard, then every square of the chessboard is uniquely determined by an ordered pair (i, j) where i denotes the column to which the square belongs and j denotes the row to which the square belongs. Any function that assigns different numbers to the rows and different numbers to the columns of a chessboard will be called a numbering of the chessboard.

We are ready to state and prove our next result, that establishes a relation between SEM labelings of 2-regular graphs and SSEM labelings of 1-regular graphs.

Theorem 3. *Every SEM labeling of a 2-regular graph of size p can be transformed into an SSEM labeling of the 1-regular graph of size p.*

Proof: Let $G \cong \bigcup_{i=1}^{k} C_i$ be a 2-regular SEM graph with k components, and assume that the number of vertices in the jth component is denoted by I_j. Define the vertex and edge sets of G in the following manner
$V(G) = \bigcup_{j=1}^{k} \left\{v_i^j \colon 1 \leq i \leq I_j\right\}$,
$E(G) = \left(\bigcup_{i=1}^{k} \left\{v_1^i v_{l_i}^i\right\}\right) \cup \left(\bigcup_{j=1}^{k} \left\{v_i^j v_{i+1}^j \colon 1 \leq i \leq l_j - 1\right\}\right)$. Consider the

vertex labeling $g\colon V(G) \to \{1, 2, \ldots, |V(G)|\}$ with the properties of the function g defined in Lemma 1. Next construct an $\sum_{i=1}^{k} l_i \times \sum_{i=1}^{k} l_i$ chessboard with the associated numbering N_1, which assigns to the columns and rows of the chessboard the numbers 1 through $\sum_{i=1}^{k} l_i$ consecutively, from left to right and from top to bottom respectively. That is to say, the column that is most to the left receives number 1, the column that is next to this one receives number 2, and so on until we get to the column that is most to the right which receives the number $\sum_{i=1}^{k} l_i$. Also, the row that is located most to the bottom receives number 1. The row immediately on top of this one receives number 2, and so on, until we reach the top row, that receives number $\sum_{i=1}^{k} l_i$. See Figure 2 for geometrical clarification.

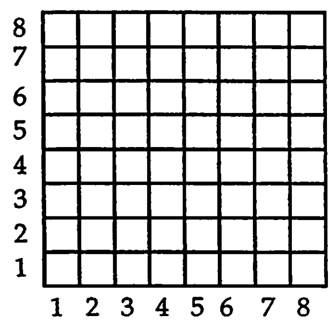

Figure 2

We now explain how we can represent the function g on the $\sum_{i=1}^{k} l_i \times \sum_{i=1}^{k} l_i$ chessboard. In order to do this, we will define the set S_1 in which each ordered pair belonging to S_1 represents a square of the representation of the function g on the $\sum_{i=1}^{k} l_i \times \sum_{i=1}^{k} l_i$ chessboard with numbering N_1

$$S_1 = \bigcup_{j=1}^{k} \left\{ \left(g(v_i^j), g(v_{i+1}^j) \right) : 1 \le i \le l_j - 1 \right\}$$
$$\bigcup \bigcup_{j=1}^{k} \left\{ \left(g(v_{l_j}^j), g(v_i^j) \right) \right\}.$$

It is easy to see that the set $A_1 = \{i + j\colon (i, j) \in S_1\}$ is a set of $|S_1|$ consecutive interges, since the function g has the property that the set $\{g(u) + g(v)\colon uv \in E(G)\}$ is a set of $|E(G)|$ consecutive intergers.

Next, define a new numbering N_2 of the $\sum_{i=1}^{k} l_i \times \sum_{i=1}^{k} l_i$ chessboard as follows. For any column C of the chessboard, let $N_2(C) = N_1(C)$ and for any row r of the chessboard, let $N_2(r) = N_1(r) + \sum_{i=1}^{k} l_i$. Then, the set S_2, which contains all ordered pairs that represent the squares of S_1, but with the numbedng N_2, is the set $S_2 = \left\{\left(x, y + \sum_{i=1}^{k} l_i\right) : (x,y) \in S_1\right\}$. Observe that the set $A_2 = \{x + y : (x,y) \in S_2\}$ is also a set of $|S_2|$ consecutive intergers since A_1, is a set $|S_1|$ consecutive intergers.

