On the Integrity of Combinations of Graphs

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Abstract

The integrity of a graph G, denoted $I(G)$, is defined by $I(G) := \min_{S \subset V(G)}\{|S| + m(G-S)\}$ where $m(G-S)$ denotes the maximum order of a component of $G-S$. In this paper we explore the integrity of various combinations of graphs in terms of the integrity and other graphical parameters of the constituent graphs. Specifically, explicit formulae and/or bounds are derived for the integrity of the join, union, cartesian and lexicographic products of two graphs. Also some results on the integrity of powers of graphs are given.

1 Introduction

Integrity was introduced by Barefoot, Entringer & Swart [1] as an alternative measure of the vulnerability of graphs to disruption caused by the removal of vertices. The motivation was that, in some respects, connectivity is oversensitive to local weaknesses and does not reflect overall vulnerability. For example, the stars $K(1, n+1)$ and the graphs $K_1 + (K_1 \cup K_n)$ are all of connectivity one but differ vastly in how much damage is done to the corresponding communications network by the removal of a cut vertex: in the former case all communications are destroyed, whereas in the latter all but two stations remain in mutual contact.

It is our aim to explore the integrity of various combinations of graphs in terms of the integrity and other graphical parameters of the constituent graphs. Specifically we give explicit formulae and/or bounds for the integrity of the join, union, cartesian and lexicographic products of two graphs. We also take a brief look at unary operations on graphs including powers of graphs. First though, we give the requisite definitions.

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2 Definitions

In this section, we define ***integrity*** and related concepts, and introduce the necessary terminology and notation. All undefined terminology and notation is taken from [3]. We use further a ***cut-set*** as any set of vertices in a graph whose removal leaves a disconnected graph. Also we use $\subset$ to denote strict inclusion.

▷▷ For any graph G, the maximum order of a component of G is denoted by $m(G)$.

The notation $m(G)$ is from [1] as is the following definition:

▷▷ For any graph G the ***integrity*** of G, denoted $I(G)$, is defined by

$$I(G) := \min_{S \subset V(G)} \{\, |S| + m(G - S)\, \}. \qquad (2.1)$$

▷▷ An ***I-set*** of G is any (strict) subset S of $V(G)$ for which $I(G) = |S| + m(G - S)$.

Further concepts that are useful computationally are now defined (see [4]):

▷▷ For any graph G,
$D_k(G) := \min\{\, |S| : S \subset V(G)\ \&\ m(G - S) \leq k\,\} \quad k = 1, 2, \ldots$
$E_l(G) := \min\{\, m(G - S) : S \subset V(G) \& |S| = l\,\} \quad l = 0, 1, \ldots, p(G) - 1.$

It is easily seen that the definition of $E_l(G)$ is unaffected by the replacement of the condition '$|S| = l$' by the inequality '$|S| \leq l$'. It is also obvious that, for any graph G,

$$D_1(G) = \alpha(G) \qquad \text{and} \qquad E_0(G) = m(G).$$

Further, the following definition is useful:

▷▷ $\theta(G) := p(G) - m(G)$.

These definitions yield the following alternative formulations for integrity:

$$\begin{aligned} I(G) &= \min_k (D_k(G) + k), \\ I(G) &= \min_{0 \le l < p(G)} (E_l(G) + l) \\ I(G) &= p(G) - \max_{H \prec G} \theta(H). \end{aligned}$$

3 Past Results

We list the following known results for reference:

Proposition 3.1 [4] *Let G be a graph of order n.*
a) $I(G) = n$ iff $G \cong K_n$
b) $I(G) = 1$ iff $G \cong \bar{K}_n$

Proposition 3.2 [1] $I(K(a_1, a_2, \ldots, a_r)) = \sum_i a_i + 1 - \max_i a_i$.

Proposition 3.3 [2]
a) $I(P_n) = \lceil 2\sqrt{n+1} \rceil - 2 \quad n = 1, 2, \ldots$
b) $I(C_n) = \lceil 2\sqrt{n} \rceil \quad n = 3, 4, \ldots$

Proposition 3.4 [4]
a) if $G \subseteq H$ then $D_k(G) \le D_k(H)$, $E_l(G) \le E_l(H)$ and $I(G) \le I(H)$.
b) If G is non-trivial then for all $v \in V(G)$, $I(G - v) \ge I(G) - 1$.
c) For all $e \in E(G)$, $I(G - e) \ge I(G) - 1$.

