On the determination of the covering numbers C(2,5,v)

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ABSTRACT

Let V be a finite set of v elements. A covering of the pairs of V by k-subsets is a family F of k-subsets of V, called blocks, such that every pair in V occurs in at least one member of F. For fixed v and k, the covering problem is to determine the number of blocks of any minimum (as opposed to minimal) covering. Denote the number of blocks in any such minimum covering by $C(2,k,v)$. Let $B(2,5,v) = \lceil v\lceil(v-1)/4\rceil/5\rceil$. In this paper, improved results for $C(2,5,v)$ are provided for the case $v \equiv 1$ or $2 \ (mod\, 4)$. For $v \equiv 2 \ (mod\, 4)$, it is shown that $C(2,5,270) = B(2,5,270)$ and $C(2,5,274) = B(2,5,274)$, establishing the fact if $v \geq 6$, and $v \equiv 2 \ mod\, 4$, then $C(2,5,v) = B(2,5,v)$. In addition, it is shown that if $v \equiv 13 \ (mod\, 20)$, then $C(2,5,v) = B(2,5,v)$ for all but 15 possible exceptions, and if $v \equiv 17 \ (mod\, 20)$, then $C(2,5,v) = B(2,5,v)$ for all but 17 possible exceptions.

1. Introduction

Let V be a finite set of v elements. A covering of the pairs of V by k-subsets is a family F of k-subsets of V, called blocks, such that every pair in V occurs in at least one member of F. For fixed v and k, the covering problem is to determine $C(2,k,v)$, the number of blocks in any minimum (as opposed to minimal) covering.

Let $B(2,k,v) = \lceil v\lceil(v-1)/(k-1)\rceil/k\rceil$. It is well known that $C(2,k,v) \geq B(2,k,v)$. We are interested here in the case $k = 5$. Gardner [6] has shown that if $v \equiv 13 \ (mod\, 20)$, then $C(2,5,v) \geq B(2,5,v)+1$. For convenience, let $C(v) = C(2,5,v)$, and let

$$B(v) = B(2,5,v) \text{ if } v \not\equiv 13 \ (mod\, 20),$$
$$= B(2,5,v)+1 \text{ if } v \equiv 13 \ (mod\, 20).$$

In [9] it is shown that if $v \equiv 2 \ (mod\, 4)$ and $v \not\equiv 270,274$, then $C(v) = B(v)$, and if $v \equiv 1 \ (mod\, 4)$ and v is moderately large, then $C(v) = B(v)$. It is the purpose of this paper to show that $C(v) = B(v)$ for $v = 270$ and 274, and to show that $C(v) = B(v)$ for several new values of $v \equiv 1 \ (mod\, 4)$.

2. Constructions for v = 270 and 274

In this section we show that $C(270) = B(270)$ and $C(274) = B(274)$. These two values are those remaining to establish in order to show that $C(v) = B(v)$ for all integers $v \geq 6$, $v \equiv 2 \ (mod\, 4)$. For definition of balanced incomplete block design (BIBD), group divisible design (GDD), resolvable balanced incomplete block design (RBIBD), pairwise balanced design (PBD); flat, and transversal design TD, the reader is referred to [13]. It is also assumed that the reader is familiar with Wilson's fundamental construction for group divisible designs [13]. Further, since there exists a BIBD(v,5,1) for all positive integers $v \equiv 1$ or $5 \ (mod\, 20)$, $v \geq 21$ (see [7]), there are group divisible designs of type 4^5 and 4^6 with blocks of size 5 obtained by deleting a point from the first two members of this series. For the existence of other BIBDs and RBIBDs, the reader is referred to [10] unless other references are given.

Theorem 2.1. The covering numbers $C(v)$ are equal to the bound $B(v)$ for $v = 270$ and $v = 274$.

Proof. There exists a resolvable BIBD(65,5,1). By adjoining s new points to this design ($s = 2,3$), one obtains a GDD of type $5^{13}s^1$ with blocks from $\{5,6\}$. By applying Wilson's fundamental theorem, giving every point weight 4, a GDD, say D, of type $20^{13}(4s)^1$ with blocks of size 5 is created. Let the groups be $G_1, G_2, \ldots, G_{14}$, with G_{14} being the group of size $4s$. Let W denote the set of points of D, and let ∞_1 and ∞_2 be points not in D. Then a minimum covering of $V = W \cup \{\infty_1, \infty_2\}$ is formed as follows.