The next step is to define the graph $H = \left(\sum_{i=1}^{k} l_i\right) - K_2$ in the following way, $V(H) = \left\{v_i : 1 \le i \le \sum_{i=1}^{k} l_i\right\} \cup \left\{u_i : 1 \le i \le \sum_{i=1}^{k} l_i\right\}$ and $E(H) = \left\{v_i u_i : 1 \le i \le \sum_{i=1}^{k} l_i\right\}$.

Let h be any bijective function from the set C of connected components of H to the set S_2 and define the bijective function $f' : V(H) \to \{1, 2, \ldots, |V(H)|\}$ as described below.

If $h(u_i v_i) = (x, y)$, then $f' = (u_i) = x$ and $f' = (v_i) = y$. The function f' is extendible to a SSEM labeling of the graph H, since f' meets the conditions of the function g described in Lemma 2. □

The next example illustrates the algorithm described above in a particular case.

Example 1. Consider the cycle C_5 with the bijective vertex labeling g shown in Figure 3.

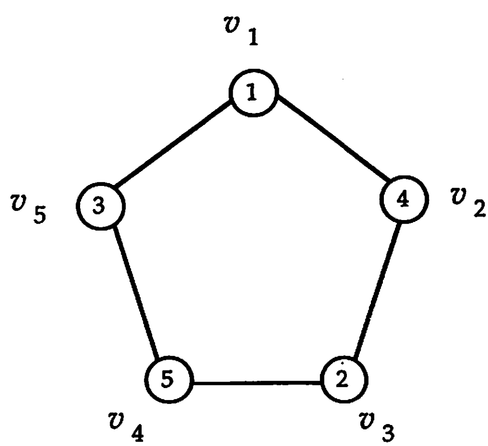

Figure 3

It is easy to see that $g : V(C_5) \to \{1, 2, 3, 4, 5\}$ is a bijective function with the property that $\{g(u) + g(v) : uv \in E(G)\} = \{4, 5, 6, 7, 8\}$. Now, we construct a 5×5 chessboard, with numbering N_1, and we represent the function g on the chessboard after obtaining the set $S_1 = \{(1,4)(4,2), (2,5), (5,3), (3,1)\}$ (See Figure 4).

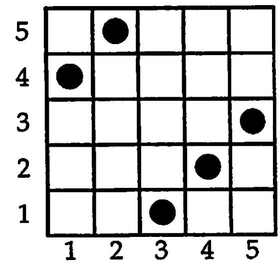

Figure 4

Next, we define the numbering N_2 on the 5×5 chessboard keeping the representation of g as shown in Figure 5.

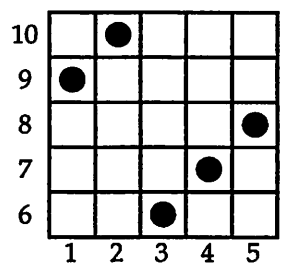

Figure 5

Now, $S_2 = \{(1,9),(2,10),(3,6),(4,7),(5,8)\}$. Consider the graph $5K_2$ represented next in Figure 6.

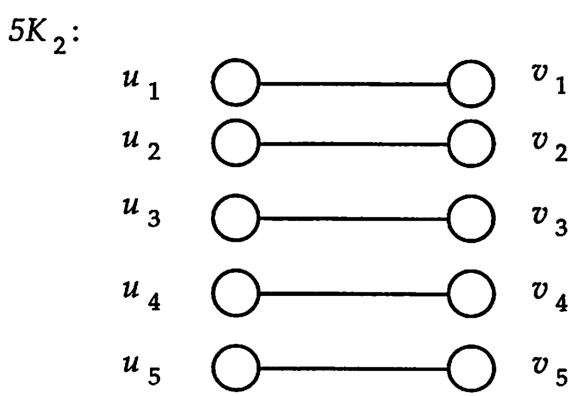

Figure 6

Let h be the bijective function from the components of nK_2 to the set S_2 defined by the rule, $h((u_1v_1)) = (1,9)$, $h((u_2v_2)) = (2,10)$, $h((u_3v_3)) = (3,6)$, $h((u_4v_4)) = (4,7)$, $h((u_5v_5)) = (5,8)$.

Hence, we get the labeling f' of the vertices of $5K_2$ defined next: $f'(u_i) = i$ and $f'(v_1) = 9$, $f'(v_2) = 10$, $f'(v_3) = 6$, $f'(v_4) = 7$, $f'(v_5) = 8$.

Since the set $\{f'(u) + f(v) \colon uv \in 5K_2\}$ is the set $\{9, 10, 11, 12, 13\}$, we can extend the function f' to an SSEM labeling of $5K_2$ as shown in Figure 7.