Proposition 3.5 [4] *For all graphs G,*
a) if G is not complete then every I-set of G is a cut-set of G and hence has cardinality at least $\kappa(G)$;
b) if S is a minimal I-set of G, then for all $v \in S$, v is a cut-vertex of $G - (S - v)$.

Proposition 3.6 [4] *For every graph G, $\delta(G) + 1 \le I(G) \le \alpha(G) + 1$.*

Proposition 3.7 [4] *For any graph G, $I(G) = p - 1$ iff $\bar{G}$ is nonempty and has girth at least 5.*

4 Powers of Graphs

In this section we consider some of the simple unary operations on graphs and investigate how the integrity of the resultant graph compares with that of the original graph.

Vertex- deleted and edge-deleted subgraphs are examined in some detail in [4]—proposition 3.4 summarises some of the results obtained; further theorems about the integrity of a graph and its complement are to be found in [5]. Focussing on the powers of graphs, we note first of all that from proposition 3.4 the following holds for all graphs G,

$$I(G) \le I(G^2) \le I(G^3) \ldots$$

Equality is possible at every step, for example, if $I(G) = m(G)$ (i.e. if the empty set is an I-set of G). Such graphs include those in which every component is complete. Further, there exist noncomplete connected graphs for which $I(G) = I(G^2)$; for example any graph which is the complement of a tree of diameter three (this follows from proposition 3.7) or the graph formed by taking two disjoint copies of K_n, $n \ge 2$, and inserting one edge between them. But for higher powers we have the following theorem:

Theorem 4.1 *For all graphs G,*

$$I(G^3) = I(G) \quad \textit{iff} \quad I(G) = m(G).$$

Proof

The 'if' part follows easily; we prove the 'only if' part and assume therefore that $I(G^3) = I(G)$. Let S be a minimal I-set of G^3; then S is an I-set of G. *Suppose* S is nonempty and let $v \in S$. Let $H_1, \ldots, H_r$ be the components of $G^3 - S$; then $k(G^3 - (S - v)) < k(G^3 - S)$ (by proposition 3.5b) so that $r \ge 2$ and there exist integers i and j, $i \ne j$ such that the vertices of H_i and H_j are contained in the same component of G^3 and hence in the same component of G. Let $P = v_1 v_2 \cdots v_n$ be a shortest path in G between vertices of two distinct components of $G^3 - S$; say $v_1 \in H_i$ and $v_n \in H_j$. Then $n \ge 5$ and $v_2, v_3, \ldots, v_{n-1} \in S$. It follows from our choice of P that

v_3 is an isolated vertex (and hence not a cut-vertex) of $G - (S - v_3)$, which contradicts proposition 3.5b. Therefore $S = \phi$ is an I-set of G and thus $I(G) = m(G)$. ⊗

From this one may derive the obvious corollary:

Corollary 4.1 *For all graphs G, if $I(G) = I(G^3)$ then $I(G) = I(G^2) = \ldots = m(G)$.*

5 The Join of Graphs

In this section we consider the integrity of the join of two graphs, inter alia.

Theorem 5.1 *For all graphs G and H,*

$$I(G+H) = \min\{I(G) + p(H), I(H) + p(G)\}.$$

Proof

We see immediately that the statement is true if $G+H$ is complete. Therefore we may assume that $G+H$ is noncomplete. Then every I-set of $G+H$ is a cut-set of $G+H$ and hence a superset of either $V(G)$ or $V(H)$. Thus $I(G+H) = \min\{f(G,H), f(H,G)\}$ where

$$f(G,H) = \min_{V(H)\subseteq S\subset V(G)\cup V(H)} \{m(G+H-S) + |S|\}$$

and $f(H,G)$ mutatis mutandis. Then

$$\begin{aligned} f(G,H) &= \min_{S'\subset V(G)} \left\{m(G-S') + |S'| + p(H)\right\} \\ &= I(G) + p(H) \end{aligned}$$

and similarly $f(H,G) = I(H) + p(G)$ so that the result follows. ⊗

Corollary 5.1 *For all graphs G, $I(G + K_s) = I(G) + s$.* ⊗

We have also the following extension of the corollary:

Theorem 5.2 *Let G be a graph and form G' from G by introducing a new vertex v and making v adjacent to r of the vertices in G. If $r \geq I(G)$ then $I(G') = I(G) + 1$.*

Proof

Certainly, $I(G') \leq 1 + I(G' - v) = 1 + I(G)$. To prove the reverse inequality, consider any $S \subset V(G')$ and the two possibilities: If $v \in S$ then

$$\begin{aligned} m(G' - S) + |S| &= m(G - (S - \{v\})) + |S - \{v\}| + 1 \\ &\geq I(G) + 1 \end{aligned}$$

while if $v \notin S$ then

$$\begin{aligned} m(G' - S) + |S| &\geq \Delta(G' - S) + 1 + |S| \\ &\geq (r - |S|) + 1 + |S| \\ &\geq I(G) + 1. \end{aligned}$$

Hence $I(G') \geq 1 + I(G' - v)$ and the theorem is proved. ⊗

Of course the converse does not hold; consider, for example, forming $G' = P_6$ from $G = P_5$. Then $r = 1$ while $I(G) = 3$ and $I(G') = 4$. Further this result is best possible in that, for example, if we form $G' = P_3$ from $G = P_2$ then $I(G) = 2$ and $I(G') = 2$ while $r = 1 = I(G) - 1$.

6 The Union of Graphs

We consider now the integrity of the union of two or more graphs. We commence with an obvious lemma:

Lemma 6.1 *Let $G = \bigcup_{j=1}^{r} G_j$. Then*

a) $D_n(G) = \sum_{j=1}^{r} D_n(G_j)$,

b) $I(G) = \min_n \left\{ n + \sum_{j=1}^{r} D_n(G_j) \right\}$,

c) $m(G) = \max_j m(G_j)$. ⊗

Further, from the above (or ab initio) we have the following:

Theorem 6.2 *For all graphs G, if $r \geq m(G) - 1$ then $I(rG) = m(G)$.* $\otimes$

There are obvious lower bounds for $I(G \cup H)$ viz. $I(G)$ and $I(H)$. For an upper bound, we have the following:

Theorem 6.3 *For all graphs G and H,*

$$I(G \cup H) \leq I(G) + I(H) - 1.$$

Proof

Let S and T be I-sets of G and H respectively. Then

$$\begin{aligned} I(G \cup H) &\leq |S \cup T| + m(G \cup H - S \cup T) \\ &= |S| + |T| + \max\left\{m(G - S), m(H - T)\right\} \\ &= \left\{|S| + m(G - S)\right\} + \left\{|T| + 1\right\} - 1, \quad \text{say} \\ &\leq I(G) + I(H) - 1 \end{aligned}$$

proving the result.

Corollary 6.3 *If $G = \bigcup_{j=1}^{r} G_j$ then*

$$I(G) \leq \sum_{j=1}^{r} I(G_j) + 1 - r. \qquad \otimes$$

For equality in the above theorem and corollary, consider the graphs $G^{(r)}$ given, for positive integers r, by

$$G^{(r)} := \bigcup_{j=1}^{r} K(2^{j-1}, 2^{j-1}).$$

Then using lemma 6.1 and noting that

$$D_k(K(a,a)) = \begin{cases} a & \text{if } k \leq a \\ 2a - k & \text{if } a \leq k \leq 2a \\ 0 & \text{if } k \geq 2a \end{cases}$$

one can easily show that $D_k(G^{(r)}) = 2^r - k$ for $k = 1, 2, \ldots, 2^r$. Thus

$I(G^{(r)}) = 2^r$ while

$$\sum_{j=1}^{r} I(K(2^{j-1}, 2^{j-1}) + 1 - r = \sum_{j=1}^{r}(2^{j-1} + 1) + 1 - r = 2^r.$$

7 The Lexicographic Product

In this section we determine the integrity of the lexicographic product of two graphs in terms of the integrity and other graphical parameters of the original graphs.