First take the blocks of size 5 of D. Then take copies of the BIBD(21,5,1) one on each of the sets $G_i \cup \{\infty_1\}$, $i = 1,2,\ldots,13$ to obtain 13×21 more blocks of size 5. To these, adjoin 65 more blocks of size 5, these blocks being obtained by partitioning the 260 points of $U = \cup_{i=1}^{13} G_i$ into blocks of size 4, and adjoining ∞_2 to each such block. The set of blocks of size 5 is completed by adjoining blocks of a minimum covering of the set $G_{14} \cup \{\infty_1, \infty_2\}$. It is easily verified that the resulting configuration is a minimum covering of v points which contains $B(v)$ blocks. □

3. Some results for $v \equiv 1 \pmod 4$

We begin this section with some general constructions for coverings for $v \equiv 1 \pmod 4$. Since there exists a BIBD(v,5,1) for all $v \equiv 1$ or $5 \pmod{20}$, $v \geq 21$, $C(v) = B(v)$ for these values.

For the definition of an incomplete transversal design $T(k,n)-TD(k,a)$, the reader is referred to [3].

Lemma 3.1. Let n and a be non-negative integers satisfying $a \leq 4n+1$. If there exists a $TD(5,12n+4+a)-TD(5,a)$, and if $C(4n+4a+1) = B(4n+4a+1)$, then $C(64n+4a+21) = B(64n+4a+21)$.

Proof. See [9, Lemma 3.1]. □

Lemma 3.2. Suppose there exists a resolvable BIBD($20m+5$,5,1). Suppose t is an integer satisfying $0 \leq t \leq 5m$ such that $C(4t+1) = B(4t+1)$, then $C(80m+21+4t) = B(80m+21+4t)$.

Proof. By adjoining t points to the resolvable BIBD, one obtains a group divisible design of type $(5)^{4m+1}(t)^1$ with block sizes in $\{5,6\}$. By applying Wilson's fundamental theorem [13] with all points having weight 4, a group divisible design of type $20^{4m+1}(4t)^1$ with blocks of size 5 is obtained. Adjoint a new point ∞ to each of the groups, replacing each block of size 21 by a copy of $PG(4,1)$ (the BIBD(21,5,1)) and the block of size $4t+1$ by a minimum covering of $4t+1$ points, a minimum covering of $v^* = 80m+21+4t$ points with $B(v^*)$ blocks is obtained. □

Lemma 3.3. If $C(4m+1) = B(4m+1)$, then $C(16m+5) = B(16m+5)$.

Proof. It is well known [7] that there exists a resolvable BIBD($12m+4$,4,1). The result is obtained by adjoining $4m+1$ new points, and replacing the block of size $4m+1$ by a minimum covering of $4m+1$ points. □

Gardner [6] has shown that $C(100m+13) = B(100m+13)$ for $m \geq 1$ and $C(100m+93) = B(100m+93)$ for $m \geq 0$.

To extend this work, incomplete transversal designs will be required below. Various constructions for these are cited below.

Lemma 3.4. If $TD(6,t)$ and $TD(5,m+m_j)-TD(5,m_j)$ all exist and if $a = \Sigma_{j=1}^{p} m_j k_j$ where k_j are positive integers satisfying $t = \Sigma_{j=1}^{p} k_j$, then there exists $TD(5,mt+a)-T(5,a)$. Further if there exists a $TD(5,a)$, and if some $m_j = 0$ or 1 in the above, then there exists a $TD(5,mt+a)-TD(5,m+m_j)$, and if there exists a $TD(5,a)$, then there exists a $TD(5,mt+a)-TD(5,t)$.

Proof See [3, Corollary 1.3]. It is easily verified that the $TD(5,mt+a)-TD(5,a)$ contains a copy of a $TD(5,t)$. If the "hole" of size a is "filled", and a copy of the $TD(5,t)$ is deleted, a $TD(5,mt+a)-TD(5,t)$ is obtained. □

Lemma 3.5. If there exists a $TD(7,m)$, and $TD(5,r)$ where $0 \leq r \leq m$, then there exists a $TD(5,7m+r+a)$ for all a satisfying $0 \leq a \leq m$.

