Repeated Edges in 3-factorizations

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ABSTRACT

The type of a 3-factorization of $3K_{2n}$ is the pair (t,s), where t is the number of doubly repeated edges in 3-factors, and $\binom{n}{2} - s$ is the number of triply repeated edges in 3-factors. We determine the spectrum of types of 3-factorizations of $3K_{2n}$ for all $n \geq 6$; for each $n \geq 6$, there are 43 pairs (t,s) meeting numerical conditions which are not types and all others are types. These 3-factorizations lead to threefold triple systems of different types.

1. The 3-factorization problem.

A *3-factor* of a multigraph $G = (V,E)$ is a spanning cubic submultigraph of G; a 3-factorization $\mathcal{F} = \{F_1,\ldots,F_d\}$ is a partition of the edges of G into 3-factors. For a graph G, the multigraph λG is formed by repeating each edge of G λ times.

In this paper, we consider 3-factorizations of $3K_{2n}$. Our primary motivation is that the standard recursive construction for producing a triple system of index three and order $2v+1$ from one of order v employs a 3-factorization of $3K_{v+1}$. Different 3-factorizations lead to different triple systems of index 3. Most importantly, any induced $2K_2$ (doubly repeated edge) in a factor leads to a triple appearing twice in the triple system, and any $3K_2$ leads to a triple appearing three times in the triple system. Hence we consider the following question: In a 3-factorization of $3K_{2n}$, how many induced $2K_2$'s and how many $3K_2$'s can appear *in total* in the collection of 3-factors?

Let G be a d-regular graph on $2n$ vertices, and let $\mathcal{F} = \{F_1,\ldots,F_d\}$ be a 3-factorization of $3G$. The *type* of $\mathcal{F}$ is (t,s), where t is the number of induced $2K_2$'s, and s is the number of edges of G *not* appearing in $3K_2$'s in factors of $\mathcal{F}$. $\mathcal{F}$ is called an (n,d,t,s) 3-factorization.

Now let $\mathcal{T}(G)$ be the set of all 3-factorizations of G, and let $\mathcal{G}_{n,d}$ be the set of all d-regular graphs on $2n$ vertices whose complements are 1-factorizable.

We are interested in possible types of 3-factorizations; hence we define $S(G) = \{(t,s)\colon (t,s) \text{ is the type of some } \mathcal{F} \in \mathcal{T}(G)\}$, and $S(n,d) = \bigcup_{G \in \mathcal{G}_{n,d}} S(G)$. Our goal is to determine $S(n,2n-1)$.

In section two, we determine necessary conditions for $(t,s) \in S(n,d)$. Let $\mathcal{A}(n,d) = \{(t,s)\colon 0 \leq t \leq s \leq nd\} \setminus \{(0,1), (0,2), (0,3), (0,4), (0,5), (0,7), (0,8), (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (1,9), (1,10), (1,11), (2,2), (2,3), (2,4), (2,5), (2,6), (2,7), (2,8), (2,9), (2,10), (3,3), (3,4), (3,5), (3,6), (3,7), (3,8), (3,10), (4,5), (4,7), (4,8), (5,5), (5,6), (5,7), (5,8), (5,9), (5,10)\}$.

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Main Theorem: For $n \geq 6$, $\mathcal{S}(n,2n-1) = \mathcal{A}(n,2n-1)$.

We prove the sufficiency by developing recursive constructions in section 3, and developing a large number of small examples in subsequent sections.

Before proving the Main Theorem, we remark on some related research. Lindner and Wallis [3] settled the analogous problem for 2-factorizations of $2K_{2n}$; in our extension to $\lambda=3$, we have chosen the most detailed generalization, by treating doubly and triply repeated edges separately. The main application of these results on factorizations is in the construction of triple systems of index λ with prescribed numbers of repeated blocks. For $\lambda=2$, this problem has been settled by Rosa and Hoffman [5]; for $\lambda=3$, two extensions have been considered. Milici and Quattrocchi [4] determined the possible numbers of triply repeated blocks, and Colbourn and Mahmoodian [1] determined the possible numbers of distinct blocks. The Main Theorem proved here is a first step in obtaining a more detailed classification which treats doubly and triply repeated blocks separately.

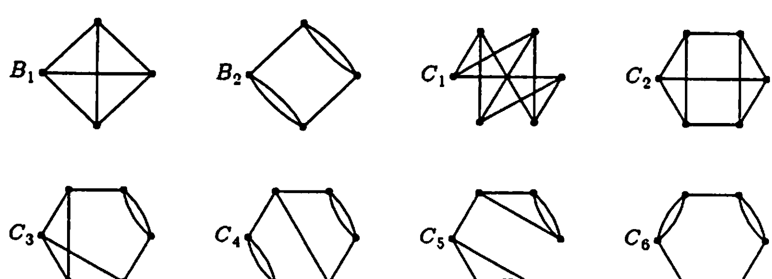
The classification of factorizations by numbers of repeated edges is also related to the neighbourhood problem [2]; in the neighbourhood problem, the isomorphism type of one neighbourhood is specified completely. Extending current results to more than one neighbourhood seems hopeless at present; however, relaxing the requirements so that the numbers of repeated edges in neighbourhoods is specified reduces to the determination of the possible numbers for repeated blocks. While a still more detailed classification may yield information about individual neighbourhoods, we do not attempt to further complicate the problem addressed here!

2. Necessary Conditions.

In this section, we establish the necessity of the conditions in the main theorem. Given a 3-factorization $\mathcal{F}$, the *remainder* of $\mathcal{F}$ is the graph with edge set $\{e$: e does not appear in a 3-times repeated edge in $\mathcal{F}\}$, and is denoted $R(\mathcal{F})$. Each edge in $R(\mathcal{F})$ appears either in two or in three distinct factors of $\mathcal{F}$. Each factor F_i of $\mathcal{F}$ induces a (possibly empty) cubic submultigraph G_i on the edges of $3R(\mathcal{F})$; each G_i naturally contains no three-times repeated edges. We call the G_i's *portions*.