Let us consider $G[H]$ where the vertex sets of graphs G and H are given by $\{v_1, v_2, \ldots, v_m\}$ and $\{w_1, w_2, \ldots, w_n\}$ respectively. Let X be an I-set of $G[H]$; then for $i = 1, 2, \ldots, m$ set $T_i := \{\, w_j \in V(H) : (v_i, w_j) \in X \,\}$ and further let $S := \{\, v_i \in V(G) : T_i = V(H) \,\}$.

If F is any component of $G - S$, let F' denote the subgraph of $G[H] - X$ induced by $\{\, (v_i, w_j) \in V(G[H]) - X : v_i \in V(F) \,\}$. We have the following two cases:

1. S is not a vertex cover of G: let F be a non-trivial component of $G - S$ so that F' is connected. Consequently, by the optimality of X, without loss of generality we may assume that $T_i = \phi$ for all $v_i \in V(F)$ and indeed we may assume that $T_i = \phi$ for all $v_i \notin S$. Therefore $m(G[H] - X) = m(G - S) \cdot p(H)$ and thus for this case,

$$\min_X\{|X| + m(G[H] - X)\} = I(G) \cdot p(H).$$

2. S is a vertex cover of G: let $F_1, F_2, \ldots, F_r$ be the trivial components of $G - S$, containing vertices $v_1, v_2, \ldots, v_r$ respectively (say). We may assume, without loss of generality, that X is such that $T_1 = T_2 = \cdots = T_r = T$ (say) while obviously $|S|$ is as small as possible viz. $\alpha(G)$. Therefore $m(G[H] - X) = m(H - T)$ and thus for this case,

$$\min_X\{|X| + m(G[H] - X)\} = \alpha(G) \cdot p(H) + I(\beta(G) \cdot H).$$

Thus we have proved the following theorem:

Theorem 7.1 *For all graphs G and H,*

$$I(G[H]) = \min\{\, I(G)\cdot p(H),\ \alpha(G)\cdot p(H) + I(\beta(G)\cdot H)\,\}. \qquad \otimes$$

Corollary 7.1 *For all graphs G and H,*
a) $I(K_m[H]) = (m-1)p(H) + I(H)$
b) $I(G[K_n]) = n \cdot I(G)$ $\otimes$

As a specific example consider the determination of the integrity of the lexicographic product of two stars. Letting $G = K(1, m-1)$ and $H = K(1, n-1)$ we have that

$$\begin{aligned} I(G[H]) &= \min\{\, I(G)\cdot p(H),\ \alpha(G)\cdot p(H) + I(\beta(G)\cdot H)\,\} \\ &= \min\{2.n, 1.n + I((m-1)H)\} \\ &= \min\{2n, n + \min\{m, n\}\} \end{aligned}$$

so that

$$I(K(1, m-1)[K(1, n-1)]) = \min\{m+n, 2n\}.$$

8 Cartesian Product

In this section, we give some results and bounds on the cartesian products and discuss the integrity of the graphs formed by the cartesian product of two complete graphs.

Unlike their lexicographic counterparts, the cartesian products seem to be much more difficult to evaluate. Indeed, as we shall show, the calculation of $I(K_m \times K_n)$ requires considerable manipulation before we get the final answer which is still an integer optimisation problem.

In general, the best we can do is to give bounds on the values. Lower bounds seem sparse; indeed we have the lower bound formed by noting that $G \times H$ contains $p(G)\cdot H$ and $p(H)\cdot G$ as subgraphs. For the upper bounds, one may use the above results on lexicographic products noting that $G \times H$ has $G[H]$ and $H[G]$ as supergraphs. These bounds appear to allow a lot of leeway.