Proof. See [3]. □

It was shown in [9] that $C(100m+33) = B(100m+33)$ for $m \geq 7$. We note also that $C(533) = B(533)$. This follows from Lemma 3.1 with $n = 7$, $a = 16$, where a $TD(5,104)-TD(5,16)$ is required. Brouwer [4] has shown that a $TD(5,10)-TD(5,2)$ exists, and a $TD(6,11)$ and $TD(5,8)-TD(5,0)$ exist. Therefore by Lemma 3.4, there exists a $TD(5,104)-TD(5,16)$, since $104 = 11.8+8.2$.

For the existence of specific transversal designs mentioned henceforth, the reader is referred to [2]. For the existence of resolvable BIBD(v,5,1), see [7].

Lemma 3.6. Let $X = \{373, 453, 473, 553, 653, 673, 773, 853, 873, 953, 973, 1053, 1073, 1153, 1173, 1253, 1273, 1253, 1353, 1373, 1453, 1573, 1953\}$. If $v \in X$, then $C(v) = B(v)$.

Proof. By applying Lemma 3.3, the result is established for $v \in \{373, 453, 773, 853, 873, 1173, 1253\}$.

For $v \in \{473, 553, 653, 673, 753, 873, 953, 973, 1053, 1073, 1153, 1273, 1373, 1453, 1573, 1953\}$ we apply Lemma 3.1 with the following parameters.

v	n	a	$12n+4+a$	Incomplete TD
473	6	17	93	(i)
553	7	21	109	5.21+4 (Lemma 3.4)
653	9	14	126	7.16+14
673	9	19	131	7.18+1.5 (Lemma 3.4)
853	12	16	164	9.16+5.4 (Lemma 3.4)
873	11	37	173	37.4+25 (Lemma 3.4)
953	14	9	181	7.23+11+9
973	14	14	186	7.23+11+14
1053	14	34	206	43.4+34 (Lemma 3.4)
1073	16	7	203	7.29
1153	17	11	219	7.29+5+11
1273	19	9	241	7.32+8+9
1373	21	2	258	7.32+32+2
1453	22	6	274	7.37+9+6
1953	29	19	371	7.49+9+19

Case (i) above requires a little more detailed explanation.

A $TD(5,93-TD(5,17)$ is required. Since there exists a BIBD(21,5,1), by the well known PBD construction for transversal designs, there exists a $TD(5,16+5)-TD(5,5)$. Also there exists a $TD(5,16)-TD(5,0)$, a $TD(5,16+1)-TD(5,1)$, a $TD(6,5)$, and a $TD(5,13)$. Therefore by Lemma 3.4, there exists a $TD(5,93)-TD(5,17)$, since $93 = 5.16+13$. □

In order to eliminate further cases, Lemma 3.2 can be generalized. To do so, we require some more group divisible designs. Since there are a $TD(5,8)$ and a $TD(5,16)$ there exist GDDs of type 8^5 and 16^5 with blocks of size 5. Also Brouwer [1] has shown that there is a group divisible design of type 8^6 with blocks of size 5. Also by applying Wilson's fundamental construction, using all weights 4 with the GDD of type 6^4 with blocks of size 5, one obtains a GDD of type 16^6 and blocks of size 5. With these observations the following is easily established.

Lemma 3.7. Suppose there exists a resolvable BIBD($20m+5,5,1$). Suppose that t is an integer satisfying $0 \leq t \leq 5m$ such that $C(2^s t+1) = B(2^s t+1)$ for $s = 3$ or $s = 4$. Then $C(2^s.20m+2^s.5+1+2^s t) = B(2^s.20m+2^s.5+1+2^s t)$.

Proof. The proof is that of Lemma 3.2, mutatis mutandis. □

The following lemma is also useful for eliminating further small cases.

Lemma 3.8. Suppose there is a BIBD(v,5,1) which contains a flat of order w. Let a be an integer satisfying $0 \leq a \leq w$. If there exists a $TD(5,v-a)-TD(5,w-a)$, and if $C(5(w-a)+a) = B(5(w-a)+a)$, then $C(5(v-a)+a) = B(5(v-a)+a)$.