Observation 2.1: A vertex v of $R(\mathcal{F})$ belongs to precisely $deg(v)$ portions in $\{G_i\}$, where $deg(v)$ is the degree of v in $R(\mathcal{F})$. □

We make frequent reference to the following portions; these are the only portions with fewer than eight vertices:



No vertex has degree 1 in $R(\mathcal{F})$ and, moreover, $R(\mathcal{F})$ has at least four vertices of nonzero degree. Furthermore, no portion can have more two-times repeated edges than singly repeated edges, and hence we have

Observation 2.2: The number of two-times repeated edges cannot exceed the number of edges in $R(\mathcal{F})$. □

A lower bound on the number of two-times repeated edges can be determined by considering vertices of degree 2:

Lemma 2.3: Let $\mathcal{T}$ be the subgraph of $R(\mathcal{F})$ induced on the vertices of degree 2. Suppose that $\mathcal{T}$ consists of c cycles of total length ℓ_c and p paths of total length ℓ_p. Then the number of two-times repeated edges is at least $\ell_c + \ell_p + 2p$.

Proof. Consider an edge e incident with a vertex in $\mathcal{T}$. The three copies of this edge in $3R(\mathcal{F})$ must be allocated so that two copies are in one portion and the remaining copy in a different portion. Hence the number of two-times repeated edges in $\mathcal{F}$ must be at least the number of edges incident with vertices of $\mathcal{T}$ in $R(\mathcal{F})$; this number is $\ell_c + \ell_p + 2p$. □

Now we can prove necessity:

Lemma 2.4. If $(t,s) \notin \mathcal{A}(n,d)$, there is no (n,d) 3-factorization of type (t,s).

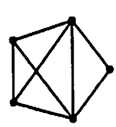
Proof. We may assume $0 \leq t \leq s \leq nd$, from elementary considerations and observation 2.2. Now let us consider cases for s.

If $s = 1,2$, or 3, there is no graph on four vertices with minimum degree 2 and s edges. For $s = 4$, the only graph permitted is C_4, and by lemma 2.3 we must have $t = 4$ as well. For $s = 5$, there are three portions each with at least four vertices for a total of at least $6 > 5$ edges.

If $s = 6$, we may only use two portions of size 6 or three portions of size 4. Since the only portion of size 6 on at most 6 edges is C_6, this requires $t = 6$. Otherwise we must use portions B_1 and B_2, and hence t must be even. For $t < 6$, we must use a portion of B_1 and hence $R(\mathcal{F})$ would be K_4; for $t \leq 2$, we must take two portions of B_1, leaving a third portion which is also B_1, and hence $t = 0$. Thus $t = 2$, $s = 6$ is impossible.

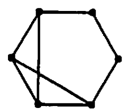
For $s = 7$, we must take one portion of size 6 and two of size 4. If either portion of size 4 is B_1, $R(\mathcal{F})$ contains a K_4 and the seventh edge must connect to a vertex in $R(\mathcal{F})$ which necessarily has degree 1, a contradiction. Hence two of the portions are B_2. The portion of size 6 can have at most 7 distinct edges and hence is C_4, C_5 or C_6; in any case, we must have $t = 6$ or 7.

For $s = 8$, we must take either four portions of size 4, or one of size 4 and two of size 6. In either case, if a portion of size 4 is B_1, then $R(\mathcal{F})$ is



Hence there can be no portions of size 6 if B_1 is a portion. Moreover, if two portions are B_1, no portion of B_2 remains. Finally, if three portions of B_1 are included, what remains is two three-times repeated edges. Hence, if B_1 is a portion, the remaining three portions are B_2 and we have $t = 6$. So let all portions of size 4 be B_2. This leads to $t = 8$ if all portions have size 4. Moreover, if the two portions of size 6 are C_4, C_5 or C_6, we have $t \geq 6$. Thus if $t \leq 5$, we must have C_3 as a portion and hence $R(\mathcal{F})$ is

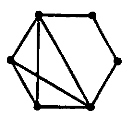
S



The removal of a portion C_3 and a portion B_2 leaves only six distinct edges, and hence any remaining portion is necessarily C_6, giving $t \geq 6$.

Now consider $s = 9$. We may have three portions of size 6, or one of size 6 and three of size 4. (By Lemma 2.3, a portion of size 8 necessitates $t \geq 8$.) To obtain $t = 1$, we must have a portion of C_3; hence $R(\mathcal{F})$ contains S, with one additional edge and hence $R(\mathcal{F})$ cannot contain C_1 or C_2. Now if $R(\mathcal{F})$ contains B_1, it is

T



This forces at least one of the portions to be B_2 and hence $t \neq 1$. To obtain $t = 2$, the same argument shows that no portion is C_3.

Now if a portion were C_1 or C_2, no portion is B_1, and so portions of size 4 lead to $t \geq 6$. Moreover, there cannot be a portion of C_1 *and* a portion of C_2. Finally, two portions of C_1 or two of C_2 force the third to be the same; hence the remaining two portions are from $\{C_4, C_5, C_6\}$; but then $t \geq 4$. At this point, no portion is C_1, C_2, or C_3. Hence some portion is C_4 or C_5, and the other portions are all B_1; this requires that $R(\mathcal{F})$ contain two K_4's intersecting in at most one edge, and hence $s > 9$. Thus t cannot be 2.

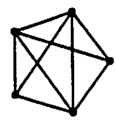
For $t = 5$, C_3 cannot be a portion since our earlier argument shows that $R(\mathcal{F})$ must be T, and taking portions C_3 and B_1 forces the selection of another portion of B_1.

