On Vertex-Transitive Graphs of Order qp

Dragan Marušič
Mathematics Department, University of California
Santa Cruz, CA 95064, USA

(Vojke Šmuc 12, 66000 Koper, Yugoslavia)

Abstract. The structure and the hamiltonicity of vertex-transitive graphs of order qp, where q and p are distinct primes, are studied. It is proved that if $q < p$ and $p \not\equiv 1 \pmod q$ and G is a vertex-transitive graph of order qp such that *Aut*G contains an imprimitive subgroup, then either G is a circulant or V(G) partitions into p subsets of cardinality q such that there exists a perfect matching between any two of them. Partial results are obtained for $p \equiv 1 \pmod q$. Moreover, it is proved that every connected vertex-transitive graph of order $3p$ is hamiltonian.

1. INTRODUCTION

All the groups and graphs considered in this paper are finite. For the group-theoretic concepts not defined here, we refer the reader to [26]. A transitive permutation group is called (m,n)*-imprimitive* if it has a complete block system consisting of m blocks of cardinality n, where $m,n \geq 2$. Such a block system will be called a *complete* (m,n)*-block system* or a *complete* n*-block system*. We shall assume familiarity with basic graph theory terminology. An n*-graph* is a graph with n vertices. In this paper, q and p will always denote distinct primes.

There has recently been a growing interest in the study of vertex-transitive graphs and the subclass of Cayley graphs. Most of this has been motivated by the intriguing conjecture made by Lovász in 1969 [13, p. 497] that every connected vertex-transitive graph has a hamiltonian path. This conjecture has been verified for graphs of order p, $2p$, p^2, p^3, and $2p^2$ (in which case the graph has a hamiltonian cycle unless it is the Petersen graph), $4p$ and $5p$ (see [1, 16, 17, 18, 19]). Other specific properties of vertex-transitive and Cayley graphs have also been studied, such as connectivity [8, 24, 25] and 1-factorizability [22]. Furthermore, considering a restricted class of vertex-transitive graphs of valency p, Lorimer [11, 12] obtained some substantial results about their automorphism groups. Of course, a graph-theoretic characterization of the entire class of vertex-transitive graphs is presently beyond us. However, since every group of order

p^k, $k \leq 3$, is of a fairly simple structure, the fact that every vertex-transitive p^k-graph, $k \leq 3$, is a Cayley graph [18] gives us considerable information about the structure of these graphs.

If we call an n-graph G having an automorphism whose sole orbit is $V(G)$ an *n-circulant*, then, of course, a p-graph is vertex-transitive if and only if it is a p-circulant. (This result was first observed by Turner [23], who gave an algebraic description of n-circulants.) An extension of this idea is the concept of a metacirculant introduced by Alspach and Parsons [2]. To rephrase their definition, an n-graph G is called a *(k,m)-metacirculant* if $n = km$ and $AutG$ contains a transitive subgroup that is a semidirect product of a cyclic group of order m by another cyclic group (whose order is necessarily a multiple of k). (A finite group Γ is a semidirect product of a group A by a group B if $A \triangleleft \Gamma$, $B < \Gamma$, $A \cap B = 1$, and $AB = \Gamma$.) We shall also say that G is an *n-metacirculant* or a *metacirculant*. Alspach and Parsons [2, 3] studied various properties of metacirculants and gave an algebraic characterization for these graphs. In [4], Alspach and Sutcliffe conjectured that every vertex-transitive $2p$-graph is a $(2,p)$-metacirculant. This conjecture was proved by the author [14] for graphs whose automorphism groups contain an imprimitive subgroup. On the other hand, using the classification of simple groups, one can deduce that no simple primitive groups of degree $2p$ exist when $p > 5$ (see [10, p. 239]). This implies that the above conjecture is true in general. However, it would be worthwhile to try to find a more self-contained proof of this fact.

In this paper, we study vertex-transitive qp-graphs, where $q,p > 2$. A vertex-transitive graph is called *(m,n)-imprimitive*, where $m,n \geq 2$, if its automorphism group contains an (m,n)-imprimitive subgroup and is called *primitive* if its automorphism group contains no imprimitive subgroup. In view of the above definitions, one easily sees that every vertex-transitive qp-graph is either (q,p)-imprimitive or (p,q)-imprimitive or primitive. (We remark that the classes of (q,p)-imprimitive and (p,q)-imprimitive graphs overlap—in fact, they both contain qp-circulants. Moreover, a $(p,2)$-imprimitive graph is necessarily $(2,p)$-imprimitive—see [14].) We know that the class of primitive qp-graphs is not empty as it contains, for example, the odd graph O_4 and its complement [15]. In a recent paper Liebeck and Saxl [10] have described all primitive groups of degree qp where q and p are distinct primes. Their result may prove useful in studying primitive qp-graphs. In this paper we shall devote our attention to the two subclasses of imprimitive graphs, investigating their hamiltonicity and studying their relationship to the metacirculants. The main results are proved in Sections 3, 4 and 5. Proposition 3.3 states that, for $q < p$, the classes of (q,p)-metacirculants and (q,p)-imprimitive graphs coincide, whereas Theorems 3.4 and 3.5 deal with the properties of imprimitive qp-graphs which are not metacirculants. We shall also give examples of such graphs. Moreover, we prove that connected (q,p)-imprimitive graphs, where $q < p$, and connected vertex-transitive $3p$-graphs are hamiltonian (Theorems 4.4 and 5.6).