Proof. As shown in [11], by an application of the singular indirect product, there exists a $PBD(5(v-a)+a,\{5,5(w-a)+a\})$ which contains precisely one block of size $5(w-a)+a$. By replacing this block by a minimum covering of $5(w-a)+a$ points, the required covering is obtained. □

Theorem 3.9. Suppose that there exists a BIBD(v,6,1) which contains a flat of order w. Let t be an integer satisfying $0 \leq t \leq w-1$. Suppose that $s = 2,3$ or 4, $C(2^s t+1) = B(2^s t+1)$. Then $C(2^s(v-w)+2^s t+1) = B(2^s(v-w)+2^s t+1)$.

Proof. Let ∞ be a distinguished point of the flat F of the BIBD, which we denote by D. By deleting ∞, a set G of blocks of size 5 in $D\backslash\{\infty\}$ is generated, namely from those blocks of D which contain ∞ but do not lie in the flat. If $(w-1-t)$ other points of F are also deleted, then the remaining t points of F, together with the blocks of G form the groups of a GDD of type $5^{|G|}t^1$ with blocks of sizes from $\{5,6\}$. Since there exist GDDs of type $(2^s)^5$ and $(2^s)^6$ with blocks of size 5 for $s = 2,3$ and 4, an application of Wilson's fundamental theorem yields a GDD of type $(2^s 5)^{|G|}(2^s t)^1$ with blocks of size 5. Adjoin a new point ∞ to all the groups, and replace all blocks of size $2^s 5+1$ by the blocks of a BIBD($2^s 5+1$,5,1) and the block of size $(2^s t+1)$ by the blocks of a minimum covering of $2^s t+1$ points. The result is a minimum covering of $v = 2^s(v-w)+2^s t+1$ points with $B(v^*)$ blocks. □

Lemma 3.10. Suppose that $v \in \{573,1353,1473\}$. Then $C(v) = B(v)$.

Proof. Since there is a $TD(5,24)$, there exists a BIBD(121,5,1) which contains a flat of order 25. (This is obtained by adjoining a point to each group of the transversal design, and replacing blocks of size 25 by copies of a BIBD(25,5,1). Apply Lemma 3.8, noting that $573 = 5(121-8)+8$, and that there exists a $TD(5,113)-TD(5,17)$ since $113 = 7.16+1$ (Lemma 3.4). Then $C(573) = B(573)$.

For $v = 1353$, proceed as follows. Since there is a $TD(6,31)$, there is a BIBD(186,6,1) which contains a flat of order 31. Taking $s = 3$ and noting that $C(113) = B(113)$, an application of Lemma 3.9 with $t = 14$ establishes the result. For $v = 1473$, apply Lemma 3.7 to a resolvable BIBD(85,5,1) with $s = 4$ and $t = 7$. □

Lemma 3.11. Suppose that v is a positive integer such that $v \equiv 53 \pmod{100}$. If v does not belong to $S = \{53, 153, 253, 353, 753\}$, then $C(v) = B(v)$.

Proof. It is shown in [9] that if $v \equiv 53\ (mod\ 100)$ and $v \geq 2753$, then $C(v) = B(v)$. It was shown in Lemmas 3.6 and 3.10 that if $v \equiv 53\ (mod\ 100)$, and if $0 \leq v \leq 1453$ and $v \notin S$, then $C(v) = B(v)$. The interval $1553 \leq v \leq 2653$ is treated below.

$80m+21+4t$	m	t	$20m+5$	$4t+1$
1553	18	23	365	93
1653	18	48	365	193
1753	18	73	365	293
1853	19	98	385	313

(By Lemma 3.6, $C(1953) = B(1953)$.)

$80m+21+4t$	m	t	$20m+5$	$4t+1$
2053	24	28	485	113
2153	24	53	485	213
2253	24	78	485	313
2353	24	103	485	413

(By Lemma 3.3, $C(2453) = B(2453)$.)

$80m+21+4t$	m	t	$20m+5$	$4t+1$
2553	27	93	545	373
2653	27	118	545	473

This establishes the lemma. □

Lemma 3.12. Suppose that v is a positive integer such that $v \equiv 73\ (mod\ 100)$. If v does not belong to $\{73,173,273\}$, then $C(v) = B(v)$.

Proof. It is shown in [9] that if $v \equiv 73\ (mod\ 100)$ and $v \geq 2273$, then $C(v) = B(v)$. For $v \leq 1473$, the result is true by Lemmas 3.6 and 3.10. The remaining cases are covered by Lemma 3.2 according to the following table.