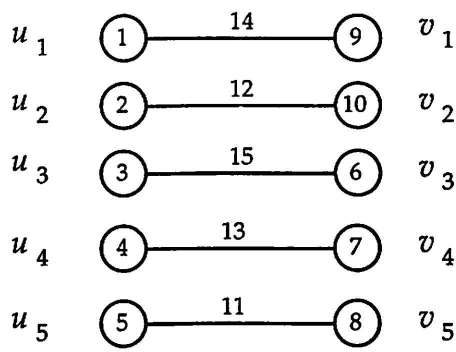

Figure 7

3 A relation between SEM labelings of graphs, pseudographs and SSEM labelings of bipartite graphs

It has become a tradition to define labelings of graphs only for simple graphs. That is to say, graphs with no loops nor multiple edges. The goal in this section is to show that it may be useful to study SEM labelings of some types of pseudographs, that is to say, graphs with loops. However, before doing this, it is necessary to extend the definition of SEM labelings of graphs to pseudographs. We do this in the obvious way.

Let P be a pseudograph. A bijective function f from $V(P) \cup E(P)$ to $\{1, 2, \ldots, |E(P)|\}$ is called a super edge magic labeling, for short, SEM labeling of P if $f(u) + f(uv) + f(v) = k$ for every uv in the set $E(P)$ and $f(V(P)) = \{1, 2, \ldots, |V(P)|\}$.

We observe that if a pseudograph has "attached" more than a single loop to any of its vertices or if the pseudo graph contains multiple edges, then the pseudograph is not SEM. Thus, the pseudographs that we will consider are basically graphs with at most one loop "attached" to as many vertices of the graph as we want. Some examples of SEM pseudographs (pseudographs that admit SEM labelings) are showed in Figure 8, with their corresponding SEM labelings.

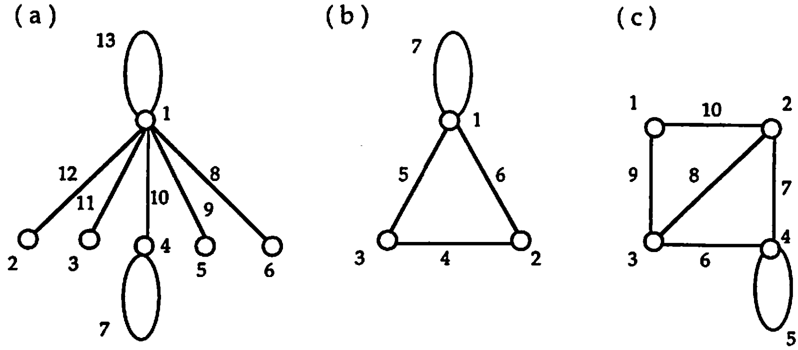

Figure 8

After looking at the previous examples, it is obvious that Lemma 1 generalizes immediately to pseudographs. Also Theorem 1, generalizes to pseudographs as shown in Corollary 1. The proof will be omitted since it is similar to the proof of Theorem 1.

Corollary 1. *If a pseudograph P is SEM, then $|E(P)| \leq 2|V(P)| - 1$.*

At this point, we are ready to establish the connection between SEM labelings of pseudographs and SSEM labelings of bipartite graphs.

Theorem 4. *Let P be either a pseudograph or a graph of order p and size q, and assume that $L\colon V(P) \to S$ (where S is a set of positive integers) is a bijective vertex labeling of G. Then, there exists a bipartite graph H of order $2p$ and size q, and a bijective vertex labeling $L^*\colon V(H) \to S \cup \{x+p\colon x \in S\}$ such that there exists a natural number γ with the property that $\{L(x)+L(y)+\gamma\colon xy \in E(P)\} = \{L^*(x)+L^*(y)\colon xy \in E(H)\}$.*

Proof: Let P be either a graph or a pseudograph of order p and size q, and let $L\colon V(P) \to S$ be a bijective function. Assume that $P = \left(\oplus_{i=1}^{k} P^i_{\alpha_i}\right) \oplus \left(\oplus_{i=1}^{r} C^i_{\beta_i}\right)$ is any edge disjoint composition of P into k paths and r cycles (each loop will be considered as a cycle), and orient each path in such a way that we can "travel" from one "end" of the path to the other "end" of the path following the sense of the arrows. Also orient each cycle either clockwise or counterdodcwise randomly.

