On Networks with Maximum Graphical Structure and tenacity T

DARA MOAZZAMI[1]

Department of Engineering Science,
Faculty of Engineering,
University of Tehran, and

Center for Theoretical Physics and Mathematics, A.E.O.I.

Abstract. In this paper, the maximum graphical structure is obtained when the number of vertices p of a connected graph G and tenacity $T(G) = T$ are given. Finally the method of constructing the sort of graphs are also presented.

Introduction:

The concept of tenacity of a graph G was introduced in reference [1,2], as a useful measure of the "vulnerability" of G. In [4], we compared integrity, connectivity, binding number, toughness, and tenacity for several classes of graphs. The results suggest that tenacity is a most suitable measure of stability or vulnerability in that for many graphs it is best able to distinguish between graphs that intuitively should have different levels of vulnerability. The tenacity of a graph G, T(G), is defined by $T(G) = min\{\frac{|A|+\tau(G-A)}{\omega(G-A)}\}$,

[1]Moazzami@shafagh.ut.ac.ir

where the minimum is taken over all vertex cutset A of G. We define G-A to be the graph induced by the vertices of V-A, $\tau(G-A)$ is the number of vertices in the largest component of the graph by G-A and $\omega(G-A)$ is the number of components of G-A. A connected graph G is called T-tenacious if $\mid A \mid +\tau(G-A) \geq T\omega(G-A)$ holds for any subset A of vertices of G with $\omega(G-A) > 1$. If G is not complete, then there is a largest T such that G is T-tenacious ; this T is the tenacity of G. On the other hand, a complete graph contains no vertex cutset and so it is T-tenacious for every T. Accordingly, we define $T(K_p) = \infty$ for every p $(p \geq 1)$. A set $A \subseteq V(G)$ is said to be a T-set of G if $T(G) = \frac{|A|+\tau(G-A)}{\omega(G-A)}$.

Theorem: Let p be a number of vertices, A be a T-set and $\tau(G-A) = \tau$. So $T = \frac{|A|+\tau(G-A)}{\omega(G-A)} = \frac{b}{a}$. Then

$$max \mid E(G) \mid = \begin{cases} \binom{p-a+1}{2} + (b-\tau)(a-1), & b > \tau, 2n-m \geq 1 \\ \binom{p-ma+1}{2} + (mb-\tau)(ma-1), & mb > \tau, 2n-m < 1. \end{cases}$$

where $m = \lfloor \frac{p+\tau}{b+a} \rfloor$, $n = \frac{1}{2a} + \frac{p+\tau}{a+2b}$ and $p \geq b+a-\tau$.

Proof: Let $A \subseteq V(G)$ be a T-set. Then $T = \frac{|A|+\tau(G-A)}{\omega(G-A)} = \frac{b}{a}$. Suppose G-A have $G_1, G_2, \cdots, G_l$ components, $\mid G_i \mid = p_i \geq 1$, $(i = 1, 2, \cdots, l)$, $\mid A \mid = x \geq 1$, so $\sum_{i=1}^{l} p_i + x = p$. Then $T = \frac{b}{a} = \frac{x+\tau}{l}$.

If we want to have maximum number of edges, $\mid E(G) \mid$, we must let

(i) G[A] to be complete,

(ii) G_i, $(i = 1, 2, \cdots, l)$, to be complete,

(iii) All vertices in A be connected in all vertices in G_i, $(i = 1, 2 \cdots, l)$.

The number of edges of a graph G can be expressed as f($p_1, p_2, \cdots, p_l, x$), if the above three conditions is satisfied.

$$f(p_1, p_2, \cdots, p_l, x) = \sum_{i=1}^{l} \binom{p_i}{2} + \binom{x}{2} + x\sum_{i=1}^{l} p_i =$$

$$\frac{1}{2}\sum_{i=1}^{l} p_i^2 - \frac{1}{2}\sum_{i=1}^{l} p_i + \binom{x}{2} + x\sum_{i=1}^{l} p_i = \frac{1}{2}(\sum_{i=1}^{l} p_i)^2 + (x - \frac{1}{2})\sum_{i=1}^{l} p_i +$$

$$\binom{x}{2}-\sum_{1\le i<j\le l} p_ip_j=\frac{1}{2}(p-x)^2+(x-\frac{1}{2})(p-x)+\binom{x}{2}-\sum_{1\le i<j\le l} p_ip_j.$$

To maximize f, it is necessary to make $\sum_{1\le i<j\le l} p_ip_j$ minimum. Here the condition is $1\le p_i\le p-x-l+1,(i=1,2,\cdots,l)$, where p_i is the positive integer number.

