Structures Within (22, 33, 12, 8, 4)-Designs

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1. Introduction

A balanced incomplete block design (BIBD) is a pair (V, B) where V is a v-set and B is a collection of b k-subsets of V, called blocks, such that each element of V is contained in exactly r blocks and any 2-subset of V is contained in exactly λ blocks. The parameters of a BIBD are denoted by (v, b, r, k, λ). Trivial necessary conditions for the existence of a BIBD (v, b, r, k, λ) are

1. $vr = bk$

2. $r(k - 1) = \lambda(v - 1)$.

Although (22, 33, 12, 8, 4) are parameters satisfying the above two conditions, it is not known at present whether or not there are any designs with these parameters. For any smaller number of elements the existence or non-existence of such a design is known. For this reason, the existence or non-existence of a (22, 33, 12, 8, 4)-BIBD is a challenging problem. If methods can be devised to settle this particular problem, it is to be expected they will have a wider impact.

A few concepts will now be defined. The incidence matrix, A, of a BIBD (v, b, r, k, λ) is the matrix whose element in the i^{th} row and j^{th} column is 1 (or 0) if element i is (or is not) in block j. (The elements and blocks are usually numbered 1, 2, 3, ...) A pairwise balanced design (PBD) is a pair (V, B) where V is a v-set and B is a collection of subsets of V such that any 2-subset of V is contained in exactly λ blocks. A sub-PBD of a BIBD is a PBD whose incidence matrix is a submatrix of the incidence matrix of the BIBD. Two designs (V_1, B_1) and (V_2, B_2) are isomorphic if there exists a bijection $\alpha: V_1 \to V_2$ such that $B_1\alpha = B_2$.

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Hamada and Kobayashi [3] have made a study of the block structure of these designs. Another detailed study, using coding theory, was made by Hall, Roth, Van Rees and Vanstone [2]. In the sequel we will again use coding theory to study these designs. However, we will use it in a different way than in the four author paper. We will use the techniques of Stinson [5] to get a short list of parameters of PBD's, some of which must be sub-PBD's of any (22, 33, 12, 8, 4)-BIBD. In section 3, a special PBD studied by Hall [1] will be examined with the aid of computers and it will be determined that no (22, 33, 12, 8, 4)-BIBD contains this special PBD.

2. The Code Over F_2

The code C of a (22, 33, 12, 8, 4)-BIBD over GF(2) is the subspace of F_2^{33} spanned by the rows of the incidence matrix A. In A, every column has eight 1's and every row has twelve 1's. As $\lambda = 4$ any two rows are 1's in exactly four columns. It follows that every codeword has weight a multiple of four and that C is self-orthogonal. As in Hall [0], we define the orthogonal code $C^{\perp}$ as consisting of the vectors y orthogonal to all of C,

$$C^{\perp} = \{y \mid (x, y) = 0 \text{ for all } x \in C\}.$$

Here the inner product $(x, y) = x_1y_1 + x_2y_2 + \ldots + x_{33}y_{33}$ modulo 2. Also $C^{\perp} \supset C$ and $C^{\perp}$ contains the vector of all 1's, since every codeword in C has even weight. Since $\dim C + \dim C^{\perp} = 33$, $\dim C \leq 16$.

Following Stinson [5], we can state that since A has 22 rows and the $\dim C \leq 16$, there must be at least 6 independent dependencies of the rows of A. A linear dependence can be written $\sum_{i \in I} r_i = 0$ for some I a subset of $\{1, 2, \ldots, 22\}$ where r_i is the i^{th} row of A. In terms of the design we have a subset $Y = \{x_i : i \in I\}$ such that $|B_j \cap Y|$ is even where B_j is the j^{th} block of the design, $1 \leq j \leq 33$. Thus there are at least 2^6 dependencies in any (22, 33, 12, 8, 4)-BIBD.

We now switch to combinatorial terminology. If $Y = \{r_i \mid i \in I\}$ corresponds to a dependence relation, we will call $\{Y, \{Y \cap B_i \mid 1 \leq i \leq 33\}\}$ an even sub-PBD. We use the word even as all the blocks of the sub-PBD have even length. Suppose an even sub-PBD has m points and b_i blocks of size i. Then by simple counting we get the following so-called B-equations.

$$\begin{aligned} \sum b_i &= b \\ \sum i b_i &= rm \\ \sum \binom{i}{2} b_i &= \lambda \binom{m}{2}. \end{aligned}$$

The vector solutions are called B-vectors.

For the (22, 33, 12, 8, 4)-BIBD case, there will be at least six independent dependencies in the rows of A. Solving the associated B-equations give the following for the m elements of the dependency.

$$\begin{aligned} b_0 + b_6 + b_8 &= 33 + 1/2\, m^2 - 8m \\ b_2 - 3b_6 - 8b_8 &= 10m - m^2 \\ b_4 + 3b_6 + 6b_8 &= 1/2\, m(m - 4). \end{aligned}$$

Since the design is an even sub-PBD itself, the B-vectors come in pairs. One for m elements and the other for 22-m elements. Here we list the 12 B-vector solutions in non-negative integers for $m \leq 11$.

CASE	b_0	b_2	b_4	b_6	b_8	m
1	9	24	0	0	0	4
2	3	24	6	0	0	6
3	2	27	3	1	0	6
4	1	30	0	2	0	6
5	0	32	0	0	1	6
6	1	16	16	0	0	8
7	0	19	13	1	0	8
8	3	0	30	0	0	10
9	2	3	27	1	0	10
10	1	6	24	2	0	10
11	0	9	21	3	0	10
12	0	8	24	0	1	10

Case 5 can immediately be eliminated as b_8 must equal 0 if $m = 6$. Case 4 can also be eliminated. Since $m = 6$, the two b_6 blocks intersect in at least six elements. From [3], we know that intersections can be at most 4. Therefore we can state the following theorem.