For $s = 10$, we may take five portions of size 4, or two of size 6 and two of size 4. (Portions of size 8 lead to $t \geq 7$ by Lemma 2.3.) In the first case, t is necessarily even since each portion is B_1 or B_2; in addition, if $t = 2$, we have 4 portions of B_1 and hence $R(\mathcal{F})$ is K_5; removing any four K_4's from K_5 leaves a fifth K_4 and hence t cannot be 2.

Now consider two portions of size 6 and two of size 4.

If C_3 is a portion, $R(\mathcal{F})$ necessarily has degree sequence $4^2 3^4$. If in addition B_1 is a portion, $R(\mathcal{F})$ must contain a K_4 involving the two vertices of degree 4. It is easily checked that no such graph exists. Similarly, neither C_1 nor C_2 is a portion provided C_3 is, since $R(\mathcal{F})$ must have the vertices of degree 4 adjacent if it contains C_1 or C_2, but they must be nonadjacent if it contains C_3. Hence including C_3 leads to $t \geq 6$.

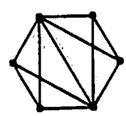
If C_1 or C_2 is a portion, B_1 is not and hence $t = 4$ or $t \geq 6$. Thus we are only allowed portions of size 6 from $\{C_4, C_5, C_6\}$ and hence $t \geq 4$. To eliminate $t = 5$, observe that we must have two portions which are B_1. If these are on the same four vertices, neither remaining portion can be C_6 and hence $t = 4$. Otherwise, $R(\mathcal{F})$ contains



since if the two K_4's intersect in two or fewer vertices, more than ten edges are needed. But then $R(\mathcal{F})$ is either K_5 or has a vertex of degree one, and both are contradictions.

Finally consider $s = 11$, $t = 1$. If there are portions of size 8, $R(\mathcal{F})$ must have a vertex of degree 2 and hence by Lemma 2.3, $t > 1$. Thus C_3 must be a portion, and the remaining portions are all simple; we have either four portions of B_1, or one portion of B_1 and two from $\{C_1, C_2\}$.

First consider four portions of B_1's. If two such portions intersect in only two vertices, $R(\mathcal{F})$ is



and the centre edge must be taken four times, a contradiction.

Thus there must be two portions of size 6, and hence the degree sequence of $R(\mathcal{F})$ is $4^4 3^2$ (since B_1 is a portion, the degree sequence $4^1 3^6$ on seven vertices is eliminated). Now the two vertices of degree 3 in $R(\mathcal{F})$ must be adjacent if $R(\mathcal{F})$ is to contain B_1. Then the doubly repeated edge in the C_3 portion cannot appear on an edge incident with one of the degree 3 vertices since each such edge appears in each of the three portions of size 6. Hence the doubly repeated edges appear on the same set of vertices as does the B_1. But then removing the portions of B_1 and C_3 leaves only ten distinct edges. Hence either both remaining portions are C_1's or both are C_2's. In either case, both portions be identical since otherwise more than eleven edges are needed. But then the two edges not in these portions lead to at least two doubly repeated edges.

In all cases forbidden by the necessary conditions, no set of suitable portions exists and hence the proof is complete. □

3. Recursive constructions.

To prove sufficiency of the conditions in the main theorem, we adopt a three stage approach. First, some special small 3-factorizations are explicitly constructed. Second, a rich set of constructions is applied to these initial structures to establish that for a suitably chosen n_0, the theorem holds for all $n_0 \leq n \leq 2n_0 - 1$. Third, a standard recursive construction is then used to establish the theorem for all $n \geq n_0$. The third stage is the most standard (and the easiest); hence we dispose with it first.

We use one very simple construction.

Construction A (basic construction). Let G be a d-regular graph on $2n$ vertices with 1-factorization $F_1,\ldots,F_d$. Then $3G$ has a 3-factorization $\mathcal{F}$ of type (tn,sn) for all $0 \leq t \leq s \leq d$ except for $(t,s) = (0,1)$, $(0,2)$, $(1,1)$, $(1,2)$ or $(1,3)$.

Proof. Form permutations ϕ and π on $\{1,\ldots,d\}$ so that the d sets $\{\{i,\phi(i),\pi(i)\}$: $i = 1,\ldots,d\}$ consist of $d - s$ sets consisting of a number repeated three times, and t of the sets consist of one element twice and a different element once. This is easily done with the exceptions listed above. Now the required 3-factorization has 3-factors

$$\{\{F_i \cup F_{\phi(i)} \cup F_{\pi(i)}\}: \ i = 1,\ldots,d\}. \quad \square$$

Lemma 3.1. If $n \geq 4$ and $S(n,2n-1) = A(n,2n-1)$, then

$$S(2n,4n-1) = A(2n,4n-1) \text{ and}$$

$$S(2n+1,4n+1) = A(2n+1,4n+1).$$

Proof. Let $(t,s) \in A(2n+x,4n-1+2x)$, $x \in \{0,1\}$. Now write

$$t = t_1 + t_2(n+x) + t_3(2n+x)$$

$$s = s_1 + s_2(n+x) + s_3(2n+x)$$

so that t_1 and s_1 are as large as possible subject to the constraints

$$0 \leq t_1 \leq s_1 \leq n(2n-1),$$

$$0 \leq t_2 \leq s_2 \leq 2n - 1, \text{ and}$$

$$0 \leq t_3 \leq s_3 \leq 2n + 2x.$$

and $(t_2,s_2),(t_3,s_3) \neq (0,1)$, $(0,2)$, $(1,1)$, $(1,2)$, $(1,3)$. Now if $s \leq n(2n-1)$, $s_1 = s$ and $s_2 = s_3 = 0$. Otherwise we may ensure that $s_1 \geq n(2n-1) - 3(n+x)$, and hence for $n \geq 4$, $(t_1,s_1) \in A(n,2n-1)$.