Built $k+r$, $P \times P$ chessboards and assign to each chessboard the numbering N_1 defined next. The numbers $1, 2, \ldots, P$ will be assigned to the columns of the chessboard in such a way that the columns placed most to the left receives number 1. The column next to this one, receives number 2, and so on until. we reach the column most to the right, which receives number p. Also N_1, assigns the numbers 1 through p to the rows of the chessboard with number 1 being assigned to the row that is placed most to the bottom. Number 2, being assigned to the row immediately on top of this one, and so on, until we reach the most top row that will receive number P.

For each path $p^j_{\alpha_j}$ $(1 \le j \le k)$ and for each cycle $C^j_{\beta_j}$ $(1 \le j \le r)$ we will represent the functions $L|_{V(C^i_{\beta_j})}$ and $L|_{V(P^j_{\beta_j})}$ on different chessboards as described next. The square $(L|_{V(x)}(u), L|_{V(x)}(v))$ will be chosen if and only if uv is an arc of x (where x is any path or cycle that shapes the decomposition of P). Thus, the set S_1 of all ordered pairs that represent the squares chosen on the chessboards with numbering N_1 is $S_1 = \left\{(|_{V(x)}(u), L|_{V(x)}(v)) : uv\right.$ is an arc of x where $x \in \left\{P^i_{\alpha_i} : 1 \le i \le k\right\} \cup \left.\left\{C^i_{\beta_i} : 1 \le i \le r\right\}\right\}$.

Define the set A_1 to be $A_1 = \{a + b\colon (a, b) \in S_1\}$. It is clear that $A_1 = \{L(u) + L(v)\colon uv \in E(P)\}$. Next, define a new numbering N_2 on the chessboards depending on the numbering N_1 as follows. For any column C of the chessboard, let $N_2(C) = N_1(C)$ and for any row r of the chessboard, let $N_2(r) = N_1(r) + p$. Thus, if we call S_2 the set of the ordered pairs that represent the squares on the chessboard, but now, with respect to N_2 we have that $S_2 = \{(a, b + p)\colon (a, b) \in S_1\}$. Therefore, we have that the set A_2 defined as $A_2 = \{x + y\colon (x, y) \in S_2\}$, is basically a shift of the set A_1 by p units. That is to say, $A_2 = \{x + p\colon x \in A_1\}$. With all this information in mind, we are now ready to define the bipartite graph H as $V(H) = \{X_i\}_{i=1}^{2P}$, and $E(H) = \{x_j x_j\colon (i, j) \in S_2\}$. Let $L^*\colon V(H) \to \{1, 2, \ldots, 2P\}$ be the biiective function defined by the rule $L^*(x_i) = i$ for every $x_i \in V(H)$. Then, L^* has the properties of the function described in the conclusion of Theorem 4. □

In order to clarify the previous proof, we will provide a particular example.

Example 2. Consider the graph G with the vertex labeling L (shown in Figure 9).

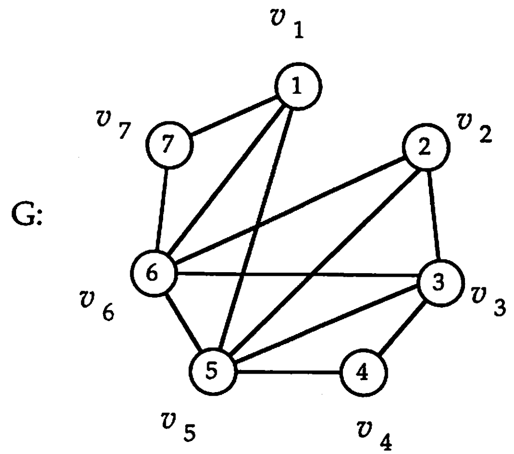

Figure 9

and let $G = C_{4,1} \oplus C_{4,2} \oplus P_{3,1} \oplus P_{1,2}$, where $C_{4,1}$, $C_{4,2}$, $P_{3,1}$, and $P_{1,2}$ are shown in the figure 10.