2. PRELIMINARIES

We start by defining a number of new concepts and then go on to prove a few lemmas that will be needed to obtain our main results.

Let V be a finite set, $W \subseteq V$, and Γ be a permutation group on V. We let $\Gamma_W = \{\gamma \in \Gamma : \gamma(W) = W\}$ and we let $\mathbf{V}(\Gamma)$ denote the set of orbits of Γ. If $\alpha \in \Gamma$, let $\mathbf{V}(\alpha) = \mathbf{V}(\langle\alpha\rangle)$. We say that α is *(m,n)-semiregular* if it has m orbits of cardinality $n \geq 2$ and no other orbits. By $[\alpha]$, we denote the subgroup of all permutations τ in Γ such that $\tau(X) \in \mathbf{V}(\alpha)$ whenever $X \in \mathbf{V}(\alpha)$. Suppose that Γ is imprimitive, with a complete block system $\mathbf{B} = \{B_1, B_2, \ldots, B_k\}$. Then, $\bar{\gamma}$ will denote the permutation on $\mathbf{B}$ induced by $\gamma \in \Gamma$ (that means $\bar{\gamma} : B_i \mapsto \gamma(B_i)$ for each $i \in (1, 2, \ldots, k)$). By [26, Proposition 7.2], $\bar{\Gamma}$, the set of all $\bar{\gamma}\,(\gamma \in \Gamma)$, is a transitive permutation group on $\mathbf{B}$, and the mapping $\gamma \to \bar{\gamma}$ is a homomorphism of Γ onto $\bar{\Gamma}$.

If G is a graph and $X, Y \subseteq V(G)$, let $X \square Y$ denote the set of edges of G having one end-vertex in X and the other end-vertex in Y and let $G[X,Y]$ denote the subgraph of G whose vertex-set is $X \cup Y$ and whose edge-set is $X \square Y$. If $\mathbf{V}$ is a partition of $V(G)$, then the *factor graph $G/\mathbf{V}$ of G* with respect to $\mathbf{V}$ has the vertex set $\mathbf{V}$ and $X, Y \in \mathbf{V}$ are adjacent if and only if $X \square Y$ contains some, but not all, of the unordered pairs $[x,y]$, $x \in X$, $y \in Y$. Let $Q \subseteq E(G)$ be an orbit of a subgroup Γ of $AutG$, where we consider the action of Γ induced on $E(G)$. We let $G\langle Q\rangle$ denote the graph induced by Q. Clearly, if Γ is a transitive subgroup of $AutG$, then $G\langle Q\rangle$ is vertex-transitive.

We let Z_n and Z_n^* denote the ring of residue classes of integers *mod n* and its group of units, respectively.

LEMMA 2.1. Let $2 \leq m < p$, where p is an odd prime, G be a (p,m)-imprimitive graph, and $\mathbf{B}$ be a complete (p,m)-block system of $\Gamma \leq AutG$. Then there exist vertices $x_1, \ldots, x_m$ of G belonging to different orbits of some (m,p)-semiregular element α of Γ such that $\mathbf{B} = \{B_i : i \in Z_p\}$ where $B_i = \{\alpha^i(x_r) : r = 1, \ldots, m\}$ for each $i \in Z_p$.

Proof. The group Γ has an element α of order p. Clearly $\bar{\alpha}$ has order 1 or p but $\bar{\alpha}$ cannot be the identity on $\mathbf{B}$ since this would imply every orbit of α has cardinality at most m. Thus, $\bar{\alpha}$ has order p which implies every orbit of α has cardinality a multiple of p and must intersect each block of $\mathbf{B}$. But since α has order p, each orbit of α has cardinality p and thus, if $B_0 = \{x_1, \ldots, x_m\}$, each x_i belongs to a different orbit and $B_i = \{\alpha^i(x_r) : r = 1, \ldots, m\}$ for each $i \in Z_p$. ■

LEMMA 2.2. Let q be an odd prime. If (W, W') is a bipartition of an edge-transitive graph G of order $2q$ and some automorphism of G interchanges W and W' and q divides $|E(G)|$, then G is regular.

Proof. Suppose that G is not regular. Then (since G is edge-transitive), there are distinct k,m such that each edge of G joins a k-valent to an m-valent vertex. For $i = k,m$, let W_i, W_i' be the sets of i-valent vertices in W,W', respectively. Since G has an automorphism that interchanges W and W', it follows that

$$|W_i| = |W_i'| \text{ for } i = k,m. \tag{1}$$

Then, (1) implies that, for $i = k,m$, the sets W_i,W_i' are nonempty proper subsets of W,W', respectively. Clearly

$$|E(G)| = |W_k \,\square\, W_m'| + |W_m \,\square\, W_k'|.$$

Counting the number of edges in $W_k \,\square\, W_m'$ and $W_m \,\square\, W_k'$ in two different ways, we obtain (in view of (1))

$$2k|W_k| = |E(G)| = 2m|W_m|. \tag{2}$$

Since $|W| = q$ is an odd prime and divides $|E(G)|$ and $\emptyset \subset W_k, W_m \subset W$, it follows from (2) that q divides both k and m, which is clearly impossible. ■

LEMMA 2.3. Let **B** be a complete p-block system of a transitive permutation group Γ on V, let $\Delta = Ker(\Gamma \to \bar{\Gamma})$, and let $B \in$ **B**. Then either $B \in$ V(Δ) or Δ fixes each point in B.