A commutative Latin square of side $4n + 2x$ with constant diagonal containing a subsquare of side $2n$ yields a 1-factorization. The 1-factors corresponding to the subsquare induce a disconnected graph $K_{2n} \cup G$, where G is a 1-factorizable graph on $2n + 2x$ vertices; the remaining 1-factors induce a disjoint graph H on $4n + 2x$ vertices.

Let $F_1 \cdots F_{2n-1}$ be a 3-factorization of type (t_1,s_1) of $3K_{2n}$. Let $G_1,\ldots,G_{2n-1}$ be a 3-factorization of type $(t_2(n+x),s_2(n+x))$ of $3G$, and let $H_1,\ldots,H_{2n+x}$ be a 3-factorization of type $(t_3(2n+x),s_3(2n+x))$ of $3H$. Then

$$\{\{F_i \cup G_i\}: i = 1,\ldots,2n-1\} \cup \{H_j: \ 1 \leq j \leq 2n+2x\}$$

is a 3-factorization of type (t,s) and hence $(t,s) \in S(2n+x,4n-1+2x)$. $\square$

Lemma 3.1 ensures that the Main Theorem holds once we have established it in an interval $n_0 \leq n \leq 2n_0 - 1$. At the same time, we want n_0 to be "small", so as to leave relatively few exceptions; hence we introduce a large set of specialized constructions, primarily designed to settle small cases.

Construction B (addition): If $(t_1,s_1) \in S(G)$ and $(t_2,s_2) \in S(H)$, where G and H are

d-regular graphs on n_1 and n_2 vertices respectively, $(t_1+t_2,s_1+s_2) \in S(G \cup H) \subseteq S(n_1+n_2,d)$, provided $\overline{G \cup H}$ is 1-factorizable.

When $n_1 = n_2$ and $d = 2n_1 - 1$ in Construction B, $G = H = K_{2n_1}$; hence $\overline{G \cup H} = K_{2n_1,2n_1}$. We therefore develop a method for 3-factorizing $K_{2n,2n}$:

Construction C (doubling): If $(t,s) \in S(n,d)$, then $(2t,2s) \in S(G)$ for some G a d-regular subgraph of $K_{2n,2n}$, and hence $(2t,2s) \in S(2n,d)$.

Proof. Let $\{F_1,\dots,F_d\}$ be a 3-factorization of type (t,s) of a graph $3H$ on vertices $\{x_1,\dots,x_n\}$. Form G on vertex set $\{y_1,\dots,y_n\} \cup \{z_1,\dots,z_n\}$; $3G$ has 3-factors

$$\{\{\{y_i,z_j\},\{z_i,y_i\}: \{x_i,x_j\} \in F_k\}: k = 1, \dots, d\}. \quad \square$$

Construction B with $n = n_1 = n_2$ yields a 3-factorization of a subgraph of $K_{2n} \cup K_{2n}$, while construction C yields a 3-factorization of a subgraph of $K_{2n,2n}$. Thus the resulting 3-factors are disjoint and can be taken together to form a 3-factorization of a subgraph of K_{4n}. We require a similar strategy for K_{4n+2}; we develop this next.

Construction D (1-splice): Let $\mathcal{F}^1$ and $\mathcal{F}^2$ be 3-factorizations of (possibly different) graphs in $\mathcal{G}_{n,d}$ of type (t_1,s_1) and (t_2,s_2) respectively. Suppose in addition that $\mathcal{F}^1 = \{F_1^1,F_2^1,\dots,F_d^1\}$ and $\mathcal{F}^2 = \{F_1^2,F_2^2,\dots,F_d^2\}$ are on the same vertex set $\{\infty v_1,\dots,v_{2n-1}\}$ and that for each $1 \le i \le 2n-1$ and $1 \le j \le d$, the edge $\{\infty v_i\}$ appears the same number of times in F_j^1 and F_j^2. Now form a 3-factorization $\mathcal{F}$ on vertex set $\{y_1,\dots,y_{2n-1}\} \cup \{z_1,\dots,z_{2n-1}\}$ with 3-factors $\{F_1,\dots,F_d\}$, where F_t is defined as follows:

(a) if edge $\{v_i,v_j\}$ appears m times in F_t^1, edge $\{y_i,y_j\}$ appears m times in F_t.

(b) if edge $\{v_i,v_j\}$ appears m times in F_t^2, edge $\{z_i,z_j\}$ appears m times in F_t.

(c) if edge $\{\infty v_i\}$ appears m times in F_t^1 (or in F_t^2), edge $\{y_i,z_i\}$ appears m times in F_t.

These d 3-factors factorize $3G$ for some G a d-regular subgraph of $(K_{2n-1} \cup K_{2n-1}) \cup \{f\}$, f a 1-factor.

Moreover, if ∞ is incident to d doubly repeated and c triply repeated edges, the type of $\mathcal{F}$ is $(t_1+t_2-d,s_1+s_2-(2n-1)+c)$, and this type is in $S(2n-1,d)$. $\square$

Construction D is supplemented with a factorization for $K_{2n-1,2n-1} - \{f\}$, f a 1-factor:

Construction E (1-double): Let $\mathcal{F}$ be a 3-factorization of type (t,s) in $S(n,d)$. Let ∞ be a vertex of $\mathcal{F}$ incident to d doubly repeated and c triply repeated edges. Then $(2t-d,2s-(2n-1)+c) \in S(G)$ for some G a d-regular subgraph of $K_{2n-1,2n-1}$; moreover, $S(G) \subseteq S(2n-1,d)$ provided $d < 2n - 1$.