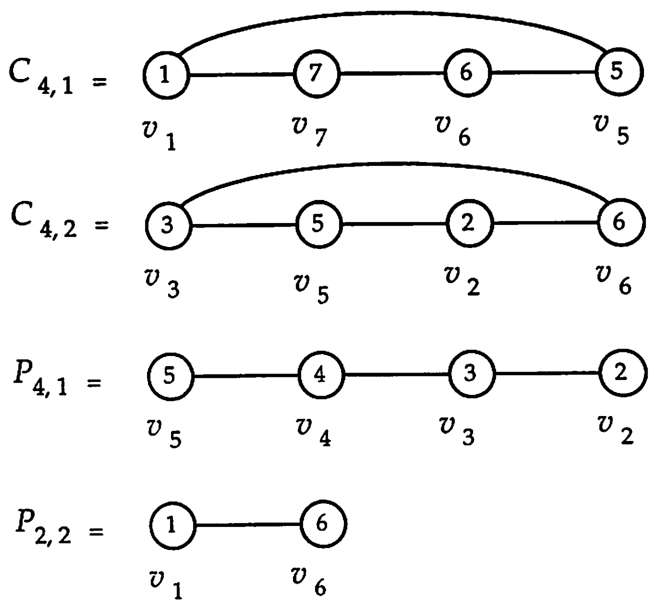

Figure 10

Next, give arbitrary orientations to the edges of G in such a way that the cycles are orientated either clockwise or counterclockwise, and the paths are oriented in such a way that we can travel from one end to the other following the sense of the arrows. We do this in Figure 11.

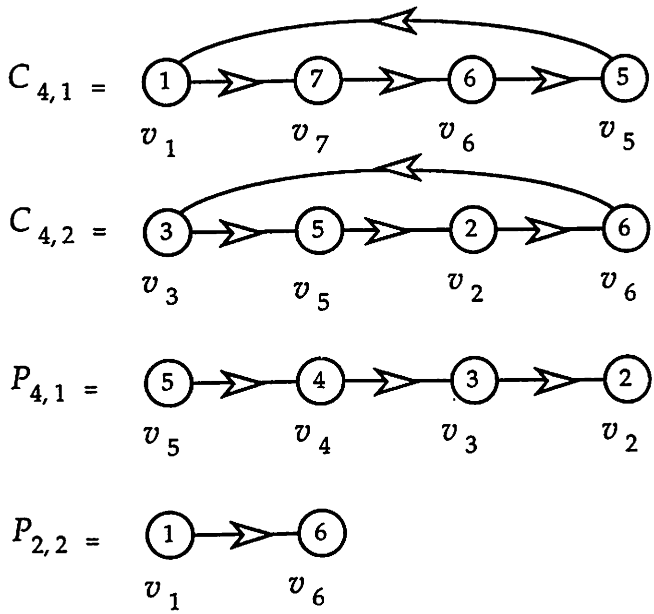

Figure 11

Next, construct 4, 7 × 7 chessboard with numbering N_1 and represent the functions $L|_{V(C_{4,1})}$, $L|_{V(C_{4,2})}$, $L|_{V(P_{3,1})}$, and $L|_{V(P_{1,2})}$ as shown in Figure 12.

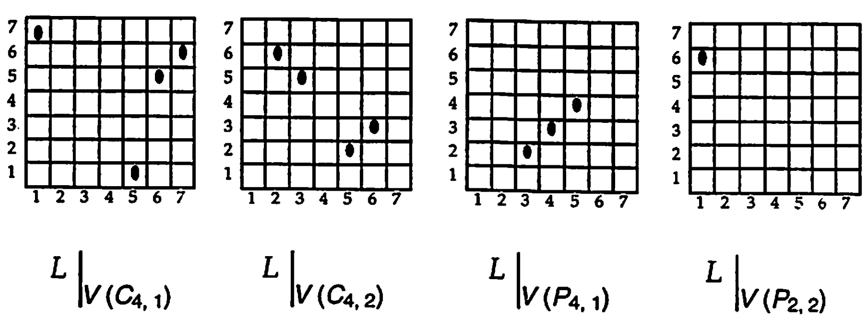

Figure 12

Now, rebuilt 4 more 7×7 chessboards and label the rows and the columns with numbering N_2, keeping the same squares choosen. See Figure 13.

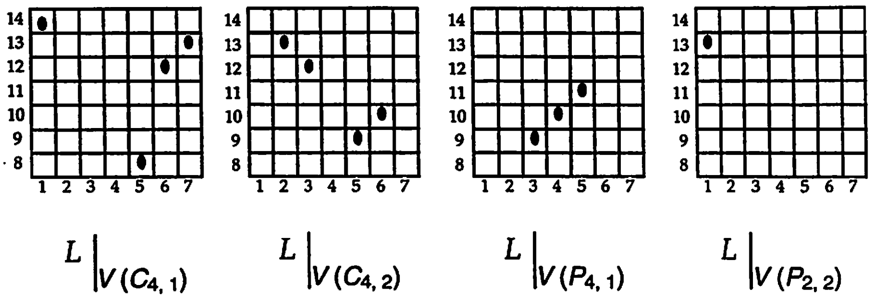

Figure 13

The following step is to construct the graph H with the desired function g as we do in Figure 14.