Proof. Let $\gamma \in \Gamma_B$. Then, $\gamma^{-1}\delta\gamma(B') = \gamma^{-1}\gamma(B') = B'$ for each $\delta \in \Delta$ and $B' \in$ **B**. Therefore, $\gamma^{-1}\Delta\gamma = \Delta$, i.e., $\Delta \triangleleft \Gamma_B$. It follows that the orbits of Δ are blocks of Γ_B. In particular, since $|B| = p$, a prime, it follows that either $B \in$ V(Δ) or Δ fixes each point in B. ■

The importance of metacirculants in the study of vertex-transitive graphs was suggested in the introduction. The following proposition summarizes some of the properties of qp-metacirculants that can be deduced from various results proved in [2].

PROPOSITION 2.4. Let p and q be distinct primes.

(i) If $p \not\equiv 1 (mod\ q)$, then a (q,p)-metacirculant is a circulant. In particular, if $p < q$, then a (q,p)-metacirculant is a circulant.

(ii) If $p \not\equiv 1 (mod\ q^2)$, then a (q,p)-metacirculant is a Cayley graph. If $p \equiv 1 (mod\ q^2)$, then there exist non-Cayley (q,p)-metacirculants. ■

Let Γ be a transitive permutation group on a finite set V. The following three group-theoretic results will be used in the proofs of Theorems 3.4 and 3.5.

PROPOSITION 2.5 ([26], Theorem 5.1). If Γ is a Frobenius group, the elements of Γ of degree $|V|$ together with 1 form a regular normal subgroup of Γ. ■

PROPOSITION 2.6 ([26], Theorem 11.6). Let $|V|$ be a prime. Then Γ is solvable if and only if $\Gamma_v \cap \Gamma_w = 1$ for $v \neq w$. ■

PROPOSITION 2.7 ([26], Theorem 11.7). If $|V|$ is a prime and Γ is nonsolvable, then Γ is 2-transitive on V. ■

3. IMPRIMITIVE *qp*-GRAPHS

We start the discussion of imprimitive qp-graphs with two simple observations on these graphs and their factor graphs.

LEMMA 3.1. Let p and q be distinct primes. Let G be a (q,p)-imprimitive graph and $\mathbf{B}$ be a complete (q,p)-block system of $\Gamma \leq AutG$ such that $G/\mathbf{B}$ is totally disconnected. Then G is a circulant.

Proof. Let $B \in \mathbf{B}$. There exists $\gamma \in \Gamma$ cyclically permuting the blocks in $\mathbf{B}$. Let $B_i = \gamma^i(B)$ $(i \in Z_q)$. The definition of the factor graph $G/\mathbf{B}$ then implies the existence of a symmetric (stable under multiplication by -1) subset of Z_q such that $G[B_i, B_j] = K_{p,p}$ if $j - i \in S$ and $G[B_i, B_j]$ is totally disconnected otherwise. Let H be the p-circulant induced by the block B and let K be the q-circulant with the symbol S. It is easily seen that G is isomorphic to the lexicographic product $K[H]$ of K by H. Therefore $AutG$ contains a transitive cyclic group of order qp and so G is a circulant. ■

LEMMA 3.2. Let p and q be distinct primes. Let an imprimitive qp-graph G have a (q,p)-semiregular automorphism α such that $[\alpha]$ is transitive and $G/\mathbf{V}(\alpha)$ is connected. Then G is a (q,p)-metacirculant.

Proof. Since $[\alpha]$ is transitive and $|\mathbf{V}(\alpha)| = q$, a prime, there exists $\gamma \in [\alpha]$, which cyclically permutes the orbits of α. The connectedness of $G/\mathbf{V}(\alpha)$ implies that $\langle\alpha\rangle$ is a Sylow p-subgroup of $\langle\alpha,\gamma\rangle$. Let Δ be the subgroup of $\langle\alpha,\gamma\rangle$ fixing each orbit of α. Since $\gamma^{-1}\langle\alpha\rangle\gamma \leq \Delta$, there exists $\delta \in \Delta$ such that $\gamma^{-1}\langle\alpha\rangle\gamma = \delta^{-1}\langle\alpha\rangle\delta$. Let $\tau = \delta\gamma^{-1}$. Then $\langle\alpha,\tau\rangle$ is a semidirect product of $\langle\alpha\rangle$ by $\langle\tau\rangle$ and a transitive subgroup of $AutG$. Therefore, G is a (q,p)-metacirculant. ■

Henceforth we shall assume that $q < p$. The following result characterizes (q,p)-imprimitive graphs.

PROPOSITION 3.3. If $q < p$ are primes, then the classes of (q,p)-imprimitive graphs and (q,p)-metacirculants coincide.

Proof. Clearly, a (q,p)-metacirculant is necessarily (q,p)-imprimitive. Conversely, let G be a (q,p)-imprimitive graph and let $\mathbf{B}$ be a complete (q,p)-block system of $\Gamma \leq AutG$. Then, by [14, Theorem 3.6], G has a (q,p)-semiregular automorphism α such that $\mathbf{V}(\alpha) = \mathbf{B}$ and $[\alpha]$ is transitive. Since $G/\mathbf{B}$ is a vertex-transitive graph of prime order q, it is either connected or totally disconnected. We then apply Lemmas 3.1 and 3.2 to deduce that G is a (q,p)-metacirculant. ■

The class of (p,q)-imprimitive graphs seems to be less tractable than the class of (q,p)-imprimitive graphs. The following two theorems provide a partial description of the structure of (p,q)-imprimitive graphs.