Proof. Let $\mathcal{F} = \{F_1,\dots,F_d\}$ have vertex set $\{\infty v_1,\dots,v_{2n-1}\}$. On vertex set $\{y_1,\dots,y_{2n-1}\} \cup \{z_1,\dots,z_{2n-1}\}$, form 3-factors $\{H_1,\dots,H_d\}$ as follows. In H_t, include edges $\{y_i,z_j\}$ and $\{z_i,y_j\}$ the same number of times that $\{v_i,v_j\}$ appears in F_t, and include edge $\{y_i,z_i\}$ the same number of times that $\{v_i,\infty\}$ appear in F_t. The factors $\{H_i\}$ factorize $3G$ for G a d-regular subgraph of $K_{2n-1,2n-1}$. If $d = 2n - 1$, $\overline{G}$ is not

1-factorizable; for $d < 2n - 1$, $\overline{G}$ is 1-factorizable and hence

$$(2t-d, 2s-(2n-1)+c) \in S(2n-1,d). \quad \square$$

We exploit one further simple construction which is not well suited as a general construction because it is difficult to predict its effect on repetition of edges; nevertheless it is extremely useful in settling small cases.

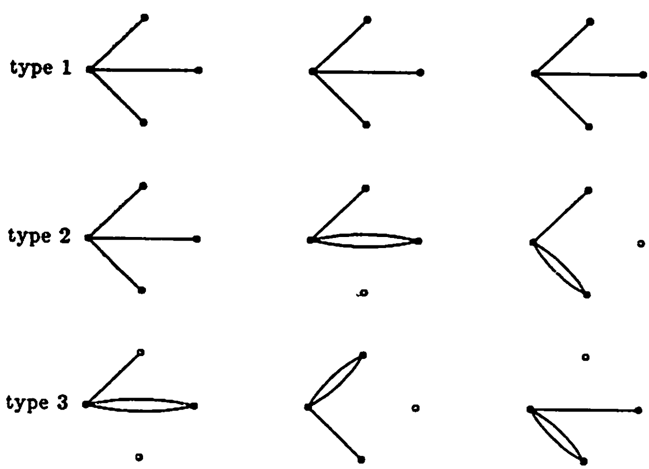
Construction F (trade): Let $\mathcal{F} = (F_1,\ldots,F_d)$ be a 3-factorization of $3G$ for $G \in \mathcal{G}_{n,d}$. Let M_i be a matching in F_i and M_j be a matching in F_j $(i \neq j)$, and suppose that M_i and M_j are the same size and on the same vertex set. Then replacing F_i by $F_i \setminus M_i \cup M_j$ and F_j by $F_j \setminus M_j \cup M_i$ yields a 3-factorization of $3G$. $\square$

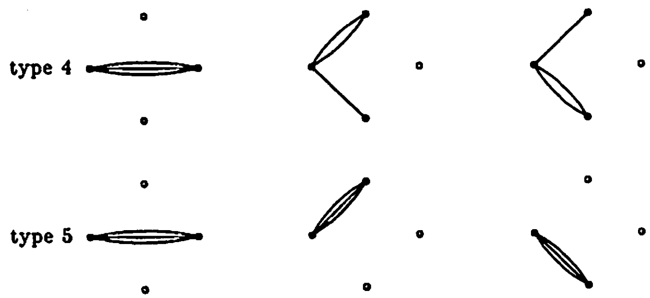
In most applications of Construction F, we take M_i and M_j to be perfect matchings (= 1-factors); in addition, Construction F is very useful in extending a 3-factorization for $G \in \mathcal{G}_{n,d-1}$ to one for $(G \cup F) \in \mathcal{G}_{n,d}$ for F a 1-factor of $\overline{G}$. We call such an application of construction F an *f-trade*.

Naturally we expect that trading would construct essentially all 3-factorizations of K_{2n} given any one; however, we use the construction sparingly because verification that a long sequence of trades constructs a desired example can be quite tedious.

Construction D can be generalized to splice factorizations of different sizes provided they have the same degree. In general, we are left with the task of determining that the complement of the result is 1-factorizable. For degree 3, however, 1-factorizability of the complement is guaranteed by a result of Rosa and Wallis [6].

Construction G (cubic expansion): Let $G \in \mathcal{G}_{n,3}$ be 1-factorizable. A vertex v of G is assigned one of the five types, based on its neighbourhood in the three 3-factors $\{F_1,F_2,F_3\}$ of $3G$. The five types are depicted here (the neighbourhood of v is shown in each factor):





The replacement of v by a triangle, and the obvious extension of the 3-factors produces factorizations for $G' \in \mathcal{G}_{n+1,3}$; if $(t,s) \in S(G)$, we obtain factorizations of G' of the following types according to the type of v:

type (v)	type of factorization in $S(n+1,3)$
1	$(t,s+3)$
2	$(t+2,s+3)$
3	$(t+3,s+3)$
4	$(t+2,s+2)$
5	(t,s)

This cubic expansion can be viewed as a 1-splice with K_4; although easily generalized, it is sufficient for our purposes.

4. Small Cases I: $n \leq 3$.

With the battery of recursive constructions in hand, we are required to construct relatively few 3-factorizations explicitly. Nevertheless, a very substantial task remains. Using constructions $A - G$ and direct constructions, we establish that $S(n,2n-1) = A(n,2n-1)$ for $6 \leq n \leq 11$; the smallest of these, $S(6,11)$ contains 2235 cases to consider. Naturally, we do not consider each case explicitly; rather we describe starting configurations and assume that constructions $A - E$, G are applied to these wherever possible. We do, however, remark explicitly when construction F is used since the verification here is most difficult.

For $n = 1$ and $n = 2$, we have $S(n,2n-1) = A(n,2n-1)$; the basic construction suffices here. Hence the first cases of interest are $S(3,d)$ for $0 \leq d \leq 5$. It is an easy exercise to verify that $S(n,0) = S(n,1) = \{(0,0)\}$. Moreover, $S(n,2) = \{(t,t)\colon 0 \leq t \leq 2n,\ t \text{ even},\ t \neq 2,2n-2\}$, since any two disjoint 1-factors in K_{2n} can be completed to a 1-factorization [6].