To solve this problem, we let $p_1=p_2=\cdots=p_{l-1}=1$, $p_l=p-x-l+1$, so $\sum_{1\le i<j\le l} p_ip_j$ will be minimum. Now substitute these values into f, we get

$$f(1,1,\cdots,1,p-x-l+1,x)=f(x)=\binom{p-x-l+1}{2}+\binom{x}{2}+x(p-x).$$

¿From $T=\frac{x+\tau}{l}$, we have $l=\frac{x+\tau}{T}=\frac{ax+a\tau}{b}$, substitute into $f(x)$, we get

$$f(x)=\binom{p-x-\frac{ax}{b}-\frac{a\tau}{b}+1}{2}+\binom{x}{2}+x(p-x)=$$
$$\frac{1}{2}\left\{\frac{a}{b}(2+\frac{a}{b})x^2-[\frac{a(2p+1)}{b}-\frac{2a}{b}(1+\frac{a}{b})\tau+2]x+(p-\frac{a\tau}{b}+1)(p-\frac{a\tau}{b})\right\}.$$

¿From $T=\frac{b}{a}$, we can easily see that $x+\tau=by$, where y is a positive integer number. Easily $y\ge 1$, also for $p_l=p-x-l+1=p-x-\frac{a}{b}x-\frac{a}{b}\tau+1=p-(b+a)y+\tau+1\ge 1$, we get $y\le\frac{p+\tau}{b+a}$. Since y is a positive integer number, we have $y\le\lfloor\frac{p+\tau}{b+a}\rfloor$. So the maximum value of $f(x)$ can be changed into the problem of finding the maximum value of $g(y)$:

$$maxg(y)=$$
$$\frac{1}{2}\left\{\frac{a}{b}(2+\frac{a}{b})(by-\tau)^2-[\frac{a(2p+1)}{b}-\frac{2a}{b}(1+\frac{a}{b})\tau+2](by-\tau)+(p-\frac{a\tau}{b}+1)\right.$$
$$\left.(p-\frac{a\tau}{b})\right\}=\ (ab+\frac{a^2}{2})y^2-(a\tau+ap+b+\frac{a}{2})y+\frac{p^2+p}{2}+\tau,$$

such that $1\le y\le\lfloor\frac{p+\tau}{b+a}\rfloor=m$, and y is a variable. Now we consider y as a real variable, and we find the $g'(y)$;

$$g'(y)=a(2b+a)y-a(\tau+p)-\frac{2b+a}{2}=0,$$

so we get

$$y = \frac{1}{2a} + \frac{p+\tau}{a+2b} = n$$

Since $g(y)$ is parabola and concave up, the symetric axis can be get from $y = n$. So when $n \geq \frac{1+m}{2}$,i.e. $(2n - m) \geq 1$, $g(y)$ is maximum for $y = 1$. So we put $y = 1$ into $g(y)$. Then we have

$$max \mid E(G) \mid = g(1) = \binom{p-a+1}{2} + (b-\tau)(a-1).$$

When $n < \frac{1+m}{2}$, i.e. $(2n - m) < 1$, $g(y)$ gets maximum value when $y = m$. Now substitute $y = m$ into $g(y)$, we get

$$max \mid E(G) \mid = g(m) = \binom{p-ma+1}{2} + (mb-\tau)(ma-1).$$

Construction and Examples:

In this section the number of vertices, p, and $T(G) = \frac{|A|+\tau}{\omega} = \frac{b}{a}$ are given, we would like to introduce a method for constructing the graph G with maximum number of edges, $\mid E(G) \mid$. We give some examples by using $m = \lfloor \frac{p+\tau}{b+a} \rfloor$ and $n = \frac{1}{2a} + \frac{p+\tau}{a+2b}$. When we found m and n, we would like to know which inequlity, $2n - m \geq 1$ or $2n - m < 1$, are fulfilled.