Theorem 1. There are at most 10 different B-vectors for the even sub-PBD's of the (22, 33, 12, 8, 4)-design.

3. Case 8

Hall [1] has studied a particularly nice subcase of Case 8. In Case 8, the even sub-PBD is a (10, 30, 12, 4, 4). Hall has considered the (10, 30, 12, 4, 4)-design formed when a tenth element has been adjoined to a Steiner Triple System on nine elements and extended to a three design, i.e. all triples of elements occur exactly once and block size is four (sometimes denoted by 3-(10, 4, 1)-design). There is only one such non-isomorphic design. Next Hall attempted to extend this design to a (22, 33, 12, 8, 4)-BIBD by adjoining additional elements. It turns out that there are several sub-cases of which Hall eliminated two.

In this section, we describe the computer results which determined that this extension was impossible. This was done twice independently to ensure correctness. We will provide some of the partial results of the first program so that the results may be verified.

An important aspect of this work is to eliminate isomorphic duplicates. This was done using Kocay's [4] algorithm and program for finding isomorphisms between graphs. The program was run on an Amdahl 350. A design is made into a bipartite graph by letting all elements and blocks be represented by points. Two points are joined only if the corresponding element occurs in the corresponding block. Two designs are isomorphic if and only if their corresponding graphs are isomorphic. The program also gives the automorphism group of the graph.

Using this program, we found that the following three-design that we used has automorphism group size 1440.

B1 :	10 1 2 3
B2 :	10 4 5 6
B3 :	10 7 8 9
B4 :	10 1 4 7
B5 :	10 2 5 8
B6 :	10 3 6 9
B7 :	10 1 5 9
B8 :	10 2 6 7
B9 :	10 3 4 8
B10:	10 1 6 8
B11:	10 2 4 9
B12:	10 3 5 7
B13:	1 2 4 8
B14:	1 2 7 9
B15:	1 2 5 6
B16:	1 3 6 7
B17:	1 3 4 5
B18:	1 3 8 9
B19:	1 4 6 9
B20:	1 5 7 8
B21:	2 3 4 6
B22:	2 3 5 9
B23:	2 3 7 8
B24:	2 4 5 7
B25:	2 6 8 9
B26:	3 4 7 9
B27:	3 5 6 8
B28:	4 5 8 9
B29:	4 6 7 8
B30:	5 6 7 9

(10, 30, 12, 4, 4)-BIBD

Let us consider what the remainder of the (22, 33, 12, 8, 4)-BIBD must look like if this 3-design is a sub-PBD of it. Each remaining element must occur in 10 of the first 30 blocks to ensure the pair count is correct. The remaining two occurrences must occur in the last three empty blocks. There is only one way to do this. If the new elements are labelled from eleven, the last three blocks may be taken as:

B31:	11 12 13 14 15 16 17 18
B32:	11 12 13 14 19 20 21 22
B33:	15 16 17 18 19 20 21 22

Let group 1 or G1 = {11, 12, 13, 14}, group 2 or G2 = {15, 16, 17, 18}, and group 3 or G3 = {19, 20, 21, 22}. Clearly the elements within a group must intersect each other twice in the first 30 blocks whereas the elements from different groups must intersect each other three times in the first 30 blocks.

So the first thing we did was to find the 900 ways to adjoin an element onto the sub-PBD. From this point on, we only work with the list of 900 sets of blocks. Using Kocay's program, we showed that there are only 8 non-isomorphic sets of blocks.

Non-Isomorphic Extensions

Case	Blocks									
1	1	2	3	4	13	19	22	23	27	30
2	1	2	3	4	14	17	22	25	27	29
3	1	2	3	7	13	16	21	23	28	30
4	1	2	3	7	13	16	24	25	26	27
5	1	2	3	7	19	20	21	22	23	29
6	1	2	4	5	18	20	21	25	26	30
7	1	2	4	8	14	18	22	27	28	29
8	1	2	4	12	13	18	22	25	29	30

TABLE 2

We note that Hall [1] has already proven that Cases 4 and 6 are not possible to complete to a (22, 33, 12, 8, 4)-BIBD.

Recall that each of the 900 sets of blocks represent one way of adjoining a block to the sub-design. So the program must find G1, G2 and G3 with the correct intersection pattern. It tried to do this by letting the first set of blocks of the case be in G1. Then all sets that intersected it in 2 or 3 varieties were recorded. Then all possible G1's were tried. Then all sets which intersected the sets of a particular G1 in 3 blocks were listed. Then G2's were formed out of these lists. Then a list of sets which intersected the sets of a particular pair, G1 and G2, were recorded. Finally, G3 was formed from this list. The results of the program are presented in the following table.

Program's Progress

Case	2i	3i	P2G	2G	3G
1	189	215	16	NO	NO
2	174	255	41	YES	YES
3	224	168	36	NO	NO
4	120	320	40	YES	NO
5	186	240	40	YES	NO
6	335	0	0	NO	NO
7	180	260	24	YES	NO
8	155	280	132	YES	YES

2i = no. of sets that intersect the first set in 2 blocks.
3i = no. of sets that intersect the first set in 3 blocks.
P2G = the size of largest number of sets that intersect each set in G1 in 3 blocks.
2G = There is a group 2 whose members intersect the sets of G1 in 3 blocks and the other sets of G2 in 2 blocks.

3G = There is a group 3 whose members intersect the sets of G1 and G2 in 3 blocks.

Unfortunately, the sets of group 3 do not intersect themselves in the correct number of blocks. So the following theorem can be stated.

Theorem 2. No (22, 33, 12, 8, 4)-BIBD contains the 3-(10, 4, 1)-design as a sub-PBD.

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