The remaining values of d are not settled so easily, even for $n = 3$. There are nine possible 3-factors:

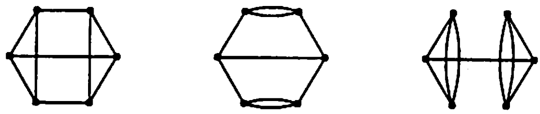
$P_0 = 3K_2 \cup 3K_2 \cup 3K_2$,

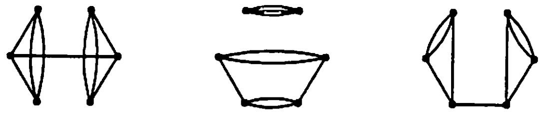
$P_i = C_i$, $i = 1,\dots,6$,

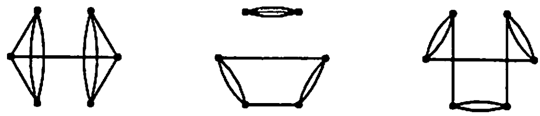
$P_7 = B_1 \cup 3K_2$, and

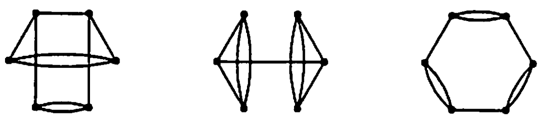
$P_8 = B_2 \cup 3K_2$.

By computer, we found 575 3-factorizations of K_6, each with a distinct subset of these nine factors. (Remarkably, this means that there are at least 575 nonisomorphic (11,3,3) designs having a (5,3,3)·subdesign, by applying the $v \rightarrow 2v + 1$ construction). Those with two factors of P_0 lead to solutions in $S(3,3)$, while those with one factor of P_0 lead to solutions in $S(3,4)$. We exhibit representative solutions here for the cases in which constructions $A - E$, G fail to provide a solution of that type.

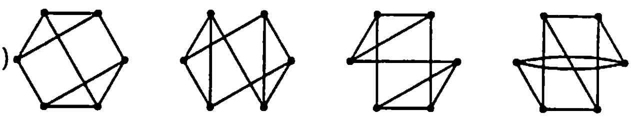
(4,9)∈$S(3,3)$ 

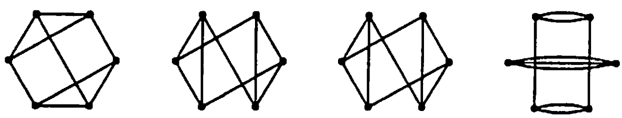
(6,8)∈$S(3,3)$ 

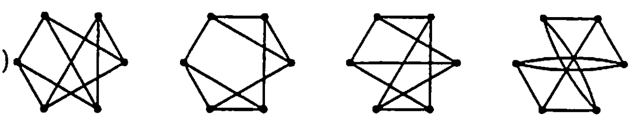
(7,8)∈$S(3,3)$ 

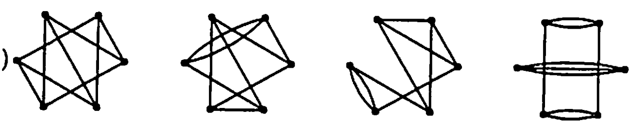
(7,9)∈$S(3,3)$ 

Next we consider $S(3,4)$:

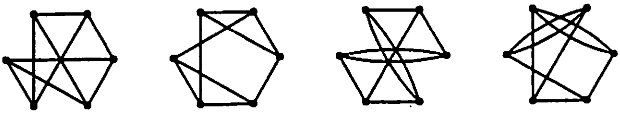
(1,12)∈$S(3,4)$ 

(2,11)∈$S(3,4)$ 

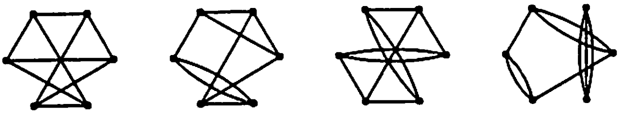
(2,12)∈$S(3,4)$ 

(4,11)∈$S(3,4)$ 

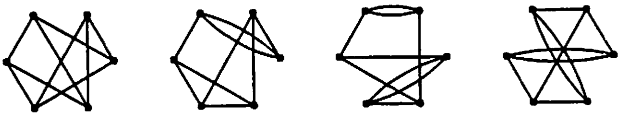
$(4,12) \in S(3,4)$



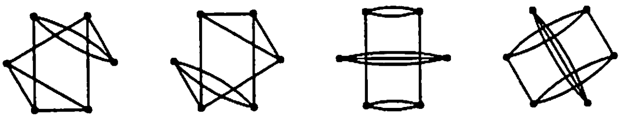
$(5,11) \in S(3,4)$



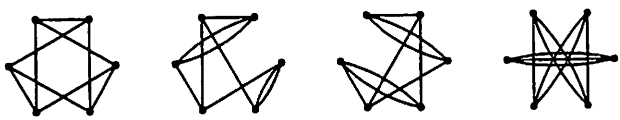
$(5,12) \in S(3,4)$



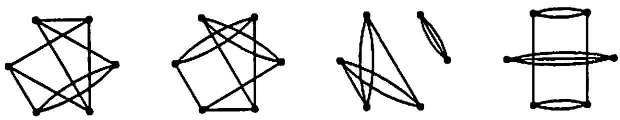
$(6,10) \in S(3,4)$



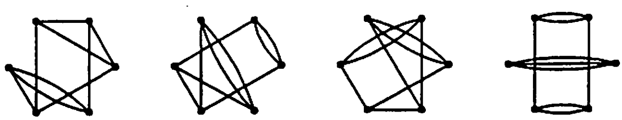
$(6,11) \in S(3,4)$



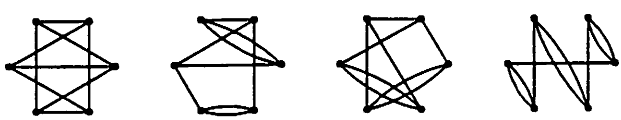
$(7,10) \in S(3,4)$



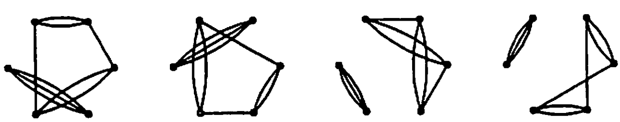
$(7,11) \in S(3,4)$



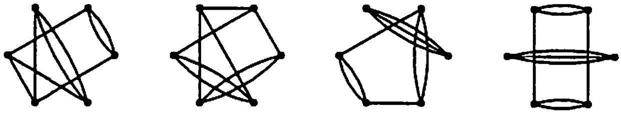
$(7,12) \in S(3,4)$



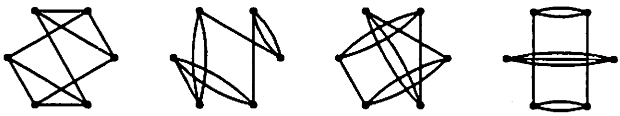
$(8,8) \in S(3,4)$



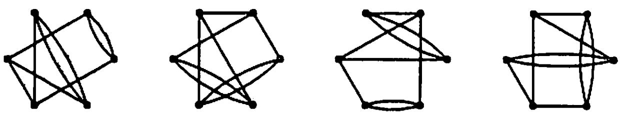
(8,10)∈$S(3,4)$



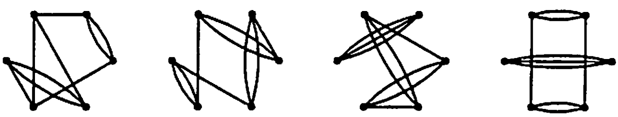
(8,11)∈$S(3,4)$



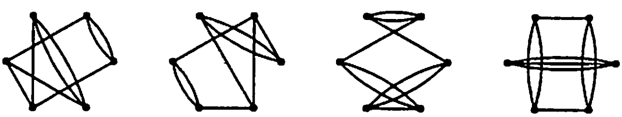
(8,12)∈$S(3,4)$



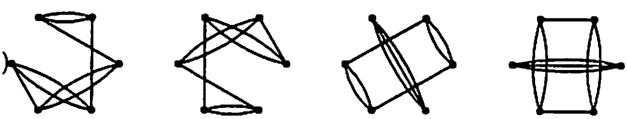
(9,10)∈$S(3,4)$



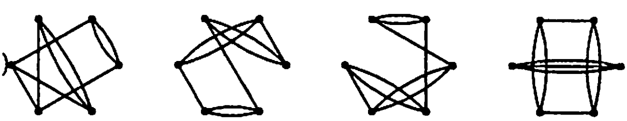
(9,11)∈$S(3,4)$



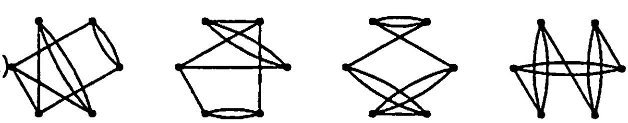
(10,10)∈$S(3,4)$



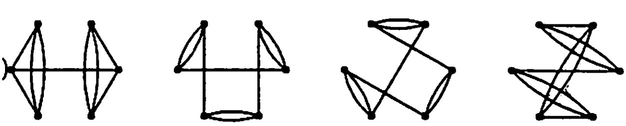
(10,11)∈$S(3,4)$



(10,12)∈$S(3,4)$



(11,12)∈$S(3,4)$

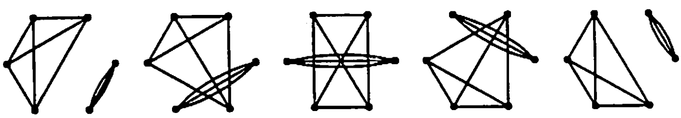


Many of the values in $S(3,5)$ are determined by applying Construction A together with factorizations in $S(3,d)$, $d < 5$. We also apply f-trades to the 3-factorizations shown above; applying one or two (obvious) f-trades yields the following:

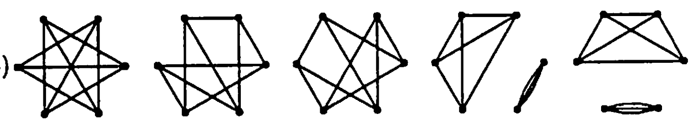
$(t,s) \in S(3,4)$	one f-trade	two f-trades
(1,12)	(4,15)	(1,15)
(2,11)	(5,14)	(2,14)
(2,12)		(2,15)
(4,12)	(5,15)	
(5,11)	(6,14)	(3,14),(4,14)
(7,10)	(8,13)	
(7,11)	(7,14),(10,14)	
(7,12)	(7,15),(8,15)	
(8,10)	(10,13)	
(8,11)	(8,14)	
(9,10)	(9,13),(11,13),(12,13)	
(9,11)	(9,14),(11,14)	
(10,10)	(13,13),(14,14)	
(11,12)	(11,15),(14,15)	

We exhibit the eleven remaining values in $S(3,5)$:

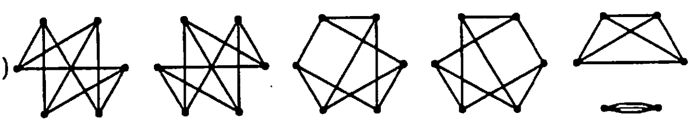
(0,10)∈S(3,5)