If $2n - m \geq 1$, then the construction is as follows:

Step 1)- Construct the complete graph K_{p-m+1};

Step 2)- Add a-1 vertices to the complete graph K_{p-m+1};

Step 3)- Joined each of these a-1 vertices to $b-\tau$ vertices of the complete graph K_{p-m+1} respectively.

If $2n - m < 1$, then construction is as follows:

Step 1)- Construct the complete graph K_{p-ma+1};

Step 2)- Add ma-1 vertices to the complete graph K_{p-ma+1};

Step 3)- Joined each of the ma-1 vertices to $mb - \tau$ vertices of the complete graph K_{p-ma+1}.

Remark: From our above proof we can see that if $2b - a = 1$, then we can use both construction.

Example 1)- Consider a graph G with $T(G) = \frac{3}{2}$, $\tau(G) = 1$ and $p = 11$. Then $b > \tau$, $m = 2$, $n = \frac{7}{4}$ and so $2n - m = \frac{3}{2} > 1$. Thus we use a first method to construct a graph with $max \mid E(G) \mid = 47$, as shown in Fig.1.

Example 2)- Consider a graph G with $T(G) = \frac{5}{2}$, $\tau(G) = 1$, and $p = 13$. Then $m = 2$, $n = \frac{17}{12}$, $mb > \tau$, and also $2n - m = \frac{5}{6} < 1$. Thus we use a second method to construct a graph G with $max \mid E(G) \mid = 72$, as shown in Fig.2.

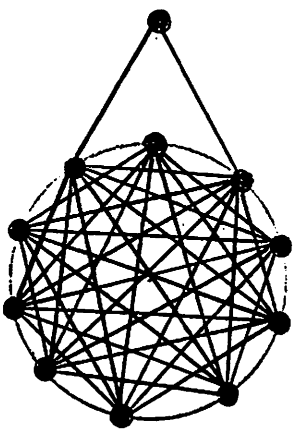

Fig.1

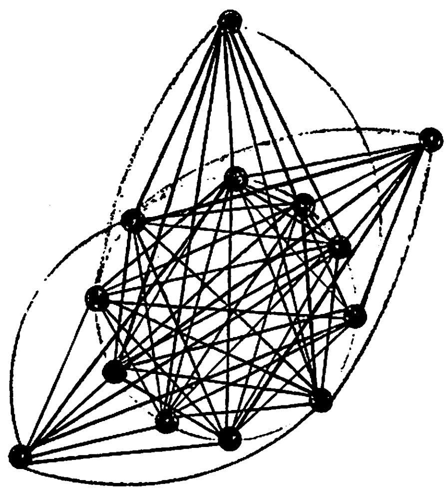

Fig.2

Example 3)- Consider a graph G with $T(G) = \frac{3}{2}$, $\tau(G) = 1$, and $p = 9$. Then $m = 2$, $n = \frac{3}{2}$, $b > \tau$, and so $2n - m = 1$. Thus we can use both method to construct a graph G with $max \mid E(G) \mid = 30$, as shown in Fig.3 and Fig.4.

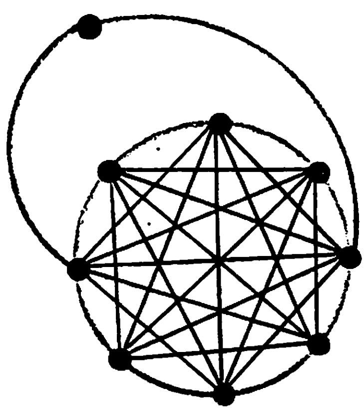

Fig.3

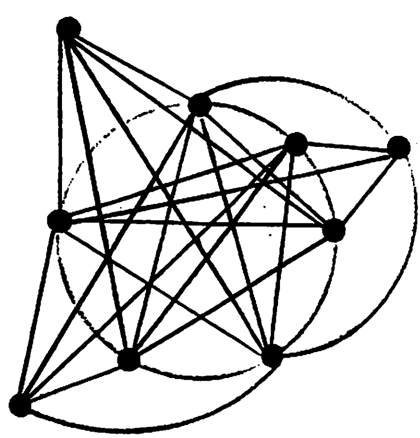

Fig.4

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