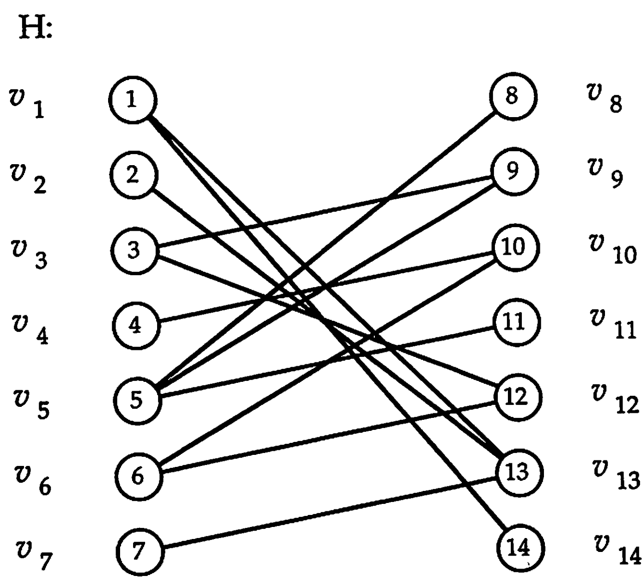

Figure 14

As an immediate consequence to Theorem 4, we get Corollary 2.

Corollary 2. *Let G be a SEM graph or pseudograph of order p and size q. Then there exists a bipartite graph H of order $2p$ and size q which is SSEM.*

From now on, we will refer to the graph H obtained in Corollary 2, as the bipartite graph H induced by G, and we will write it as H_G. A natural question to ask is whether we can say anything about the bipartite graph H_G, from the properties of G. Although the study of the structure of H_G has not been done very carefully yet, the following observations are immediate from Theorems 3 and 4.

Theorem 5. *If a digraph G contains an eulerian cycle which is oriented either clockwise or counterclockwise, then H_G contains a perfect matching.*

Theorem 6. *If the digraph G contains two edge disjoint eulerian cycles which are oriented both clockwise, counterclockwise, or one clockwise and the other counterclockwise, then H_G contains a 2-regular spanning subgraph.*

Theorem 7. *Let G be a graph with vertex set $V(G) = \{v_i : 1 \leq i \leq P\}$ with an orientation, and denote by $out(v)$ and $in(v)$ the outdegree and the indegree of any vertex v of $V(G)$ respectively. Then if V_1 and V_2 are the bipartite sets of the graph H_G, the degree sequence of the vertices of V_1 is $out(v_1), out(v_2), \ldots, out(v_p)$ and the degree sequence of the vertices of V_2 is $in(v_1), in(v_2), \ldots, in(v_p)$.*

The next theorem provides an interesting corollary to Theorem 7.

Theorem 8. *If G is a graph with degree sequence $2k_1, 2k_2, \ldots, 2k_p$, then there exists an orientation of the edges of G such that the graph H_G is a bipartite graph with bipartite set V_1 and V_2 and the degree sequence of vertices of V_1 is equal to the degree sequence of the vertices of V_2 and equal to $k_1, k_2, \ldots, k_p$.*

Proof: Since all the degrees of the vertices of G are even, then G is decomposable into cycles (see [8]). Orient each cycle either clockwise or counterclockwise and apply theorem 7. □

4 Conclusions

This work has been focused to generate SSEM labelings of bipartite graphs from preexisting labelings of graphs and some specific types of pseudographs. This is interesting, since Figueroa *et al.* [4] have shown that if a graph G is a bipartite SEM graph, then the graph shaped by an odd number of copies of G is also SEM.

Also, Kotzig and Rosa [7] proposed the problem of characterizing the set of 2-regular edge magic graphs. Motivated by this problem, we propose the problem of characterizing 2-regular super edge magic graphs. We have done some work in this direction by proving that every SEM labeling of a 2-regular graph "come from" an SSEM labeling of a 1-regular graph Some other work on this problem has been done by Figueroa *et al.* and can be found in [5] and [6].

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