THEOREM 3.4. Let $q<p$ be primes. Let G be a (p,q)-imprimitive graph which is not a metacirculant and let Γ be a (p,q)-imprimitive subgroup of $AutG$. Then, the mapping $(\Gamma \to \bar{\Gamma})$ is an isomorphism and $\Gamma = \bar{\Gamma}$ is nonsolvable.

Proof. Let $\mathbf{B}$ be a complete (p,q)-block system of Γ. By Lemma 2.1, there are vertices $x_1, \dots, x_q$ of G belonging to different orbits of some (q,p)-semiregular element α of Γ such that $\mathbf{B} = \{B_i : i \in Z_p\}$, where $B_i = \{\alpha^i(x_r) : r = 1, \dots, q\}$. In view of Lemma 3.1, we may assume that $G/\mathbf{B}$ is connected. Let us assume that $\Delta = Ker(\Gamma \to \bar{\Gamma})$ is non-trivial. Then, by Lemma 2.3, there exist $\beta \in \Delta$ and $i \in Z_p$ such that $B_i \in V(\beta)$. Since $G/\mathbf{B}$ is connected, it follows by [14, Lemma 3.5] that $V(\beta) = \mathbf{B}$. Therefore, $\langle\alpha,\beta\rangle \le [\beta]$ and so $[\beta]$ is transitive. Therefore Lemma 3.2 (with α replaced by β) implies that G is a (q,p)-metacirculant, a contradiction. This proves that Δ is trivial, i.e., $(\Gamma \to \bar{\Gamma})$ is an isomorphism. Suppose now that $\Gamma = \bar{\Gamma}$ is solvable. Since $|\bar{\Gamma}| = |\Gamma| \geq qp$, $\bar{\Gamma}$ cannot be regular, so its minimal degree is not p. Therefore, by Proposition 2.6, the minimal degree of $\bar{\Gamma}$ is $p-1$ and, thus, is a Frobenius group. By Proposition 2.5, the elements of $\bar{\Gamma}$ of degree p together with 1 form a regular normal subgroup $\bar{\Pi}$ of $\bar{\Gamma}$. Since $\bar{\Pi}$ is regular, $|\bar{\Pi}| = p$. Therefore $\bar{\Pi}$ is the image of $\langle\alpha\rangle$ under the isomorphism $(\Gamma \to \bar{\Gamma})$. From this and the fact that $\bar{\Pi} \lhd \bar{\Gamma}$, we infer that $\langle\alpha\rangle \lhd \Gamma$. Hence, $\Gamma \le [\alpha]$. Therefore, $[\alpha]$ is transitive. Hence $G/V(\alpha)$ is either connected or totally disconnected. Lemmas 3.1 and 3.2 then imply that G is a metacirculant, a contradiction. Therefore, $\Gamma = \bar{\Gamma}$ is nonsolvable. This concludes the proof of Theorem 3.4. ■

THEOREM 3.5. Let $q<p$ be primes, $p \not\equiv 1 (mod\ q)$, and let G be a (p,q)-imprimitive graph which is not a circulant. Then $V(G)$ can be partitioned into p subsets of cardinality q such that there exists a perfect matching between any two of them.

Proof. Let Γ be a (p,q)-imprimitive subgroup of $AutG$ and $\mathbf{B}$ be a complete (p,q)-block system of Γ. As in the proof of Theorem 3.4, we let the vertices $x_1, \ldots, x_q$ of G belong to different orbits of some (q,p)-semiregular element α of Γ such that $\mathbf{B} = \{B_i : i \in Z_p\}$, where $B_i = \{\alpha^i(x_r) : r = 1, \ldots, q\}$. Moreover, we may assume that $G/\mathbf{B}$ is connected and that $\Gamma = \bar{\Gamma}$ is nonsolvable and thus, by Proposition 2.7, it is 2-transitive on $\mathbf{B}$. Therefore, the subgraphs $B(i,j) = G[B_i, B_j]$ $(i \neq j)$ of G are all isomorphic bipartite graphs, and since $G/\mathbf{B}$ is connected, they are not totally disconnected. Further, the 2-transitivity of $\bar{\Gamma}$ implies that, for each edge orbit Q of Γ, the graphs $G\langle Q\rangle \cap B(i,j)$ $(i \neq j)$ are all isomorphic. Let $i,j \in Z_p$ be distinct and Q_0 be an edge orbit of Γ. Then the graph $G_0 = G\langle Q_0\rangle \cap B(i,j)$ is not totally disconnected. Clearly, G_0 is edge-transitive. An element of Q_0 must have either one or no end-vertex in B_i. Therefore, the number of edges of $G\langle Q_0\rangle$ with one end-vertex in B_i is

$$\sum_{a \in Z_p \setminus \{i\}} |E(B(i,a) \cap G\langle Q_0\rangle)| = (p-1)|E(G_0)|. \qquad (3)$$

Clearly, $G\langle Q_0\rangle$ is vertex-transitive and, therefore, k-regular for some k. Since $|B_i| = q$, it follows by (3) that $(p-1)|E(G_0)| = kq$. Therefore, since $p \not\equiv 1 \ (mod\ q)$, q divides $|E(G_0)|$. The 2-transitivity of $\bar{\Gamma}$ implies the existence of an element of Γ, which interchanges B_i and B_j. Therefore, the graph G_0 satisfies all the assumptions of Lemma 2.2 and, hence, is regular of valency greater than or equal to 1. Hence, by [5, Theorem 8.7], G_0 has a perfect matching, and so $B(i,j)$ has a perfect matching. Since i,j were arbitrary, distinct elements of Z_p, it follows that $\mathbf{B}$ is the desired partition of $V(G)$. ■

COROLLARY 3.6. Let $q < p$ be primes, $p \not\equiv 1 (mod\ q)$ and let G be an imprimitive qp-graph with valency strictly less than $p-1$. Then G is a circulant.