However a way of combining subsets of the vertex sets of the original graphs to get a suitable (i.e. near optimal) subset of the vertex set of the product, appears more promising:

Let G and H have orders m and n respectively. Let $S \subseteq V(G)$ and $T \subseteq V(H)$ of cardinality s and t respectively. Then set

$$X := S \times V(H) \cup T \times V(G) - S \times T. \tag{8.1}$$

Here $|X| = sn + tm - 2st$ and

$$\begin{aligned} m(G \times H - X) &= m((G - S) \times (H - T) \cup \langle S \times T \rangle) \\ &= \max\{m(G - S) \cdot m(H - T), m(\langle S \rangle) \cdot m(\langle T \rangle)\} \end{aligned}$$

where the null graph K_0 is assumed to have $m(K_0) = 0$ and where the last line follows from noting the general result that $m(F_1 \times F_2) = m(F_1) \times m(F_2)$.

We consider a few specific examples. Consider firstly, the case where G and H are complete. Then, as it is shown in the following section, suitably chosen sets X are (among the) I-sets of the product.

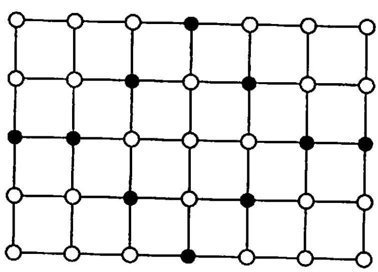

Figure 1: The I-set of $P_5 \times P_7$

As a second example, consider the case where G and H are paths. Specifically, say $G \cong P_5$ and $H \cong P_7$; then the smallest value of $|X| + m(G \times H - X)$, where X is constructed by the above method, is 16 while $I(P_5 \times P_7) = 15$ with the unique I-set given in figure 1. It seems feasible that this is indicative of the determination of $I(P_m \times P_n)$ for m and n of the same order of magnitude. On the other hand, for small values of m and n one does get I-sets of the form given in equation 8.1 above. For example, $I(P_2 \times P_8) = 8$ and the unique I-set is given by X above where S is empty and T is the I-set of P_8.

For our third example we need to determine the integrity of the cartesian product of two stars:

Theorem 8.1 *For all $m \geq n \geq 2$, it holds that $I(K(1,m-1) \times K(1,n-1)) = \min\{m+n-1, 2n\}$.*

Proof

Let P be an I-set of the product $G = K(1,m-1) \times K(1,n-1)$. Then $\alpha(G) = m+n-2$ so that $I(G) \leq m+n-1$. Now suppose that $I(G) < m+n-1$. Let the vertices of both stars be labelled such that the central vertices of each receive the labelling 0; then we note the following observations:

- $(0,0) \in P$ (else $m(G-P) + |P| \geq |N[(0,0)]| = n+m-1$)
- *If $(i,0) \notin P$ and $(0,j) \notin P$ then $(i,j) \in P$;* (else $m(G-P) + |P| \geq |N[V(F)]| = n+m$ where F is the (connected) subgraph induced by the vertices $(i,0)$, $(0,j)$ and (i,j))

Now let $q = |\{(i,0) : (i,0) \notin P\}|$ and $r = |\{(0,j) : (0,j) \notin P\}|$. Then by the above observations, at least $B = 1+(m-1-q)+(n-1-r)+qr$ vertices are contained in P; further depending on the degrees of the vertices not in P we have the following lower bounds for $|P| + m(G-P)$:

$$
\begin{array}{ll}
B+1 & \text{if } r = q = 0, \\
B+n & \text{if } r = 0 \text{ and } q \neq 0, \\
B+m & \text{if } q = 0 \text{ and } r \neq 0, \\
B+\max\{m-q, n-r\} & \text{if } q \neq 0 \text{ and } r \neq 0.
\end{array}
$$

Minimising these bounds over q and r one obtains an overall minimum of precisely the RHS of the equation in the statement of the theorem. This result together with the initial comments proves the theorem. ⊗

Thereafter it may be shown that all I-sets are of the form described by equation 8.1 above where S consists of a single central vertex of $K(1,m-1)$ and T is empty or consists of a single central vertex of $K(1,n-1)$.

We conclude this section with a conjecture on cubes:

Conjecture 8.2 *For all $r \geq 1$, $I(Q_r) = 2^{r-1} + 1$.*

Certainly the value given is an upper bound by proposition 3.6. The conjecture is easily verified for $r = 1$, 2 and 3. Furthermore, the results for $r = 2$ and $r = 3$ suggest that the following stronger conjecture may be true:

Conjecture 8.3 *For all $r \geq 2$, the I-sets of Q_r are the two minimum vertex covers.*

9 The Integrity of $K_m \times K_n$

In this section we investigate the integrity of the cartesian product of two complete graphs.