$80m+21+4t$	m	t	$20m+5$	$4t+1$
1573	18	28	365	113
1673	18	53	365	213
1773	18	78	365	313
1873	22	23	445	93
1973	22	48	445	193
2073	22	73	445	293
2173	22	98	445	393

This establishes the lemma. □

The foregoing can be summarized as follows.

Theorem 3.13. Let v be a positive integer such that $v \equiv 13\ (mod\ 20)$. Let $S = \{13, 33, 53, 73, 133, 153, 173, 233, 253, 273, 333, 353, 433, 633, 753\}$. If $v \notin S$, then $C(v) = B(v)$.

Next we consider the case of $v \equiv 17\ (mod\ 20)$. Gardner [6] has shown that if $v = 100m+17$, $m \geq 1$ or $v = 100m+97$, $m \geq 0$, then $C(v) = B(v)$. It was shown in [9] that $C(100m+37) = B(100m+37)$ for $m \geq 7$. We note also that $C(537) = B(537)$. This follows from Lemma 3.1 with $n = 7$, $a = 17$. A $TD(5,105)-TD(5,17)$ is required. As noted in the case of $v = 533$, there exists a $TD(5,10)-TD(5,2)$, a $TD(5,8)-TD(5,0)$, and a $TD(6,11)$. There is also a $TD(5,9)-TD(5,1)$. Therefore since $105 = 11.8+8.2+1$, a $TD(5,105)-TD(5,17)$ also exists.

Lemma 3.14. Let $S = \{477, 557, 657, 677, 857, 877, 957, 977, 1057, 1077, 1257, 1277, 1357, 1377, 1477, 1957, 2557, 2757, 2777, 2857, 2877, 3157, 3257\}$. If $v \in S$, then $C(v) = B(v)$.

Proof. Apply Lemma 3.1 with the following parameters.

v	n	a	$12n+4+a$	Incomplete TD
477	6	18	94	19.4+18 (Lemma 3.4)
557	7	22	110	5.22 (Lemma 3.4)
657	9	15	127	7.16+15 (Lemma 3.4)
677	9	20	132	*7.16+4.5 (Lemma 3.4)
857	12	17	165	*9.16+4.5+1 (Lemma 3.4)
877	11	38	174	37.4+26 (Lemma 3.4)
957	14	10	182	7.23+11+10
977	14	15	187	7.23+11+15
1057	14	35	207	43.4+35
1077	16	8	204	7.27+7+8
1257	19	5	237	7.32+8+5
1277	19	10	242	7.32+8+10
1357	19	30	262	7.32+8+30
1377	21	3	259	7.32+32+3
1477	21	28	284	7.32+32+28
1957	29	20	372	7.49+9+20
2557	39	10	482	7.61+45+10
2757	42	12	520	7.71+11+12
2777	41	33	529	7.67+27+33
2857	44	5	537	7.73+21+5
2877	44	10	542	7.73+21+10
3157	49	0	592	∃TD(5,592)
3257	49	25	617	7.83+11+25

*(These incomplete transversal designs use a $TD(5,21)-TD(5,5)$ as ingredients).

This establishes the lemma. □

The following lemma is useful in creating balanced incomplete block designs with $k = 6$ and $\lambda = 1$ which contain large flats.

Lemma 3.15. If there exists D, a BIBD(v,6,1), then there exists a BIBD($5v$+1,6,1) which contains a flat of order v.

Proof. By deleting a point from $PG(2,5)$ (the BIBD(31,6,1)), a group divisible design, of type 5^6 and blocks of size G is obtained. This design, G, contains a block which meets every group in one point (that is, a transversal). A group divisible design of type v^1 and blocks of size 6 is obtained by considering each point of D as a group. By inflating D by a factor of 6 (using G, which contains a transversal), then adjoining an ideal point ∞ to each group, the required BIBD($5v$+1,6,1) is obtained. □

Lemma 3.16. If $v = 577$ or $v = 2457$, then $C(v) = B(v)$.