Proof. If G is (q,p)-imprimitive, then G is a circulant by Propositions 2.4 and 3.3. If G is (p,q)-imprimitive, then G is a circulant by Theorem 3.5. ■

We know of four imprimitive qp graphs—$L(O_3)$, $L(K_6)$, and their complements—that are not metacirculants. All four have order 15. Since $5 \not\equiv 1$ $(mod\ 3)$, these graphs must satisfy the conditions of Theorem 3.5. In fact, the vertex set of $L(O_3)$ partitions into five blocks $B_i (i = 1,2,3,4,5)$ of cardinality three, each of which is an independent set and, furthermore,

$L(O_3)[B_i, B_j] = 3K_2$ for $i \neq j$. Thus, $L(O_3)$ is 4-regular (see [14] for a detailed discussion of $L(O_3)$). If we add all the edges within each of the five blocks, we get a 6-regular graph, which turns out to be $L(K_6)^c$.

We would like to get more information about imprimitive qp-graphs for the case $p \equiv 1 \ (mod \ q)$. Use of Theorem 3.4 and the classification of 2-transitive finite permutation groups (see [7]) might contribute to a classification of imprimitive qp-graphs.

In the last two sections we study the hamiltonicity of vertex-transitive qp-graphs.

4. HAMILTONICITY OF IMPRIMITIVE *qp*-GRAPHS

If $P = v_1 v_2 \cdots v_n$ is an n-path of a graph G we call $v_1 = o(P)$ and $v_n = t(P)$ respectively the *origin* and *terminus* of P. We shall sometimes call P a $v_1 v_n$*-path* or a UW*-path* or a $v_1 W$*-path* or a $U v_n$*-path* if U, W are any sets such that $v_1 \in U$ and $v_n \in W$. If $\alpha \in AutG$, we let $\alpha(P) = \alpha(v_1)\alpha(v_2) \cdots \alpha(v_n)$. Similarly, if $u_1 u_2 \cdots u_n u_1$ is an n-cycle of G we let $\alpha(C) = \alpha(u_1)\alpha(u_2) \cdots \alpha(u_n)\alpha(u_1)$.

Let α be an (m, p)-semiregular automorphism of a graph G. The *quotient graph* G/α *of* G *w.r.t.* α is the graph with vertex set $V(\alpha)$ and $X, Y \in V(\alpha)$ adjacent if $X \square Y \neq \emptyset$. If $\bar{C} = X_0 X_1 \cdots X_{n-1} X_0$ is a cycle of G/α, we let $\langle \bar{C} \rangle$ denote the subgraph of G with the vertex set $\bigcup_{i \in Z_n} X_i$ and the edge set $\bigcup_{i \in Z_n} X_i \square X_{i+1}$. We let $\langle G, \alpha \rangle$ be the spanning subgraph of G whose edge set is $E(G) \setminus \bigcup_{X \in V(\alpha)} X \square X$. A *spiral path* of $\langle \bar{C} \rangle$ is a path $x_0 x_1 \cdots x_{n-1} \beta(x_0)$ where $x_r \in X_r$ $(r \in Z_n)$ and $\beta \in \langle \alpha \rangle \setminus \{1\}$. If $X, Y \in V(\alpha)$, then $G[X, Y]$ is regular of some valency $d(X, Y) = d(Y, X)$ and $G[X] = G[X, X]$ is regular of some valency $d(X)$.

LEMMA 4.1. If $\langle \bar{C} \rangle$ has a spiral path, then $\langle \bar{C} \rangle$ is hamiltonian.

Proof. Let $x_0 x_1 \cdots x_{n-1} \beta(x_0)$ be a spiral path of $\langle \bar{C} \rangle$ and let $P = x_0 x_1 \cdots x_{n-1}$. Then $P \beta(P) \cdots \beta^{p-1}(P) x_0$ is a hamiltonian cycle of $\langle \bar{C} \rangle$. ■

LEMMA 4.2. If $\langle \tilde{C} \rangle$ is non-hamiltonian, then $d(X_r, X_{r+1}) = 1$ for each $r \in Z_n$ and there exists a cycle $C = x_0 x_1 \cdots x_{n-1} x_0$ such that $x_r \in X_r$ for each $r \in Z_n$ and $\langle \tilde{C} \rangle$ is the union of the n-cycles $\alpha^j(C)$ ($j \in Z_p$).