9.1 Formulation

Let $G \cong K_m$ and $H \cong K_n$ with $m \geq n \geq 2$, and let the vertex sets $V(G)$ and $V(H)$ be given by $\{v_1, v_2, \ldots, v_m\}$, $V(H) = \{w_1, w_2, \ldots, w_n\}$ and $F = G \times H$. Then necessarily F is noncomplete. Let S be a cut-set of F and let $F_1, F_2, \ldots, F_k$ be the components of $F - S$. Denote for $i = 1, 2, \ldots, k$,

$$\begin{aligned} V_i &:= \{v_j \in V(G) : \exists w_l : (v_j, w_l) \in F_i\} \\ W_i &:= \{w_l \in V(H) : \exists v_j : (v_j, w_l) \in F_i\}. \end{aligned}$$

Then, by the definition of cartesian products, $V_1, V_2, \ldots, V_k$ are disjoint as are $W_1, W_2, \ldots, W_k$. Now form

$$\begin{cases} X_1 = V(G) - \bigcup_{j=2}^{k} V_j \\ X_i = V_i \quad i = 2, 3, \ldots, k \end{cases}$$

and form sets Y_i from the W_i similarly. Further define

$$T = V(F) - \bigcup_{i=1}^{k} X_i \times Y_i.$$

Then every component of $F - T$ is a supergraph of exactly one component of $F - S$ so that $\theta(F - T) \geq \theta(F - S)$. Therefore, in our search for I-sets, we need only consider sets of the form given by T.

Now

$$\theta(F-T)=\sum_i |X_i||Y_i| - \max_i |X_i||Y_i|.$$

Let us denote for integral $r \geq k \geq 2$:

▷▷ $\Pi(k,r)$ is the set of all vectors $\underline{z}=(z_1,z_2,\ldots,z_k)$ such that $\sum_{i=1}^k z_i = r$, and $z_1,z_2,\ldots,z_k$ are positive integers.

Hence we may write

$$I(F)=mn-\max_{2\leq k\leq n}\theta_k(m,n) \tag{9.1}$$

where

$$\theta_k(m,n)=\max_{\substack{\underline{x}\in\Pi(k,m)\\ \underline{y}\in\Pi(k,n)}}\left\{\sum_{i=1}^k x_iy_i-\max_i x_iy_i\right\}. \tag{9.2}$$

9.2 The case when $k \geq 3$

We now show that the maximum on the RHS of equation 9.2 is attained when $k=2$ (but not necessarily only for $k=2$). Elementary optimisation techniques yield the following lemma:

Lemma 9.1 *For all $m \geq n \geq 2$, $\theta_2(m,n) \leq mn/4$ with equality iff m and n are both even.* ⊗

This lemma is used in the proof of the following:

Lemma 9.2 *For all $m \geq n \geq k \geq 3$,*

$$\theta_k(m,n) \leq \max\{1+(m-1)(n-1)/4, 2mn/9\}.$$

Proof

Let $k=3$ and let the component of $F-T$ of smallest order be of the form $K_x \times K_y$. Then the order of the 'middle' component of $F-T$ is bounded above by $\theta_2(m-x,n-y)$. Consequently, by lemma 9.1

$$\theta_3(m,n) \leq max_{x,y}xy+(m-x)(n-y)/4$$

$$\text{subject to} \begin{cases} xy \leq (m-x)(n-y)/4 \\ x \geq 1,\ y \geq 1,\ x, y \in \mathbb{R}. \end{cases}$$

Using elementary calculus, we obtain the stated result and by similar techniques all the required bounds may be derived. ⊗

This lemma is then used in the proof of:

Lemma 9.3 *For all $n \geq k \geq 3$, $\theta_2(m,n) \geq \theta_k$.*

Proof

We prove the following result in three cases:

$$\theta_2(m,n) \geq \max\left\{1 + \lfloor (m-1)(n-1)/4 \rfloor, \lfloor 2mn/9 \rfloor\right\}. \qquad (i)$$

so that by lemma 9.2 the lemma follows.