Proof. For $v = 577$, proceed as follows. Let T be a TD(5,24) with groups $G_1, G_2, \ldots, G_5$. Let ∞ be a point not in T. By replacing G_i by a copy of a BIBD(25,5,1), for $i = 1,2,\ldots,5$, a BIBD(121,5,1) which contains a flat of order 25. Further, there is a TD(5,114)–TD(5,18) since $114 = 4.24+18$ (Lemma 3.4). Since $577 = 5(121-7)+7$, then $C(577) = B(577)$. For $v = 2457$, note that there exists a BIBD(331,6,1) containing a flat of order 66, by Lemma 3.15. Apply Lemma 3.9 with $s = 3$ and $t = 42$, noting that $2457 = 8.265+8.42+1$. □

Lemma 3.17. Let $S = \{v{:}v \equiv 57$ or $77\ (mod\,100),\ 1557 \leq v \leq 3277\}$ and $T = \{777, 1157, 1177, 1457\}$ If $v \in S \cup T$, then $C(v) = B(v)$.

Proof. For $v = 777$, we use Lemma 3.7. There is a resolvable BIBD(85,51). Using $s = 3$ and $t = 12$ yields the result. For $v = 1457$, we use the resolvable BIBD(85,5,1) and Lemma 3.7 with $s = 4$ and $t = 6$. The remaining cases follow from Lemma 3.3 with the following parameters.

$80m+21+4t$	m	t	$20m+5$	$4t+1$	$80m+21+4t$	m	t	$20m+5$	$4t+1$
1157	13	24	265	97	2377	27	49	545	197
1177	13	29	265	117	2457	(see Lemma 3.16)			
1557	18	24	365	97	2477	27	74	545	297
1577	18	29	365	117	2557	(see Lemma 3.14)			
1657	18	49	365	197	2577	27	99	545	397
1677	18	54	365	217	2657	27	119	545	477
1757	18	74	365	297	2677	27	124	545	497
1777	18	79	365	317	2757	(see Lemma 3.14)			
1857	19	79	385	317	2777	(see Lemma 3.14)			
1877	22	24	445	97	2857	(see Lemma 3.14)			
1957	(see Lemma 3.14)				2877	(see Lemma 3.14)			
1977	22	49	445	197	2957	30	134	605	537
2057	24	29	485	117	2977	30	139	605	557
2077	22	74	445	297	3057	31	139	625	557
2157	24	54	485	217	3077	37	24	745	97
2177	22	99	445	397	3157	(see Lemma 3.14)			
2257	24	79	485	317	3177	37	49	745	197
2277	27	24	545	97	3257	(see Lemma 3.14)			
2357	24	104	485	417	3277	37	74	745	297

This establishes the lemma. □

Since it is shown in [9] that if $v \equiv 57$ or $77 (\mathrm{mod}\ 100)$ and $v > 3277$, then $C(v) = B(v)$, the preceding results can be summarized as follows.

Theorem 3.18. Let $S = \{17, 37, 57, 77, 137, 157, 177, 237, 257, 277, 337, 357, 377, 437, 457, 637, 757\}$. If v is a positive integer such that $v \equiv 17\ (mod\,20)$ and $v \notin S$, then $C(v) = B(v)$.

Let us consider the case of $v \equiv 9\ (mod\,20)$. Applying Lemma 3.3 to an integer congruent to 17 $(mod\,20)$ yields a result for an integer congruent to 9 $(mod\,20)$. We observe that the results of the previous theorem, together with Lemma 3.3, show that for $v \equiv 69\ (mod\,80)$ and $v \geq 3109$, then $C(v) = B(v)$. It is also shown in [9] that if $v \equiv 9\ (mod\,20)$ and $v \geq 13469$, then $C(v) = B(v)$. The following lemma is useful in improving these results.

Lemma 3.19. Let $u = 20m+17$ where $m \geq 4$ and $C(u) = B(u)$. If $v \equiv 20m+29 \ (mod\, 60)$, and v satisfies $320m+329 \leq v \leq 1280m+1109$, then $C(v) = B(v)$.

Proof. Let n be an integer satisfying $4m+4 \leq n \leq u$, and let $a = u-n$. Apply Lemma 3.1. Since $C(u) = B(u)$, then $C(4u+1) = B(4u+1)$ by Lemma 3.3. The condition $n \geq 4m+4$ and $n \leq u$ guarantees that $0 \leq a \leq 4n+1$.