Proof. Since $\tilde{C}$ is a cycle of G/α, it follows that $d(X_r, X_{r+1}) \geq 1$ for each $r \in Z_n$. Thus there exists a path $x_0 x_1 \cdots x_{n-1}$ where $x_r \in X_r$ for each $r \in Z_n$ and a vertex $y_0 \in X_0$ is adjacent to x_{n-1}. Since $\langle \tilde{C} \rangle$ is non-hamiltonian, $y_0 = x_0$ by Lemma 4.1. If $d(X_r, X_{r+1}) > 1$ for some $r \in Z_n$, then there exists $j \in Z_p^*$ such that $x_r \sim a^j(x_{r+1})$ and therefore $x_0 x_1 \cdots x_r \alpha^j(x_{r+1}) \cdots \alpha^j(x_{n-1}) \alpha^j(x_0)$ is a spiral path of $\langle \tilde{C} \rangle$, a contradiction. Thus $d(X_r, X_{r+1}) = 1$ for each $r \in Z_n$. Hence $\langle \tilde{C} \rangle$ is the sum of the n-cycles $\alpha^j(x_0 x_1 \cdots x_{n-1} x_0)$ ($j \in Z_p$). ■

PROPOSITION 4.3 ([1], Lemma). Let $p \geq 5$ and G be a vertex-transitive p-graph with valency at least 4 and $u, v \in V(G)$ be distinct. Then G has a hamiltonian u,v-path.

We begin the discussion of hamiltonian properties of vertex-transitive qp-graphs with the following theorem.

THEOREM 4.4. If a (q,p)-imprimitive graph G, where $q < p$, is connected and not isomorphic to O_3, then G is hamiltonian.

The above theorem was first proved in [15]. However, as Alspach and Parsons proved that every connected (m,p)-metacirculant is hamiltonian for m odd [3, Theorem 1], we can deduce Theorem 4.4 from Proposition 3.3. For the sake of completeness, we sketch the proof of Theorem 4.4. Let $\alpha \in AutG$ be (q,p)-semiregular with $[\alpha]$ transitive. Then the induced subgraphs $G[X]$, $X \in V(\alpha)$ are all isomorphic vertex-transitive p-graphs. Note that G/α is a connected vertex-transitive q-graph and therefore hamiltonian. Using Lemma 4.2 one can prove that G/α contains a hamiltonian cycle $\tilde{C}$ such that either $\langle \tilde{C} \rangle$ has a spiral path (in which case G is hamiltonian) or $\langle \tilde{C} \rangle$ is a sum of q cycles of length p and $d(X) \geq 2$ for each $X \in V(\alpha)$. In the case $d(X) \geq 4$, a hamiltonian cycle of G is easily found by Proposition 4.3. Since $|X| = p > q \geq 2$ is odd, it follows that $d(X)$ is even and so we are left with the case $d(X) = 2$ for each $X \in V(\alpha)$. The transitivity of $[\alpha]$ implies the existence of $\tau \in [\alpha]$ such that $\langle\alpha,\tau\rangle$ is transitive and $\tau^{-1}\alpha\tau = \alpha^s$ for some integer s. If $s = \pm 1$, then G contains a subgraph isomorphic to the cartesian product of K_2 with C_p, if $q = 2$, and a subgraph isomorphic to the cartesian product of C_q with C_p, if $q \neq 2$. In both cases, G is clearly hamiltonian. If $s \neq \pm 1$, then one can construct a hamiltonian cycle of G which uses all but one edge in each of the graphs $G[X]$, $X \in V(\alpha)$ and q edges of $\langle G,\alpha\rangle$. ■

We believe that Theorem 4.4 can be generalized to the class of all connected vertex-transitive qp-graphs. In Section 5 we prove this for $q = 3$. Therefore the four imprimitive 15-graphs which are not metacirculants, given in the previous section, are hamiltonian. Hopefully, Theorems 3.4 and 3.5 will prove useful in generalizing Theorem 4.4. In fact, Theorem 4.4 and Theorem 3.5 together imply that for $p \not\equiv 1 \pmod{q}$ a connected non-hamiltonian imprimitive qp-graph would have valency greater than $p-1$. It is unlikely that such a graph exists.

To conclude this section we remark that O_4 and O_4^c, both primitive 35-graphs, are also hamiltonian. The proof that O_4 is hamiltonian is given in [16] and it also implies the hamiltonicity of O_4^c.

5. HAMILTONIAN CYCLES IN VERTEX-TRANSITIVE *3p*-GRAPHS

A number of fairly deep group-theoretic results will be needed to prove that every connected vertex-transitive $3p$-graph is hamiltonian.

Let Γ be a transitive permutation group on a finite set V [20, Theorems 2, 5], [20, Theorem 10] and [21, §11] contain the following three propositions respectively.

PROPOSITION 5.1. Let Γ be primitive with subdegrees $n_0 \leq n_1 \leq \cdots \leq n_k$. If $n_1 = 1$, then Γ is regular of prime degree and order. If $n_1 > 1$, then $n_i \le (n_1-1)n_{i-1}$ for all $i \ge 2$. ■

PROPOSITION 5.2. If $|V| = 3p$ and Γ is primitive, then the rank of Γ is at most 4 and either

(i) $p = 3r^2 + 3r + 1$ for some $r \in Z^+$ or

(ii) $p = 192r^2 + 60r + 5$ for some $r \in Z^+ \cup \{0\}$. ■

PROPOSITION 5.3. If $p > 3$ and Γ is $(p,3)$-imprimitive and nonsolvable, then its subdegrees are

(i) $1, 2, 3(p-1)$ or

(ii) $1, 2, p-1, 2(p-1)$ or

(iii) $1, 1, 1, 3(p-1)$ or

(iv) $1, 1, 1, p-1, p-1, p-1$ or

(v) $1, 2, 2, 4, 4, 8$ with $p = 7$. ■

Let $p \geq 3$ and $r, s, t \in Z_p^*$. Any graph isomorphic to the graph with the vertex set $\{x_i : i \in Z_p\} \cup \{y_i : i \in Z_p\} \cup \{w_i : i \in Z_p\}$ and the edges $[x_i, x_{i+1}]$, $[y_i, y_{i+t}]$, $[x_i, w_{i+r}]$, $[x_i, w_{i-r}]$, $[y_i, w_{i+s}]$, $[y_i, w_{i-s}]$ $(i \in Z_p)$ will be denoted by $G(p, r, s, t)$.