1. *m, n even*: This case follows directly from lemma 9.1.
2. *m odd, n even*: Here $m \to ((m-1)/2, (m+1)/2)$ and $n \to (n/2, n/2)$ represent a valid partition where $\theta = (m-1)n/4$. Then it may easily be shown that $(m-1)n/4$ is at least as large as the RHS of equation (i).
3. *n odd*: In this case $n \to ((n-1)/2, (n+1)/2)$, $m \to (a, m-a)$, where $a = \lfloor m(n+1)/(2n) \rfloor$, represents a valid partition. Then it can be shown that for this partition

   $$\theta = \frac{n-1}{2} \left\lfloor \frac{n+1}{2} \frac{m}{n} \right\rfloor.$$

   Using some manipulation it may be shown that this value too is at least as large as the RHS of equation (i) so that the third and final case is verified. ⊗

9.3 The case when $k = 2$

We have thus shown that we must determine the value of θ_2 in order to determine the integrity. For even m and n this has been done. Using lemmas 9.1 and 9.3 we may therefore state:

Theorem 9.4 *If m and n are even then*

$$I(K_m \times K_n) = \frac{3}{4}mn. \qquad \otimes$$

We consider now general m and n. We may write

$$\theta_2 = \max_{x\in\{0,1,\dots,m\}} F(x)$$

where

$$F(x) := \max_{y\in\{0,1,\dots,n\}} f(x,y)$$

and

$$f(x,y) := \min\{xy, (m-x)(n-y)\}.$$

If we allow $f(x,y)$ for a fixed x to range over the reals as a function in y, then it is maximised at $y^* = n(m-x)/m$. Thus

$$F(x) = \max\{x\lfloor y^*\rfloor, (m-x)(n-\lceil y^*\rceil\}$$

and

$$\theta_2 = \max\{E_1, E_2\}$$

where

$$E_1 = \max_x x\cdot\lfloor (m-x)n/m\rfloor$$

and

$$\begin{aligned} E_2 &= \max_x (m-x)\,(n-\lceil n(m-x)/m\rceil) \\ &= \max_{x'} x'\,(n-\lceil nx'/m\rceil) \\ &= \max_{x'} x'\,\lfloor n-nx'/m\rfloor \\ &= E_1. \end{aligned}$$

Hence we have proved the following theorem:

Theorem 9.5 *If $m \geq n \geq 2$ then*

$$I(K_m \times K_n) = mn - \max_{j=1,2,\dots,m-1} j\cdot\left\lfloor\frac{(m-j)n}{m}\right\rfloor. \qquad \otimes$$

Now one cannot explicitly evaluate the RHS except for m and n even. It is possible though to determine its behaviour for $m \gg n$. Basically, the task is to optimise the real function defined by $h(j) := j \cdot \lfloor n(m-j)/m \rfloor$ over integral values. For sufficiently large values of m, one can guarantee that a certain value of j is the maximum, thus solving the problem. For h even, this value is $\lfloor m/2 \rfloor$, while for h odd this value is one of $\lfloor m(n \pm 1)/(2n) \rfloor$ depending on the parity of the remainder when n is divided into m. The arithmetical details are omitted and thus we reach our final theorem, where we note that the conditions that are given are not necessarily best possible.

Theorem 9.6 *For all positive integers m and n,*
a) if n is even and $m \geq n^2/4$ then $I(K_m \times K_n) = mn - \lfloor m/2 \rfloor \cdot n/2$;
b) if n is odd, $m \geq (n-1)^2/4$ and $m = rn + q$ where r and q are integral and $0 \leq q < r$ then

$$I(K_m \times K_n) = \begin{cases} mn - \dfrac{n+1}{2}\left\lfloor \dfrac{n-1}{2} \cdot \dfrac{m}{n} \right\rfloor & \text{if } q \text{ is odd} \\ mn - \dfrac{n-1}{2}\left\lfloor \dfrac{n+1}{2} \cdot \dfrac{m}{n} \right\rfloor & \text{if } q \text{ is even.} \end{cases}$$

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