First consider the set of n satisfying $5m+4 \leq n \leq u$. Since $m \geq 4$, then $3n+1 > 52$, and there exists a $TD(6,3n+1)$. Since $n \geq 5m+4$, then $a \leq 3n+1$. Apply Lemma 3.4 (using the fact that there exist a $TD(5,4)$–$TD(5,0)$ and a $TD(5,5)$–$TD(5,1)$) to obtain a $TD(5,4(3n+1)+a)$–$TD(5,a)$. Then Lemma 3.1 states that for $v \equiv 20m+29 \ (mod\, 60)$ and v lying in the interval $380m+329 \leq v \leq 1280m+1109$, we have $C(v) = B(v)$.

Now consider n satisfying $4m+4 \leq v \leq 5m+4$. Let $z = 12n+4-3a$. For v in this range, we have $0 \leq z \leq a$; and since $m \geq 4$, then $a > 52$, so there exists a $TD(6,a)$. Since $4a+z = 12n+4+a$, there exists a $TD(5,12n+4+a)$–$TD(5,a)$ provided that there is a $TD(5,z)$. However, there exists a $TD(5,z)$ for all non-negative integers z except for z in $\{2,3,6,10\}$. (The existence of a $TD(5,13)$ was shown by Todorov, [12]). But by definition, $z \equiv 1 \ (mod\, 3)$. Hence we need only consider the case of $z = 10$. In this case, $4n-a = 2$. However $n+a = 20m+17$, so $5n = 20m+19$, so no such integer n exists. □

Lemma 3.20. Let v be an integer satisfying $v \equiv 49 \ (mod\, 60)$. If $v \geq 1609$, then $C(v) = B(v)$.

Proof. For $v \geq 13469$, the result is established in [9]. For the remaining values we use Lemma 3.18 with parameters as in the following table.

u	values covered
97	1609-6229
217	3529-13909

This establishes the lemma. □

Lemma 3.21. Let v be an integer satisfying $v \equiv 9 \ (mod\, 60)$. If $v \geq 1929$, then $C(v) = B(v)$.

Proof. For $v \geq 13467$, the result is established in [9]. For the remaining values we use Lemma 3.18 as in the following table.

u	values covered
117	1929-7509
297	4809-19029

This establishes the lemma. □

Lemma 3.22. Let v be an integer satisfying $v \equiv 29 \ (mod\, 60)$. If $v \geq 3209$, then $C(v) = B(v)$.

Proof. For $v \geq 13469$, the result is established in [9]. For the remaining values, we use Lemma 3.19 with the parameters in the following table.

u	values covered
197	3209-12629
317	5129-20309

This establishes the lemma. □

As a result of the above, we have the following.

Lemma 3.23. Let v be an integer congruent to 9 (mod 20). If $v \geq 3209$, then $C(v) = B(v)$. Values below 3209 are treated below.

Theorem 3.24. Let $S = \{389, 469, 789, 869, 1189, 1269, 1589, 1609, 1669, 1729, 1789, 1849, 1909, 1929, 1969, 1989, 2009, 2029, 2049, 2069, 2089, 2109, 2149, 2169, 2209, 2229, 2269, 2289, 2309, 2349\}$. Then $C(v) = B(v)$.

Proof. If $v \neq 2009$, then the results follow from Lemmas 3.3, 3.20 and 3.21. For $v = 2009$, we use Lemma 3.2, noting that there is resolvable BIBD(405,5,1), and $C(389) = B(389)$. □

Theorem 3.24. Suppose v is congruent to 9 (mod 20) and $v \geq 2369$. Let $S = \{2369, 2429, 2669, 2729, 2849, 3029, 3149\}$. Then $C(v) = B(v)$ with the possible exception of v in S.

Proof. In [9], it is shown that if $v \equiv 89$ (mod 100) and $v \geq 2389$, or if $v \equiv 9$ (mod 100) and $v \geq 2509$, or if $v \equiv 69$ (mod 100) and $v \geq 2869$, then $C(v) = B(v)$. These results, together with Lemmas 3.20, 3.21 and 3.23 establish the result for all v except for $v = 2549$. This case is treated by Lemma 3.8. Since there exists a $TD(5,104)$, there exists a BIBD(521,5,1) which contains a flat of order 105. Note that by Lemma 3.4, there is a $TD(5,507)-TD(5,91)$ since $507 = 104.4+91$. Also $469 = 5(105-14)+14$, and since $C(469) = B(469)$, then $C(2549) = B(2549)$, as required. □

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