It is easily seen that

$$G(p, r, s, t) \cong G(p, r', s', t') \text{ if } r' = \pm r,\ s' = \pm s \text{ and } t' = \pm t \tag{4}$$

and

$$G(p, t^{-1}s, t^{-1}r, t^{-1}) \cong G(p, r, s, t). \tag{5}$$

LEMMA 5.4. Let $p \geq 3$ and α be a $(2,p)$-semiregular automorphism of a graph G, let W and X be the orbits of α such that $d(X) > 0$, $d(W, x) \geq 2$ and let $w \in W$. Then $G[W, X]$ has a hamiltonian wW-path.

Proof. Clearly $d(X)$ is even and thus at least 2. Let $x \in N(w) \cap X$ where $N(w)$ is the neighborhood of w. Then there are $\beta \in \langle\alpha\rangle \setminus \{1\}$ and $r \in Z_p^*$ such that $\{w, \beta(w), \beta^r(x), \beta^{-r}(x)\} \subseteq N(x)$. This implies that

$$wx\beta(w)\beta(x) \cdots \beta^{r-1}(w)\beta^{r-1}(x)\beta^{p-1}(x)\beta^{p-1}(w)\beta^{p-2}(x)\beta^{p-2}(w) \cdots \beta^r(x)\beta^r(w)$$

is a hamiltonian wW-path of G with the desired property. ■

B. Jackson [9] proved that every 2-connected k-regular graph of order not greater than $3k$ is hamiltonian. Since every connected vertex-transitive graph is 2-connected, Jackson's result implies

PROPOSITION 5.5. If a vertex-transitive n-graph G is connected and $valG \geq n/3$, then G is hamiltonian. ■

We can now prove the main result of this section.

THEOREM 5.6. A connected vertex-transitive $3p$-graph is hamiltonian.

Proof. The assertion of Theorem 5.6 is trivially true when $p = 2, 3$ by Proposition 5.5. We may thus assume that $p \geq 5$. Let G be a connected vertex-transitive $3p$-graph. If G is $(3,p)$-imprimitive, then it is hamiltonian by Theorem 4.4. Therefore we can assume that

$$G \text{ is not } (3,p)\text{–imprimitive.} \tag{6}$$

By [14, Theorem 3.4] G has a $(3,p)$-semiregular automorphism α with orbits W, X and Y (say). Let $d_0 = valG$. Since $|V(G)| = 3p$ and $|W| = |X| = |Y| = p$ are odd numbers, it follows that

$$d_0, d(W), d(X), d(Y) \text{ are even numbers.} \tag{7}$$

If $d_0 = 2$, then G is a cycle. We may therefore assume that $d_0 \geq 4$. Since G is connected, G/α is either K_3 or P_3.

Case 1. $G/\alpha = K_3$.

By Lemma 2.2 we may assume that $d(W,X) = d(X,Y) = d(Y,W) = 1$ and that there is a cycle $C = wxyw$ such that $w \in W, x \in X, y \in Y$ and $\langle G, \alpha \rangle$ is the sum of p cycles of length 3, viz.

$$\langle G, \alpha \rangle = C + \alpha(C) + \cdots + \alpha^{p-1}(C). \tag{8}$$

Since $d_0 \geq 4$, it follows that $2 \leq d(W) = d(X) = d(Y) = d$, say. Suppose that $d = 2$. Then, $G[W]$, $G[X]$ and $G[Y]$ are p-cycles. From this and (8) we see that every edge in $(W \square X) \cup (X \square Y) \cup (Y \square W)$ lies in a 3-cycle and no edge in $(W \square W) \cup (X \square X) \cup (Y \square Y)$ lies in a 3-cycle. Hence every automorphism of G maps $G[W] + G[X] + G[Y]$ onto itself and so maps each of W, X, Y into one of W, X, Y. Therefore $\{W, X, Y\}$ is a complete $(3,p)$-block system of $AutG$ and so G is $(3,p)$-imprimitive which contradicts (6). Therefore $d \geq 4$ by (7). By Proposition 4.3 there are hamiltonian paths P_W, P_X, P_Y in $G[W]$, $G[X]$, $G[Y]$, respectively, whose origins and termini are w and $\alpha(w)$, $\alpha(x)$ and $\alpha^2(x)$, $\alpha^2(y)$ and y, respectively. Then $P_W P_X P_Y$ is a hamiltonian cycle of G.

Case 2. $G/\alpha = P_3$.

By suitable choice of notation we may assume that

$$d(X,Y) = 0, \tag{9}$$

$$d(W,X) \geq 1,\ d(W,Y) \geq 1, \tag{10}$$

and

$$d(Y) \geq d(X). \tag{11}$$

It follows from (9) that

$$d(X)+d(W,X)=d(Y)+d(W,Y)=d(W)+d(W,X)+d(W,Y)=d_0. \qquad (12)$$

By (7) and (12), $d(W,X)$ and $d(W,Y)$ are even and so by (10)

$$d(W,X)\geq 2,\ d(W,Y)\geq 2. \qquad (13)$$

Subcase 2.a. $d_0 \geq 6$.

By (12) and (13),

$$d(X)=d(W)+d(W,Y)\geq d(W,Y)\geq 2. \qquad (14)$$

By (13) and (14) and Lemma 5.4 there exists a hamiltonian ww'-path P in $G[W \cup X]$ where $w, w' \in W$. By (13) there are distinct $y, y' \in Y$ such that $w \sim y$, $w' \sim y'$. By (11) and (14), $d(Y) \geq d(W,Y)$ and so $2d(Y) \geq d(Y)+d(W,Y) = d_0 \geq 6$, by (12) and therefore $d(Y) \geq 4$ by (7). Therefore by Proposition 4.3, there is a hamiltonian $y'y$-path Q in $G[Y]$ and hence PQw is a hamiltonian cycle in G.

Subcase 2.b. $d_0 \leq 4$.

By (11), (12), (13) and (14), $d(X)=d(Y)=d(W,X)=d(W,Y)=2$ and $d(W)=0$. Therefore there are $w \in W, x \in X, y \in Y, \beta \in \langle\alpha\rangle \setminus \{1\}$ and $r, s, t \in Z_p^*$ such that G has the edges $[x_i, x_{i+1}]$, $[y_i, y_{i+t}]$, $[x_i, w_{i+r}]$, $[x_i, w_{i-r}]$, $[y_i, w_{i+s}]$, $y_i, w_{i-s}]$, $(i \in Z_p)$ where $w_i = \beta^i(w)$, $x_i = \beta^i(x)$ and $y_i = \beta^i(y)$ $(i \in Z_p)$. Thus $G = G(p, r, s, t)$.

Suppose first that *AutG* is imprimitive. Then *AutG* is $(p, 3)$-imprimitive by (6). Moreover, by Theorem 3.4, *AutG* is nonsolvable and thus (since *valG* $= 4$) it follows, by Proposition 5.3, that $p \in \{5, 7\}$.

Suppose now that *AutG* is primitive. Then it has rank at most 4, by Proposition 5.2. Let $n_0 \leq n_1 \leq \cdots \leq n_k$ $(k \leq 3)$ by the subdegrees of *AutG*. Then clearly $n_0 = 1$ and $n_1 \leq d_0 = 4$ and since the degree of *AutG* is not prime, it follows, by Proposition 5.1, that $n_1 > 1$ and $n_i \leq (n_1-1)n_{i-1}$ for $2 \leq i \leq k$. Therefore $3p = |V(G)| = \sum_{i=0}^{k} n_i \leq 1 + n_1 + (n_1-1)n_1 + (n_1-1)^2 n_1$ $\leq 1 + 4 + 12 + 36 = 53$ which implies that $p \leq 17$ and therefore, by Proposition 5.2, $p \in \{5, 7\}$.

Suppose that $t \in \{1,-1\}$. Let $j=(2s)^{-1}$. Since $y_{2ps}=y_0-y_1=y_{2js}$, it follows that $w_{s+1}x_{s+r+1}x_{s+r+2} \cdots x_{s+r-1}x_{s+r}w_s y_{2s} w_{3s} y_{4s} w_{5s} y_{6s} \cdots y_{(2j-2)s} w_{(2j-1)s} y_{2js} y_{2ps} \cdot w_{(2p-1)s} y_{(2p-2)s} w_{(2p-3)s} \cdots y_{(2j+2)s} w_{(2j+1)s}$ is a hamiltonian cycle of G. We may therefore assume that $t \in \{1,-1\}$.

If $p=7$ it follows from (4) and (5) that G is one of the following nine graphs: $G(7,1,1,2)$, $G(7,2,1,2)$, $G(7,3,2,2)$, $G(7,3,3,2)$, $G(7,2,2,2)$, $G(7,1,3,2)$, $G(7,1,2,2)$, $G(7,3,1,2)$, or $G(7,2,3,2)$. Among these graphs $G(7,2,3,2)$ is the only one in which any two vertices lie in the same number of 3-cycles, 4-cycles and 5-cycles. Therefore, since vertex-transitivity of G implies that any two of its vertices lie in the same number of n-cycles for all n, we deduce that $G=G(7,2,3,2)$ and therefore it has a hamiltonian cycle

$x_1w_3x_5x_6w_4x_2w_0y_3y_1y_6y_4y_2y_0y_5w_1x_3w_5x_0w_2x_4w_6x_1$.

If $p=5$, it follows from (4) and (5) that G is either $G(5,1,2,2)$, $G(5,1,1,2)$ or $G(5,2,1,2)$. The same cycle argument as in the case $p=7$ implies that $G=G(5,2,1,2)$ and it has a hamiltonian cycle $x_0x_1w_3y_4y_1w_0x_3x_2w_4y_3y_0y_2w_1x_4w_2x_0$. This completes the proof of Theorem 5.6.

■

In fact, it can be proved that $G(5,2,1,2)$ is the only vertex-transitive graph among the graphs $G(p,r,s,t)$ and it transpires that $L(O_3)=G(5,2,1,2)$.

It is possible that a more self-contained proof of Theorem 5.6 could be obtained by showing that each graph $G(p,r,s,t)$ is hamiltonian, which we believe is the case.

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