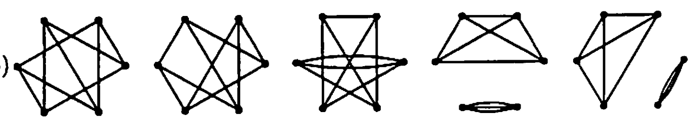
(0,13)∈S(3,5)



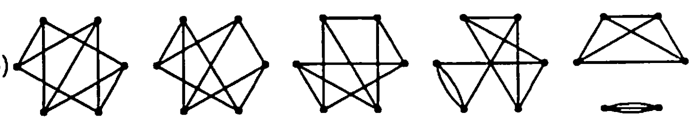
(0,14)∈S(3,5)



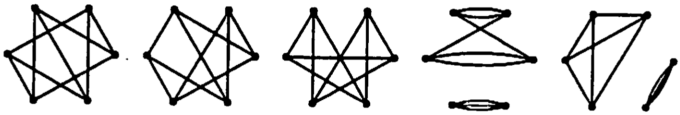
(1,13)∈S(3,5)



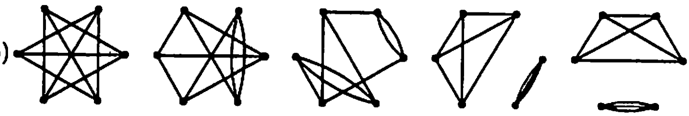
(1,14)∈S(3,5)



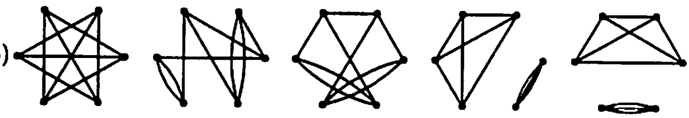
$(2,13) \in \mathcal{S}(3,5)$



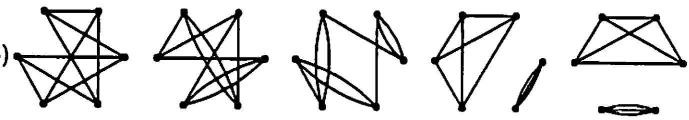
$(3,13) \in \mathcal{S}(3,5)$



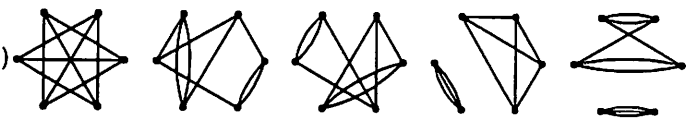
$(4,13) \in \mathcal{S}(3,5)$



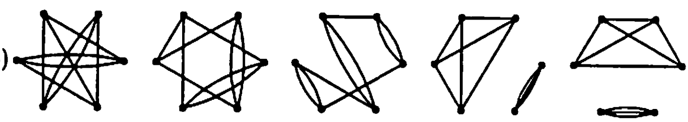
$(5,13) \in \mathcal{S}(3,5)$



$(6,13) \in \mathcal{S}(3,5)$



$(7,13) \in \mathcal{S}(3,5)$



The sequence of constructions given here shows that $\mathcal{A}(3,5) \backslash \{(0,6), (0,11), (3,9), (3,11), (4,4), (4,6), (4,10), (6,7), (7,7), (8,9), (11,11)\} \subseteq \mathcal{S}(3,5)$. In fact, an exhaustive computer search shows that equality holds here. This large number of exceptions for $n = 3$ makes our problem more complicated since they leave "holes" in the recursion which must be handled separately.

5. Small Cases II: $n = 4$.

First we consider $\mathcal{S}(4,3)$. From constructions A,B and C, we obtain

$$\{(0,0),(0,6),(0,12),(4,4),(4,6),(4,10),(4,12),$$

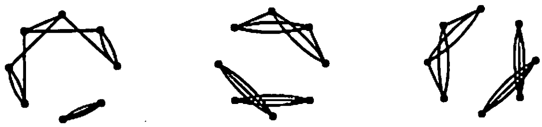
$$(6,6),(6,12),(8,8),(8,10),(8,12),(10,10),(10,12),(12,12)\} \subseteq \mathcal{S}(4,3).$$

In addition, applying cubic expansion to a factorization of $G \in \mathcal{G}_{6,3}$ establishes that

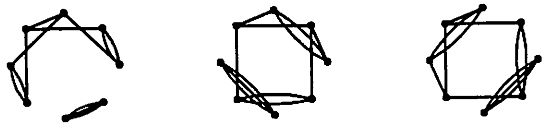
$$\{(8,11),(9,9),(9,10),(9,11),(9,12),(10,11)\} \subseteq \mathcal{S}(4,3).$$

We also employ some direct constructions:

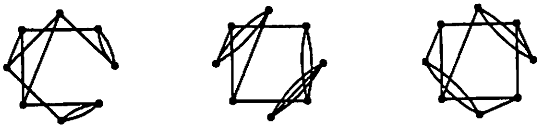
(6,7)∈$S(4,3)$



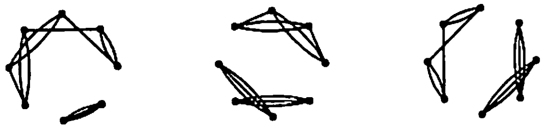
(6,9)∈$S(4,3)$



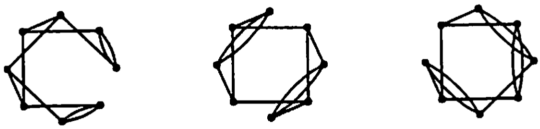
(6,11)∈$S(4,3)$



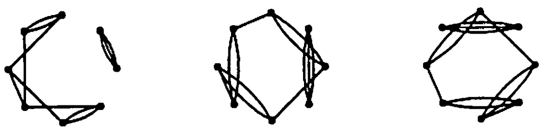
(7,7)∈$S(4,3)$



(7,12)∈$S(4,3)$



(8,9)∈$S(4,3)$

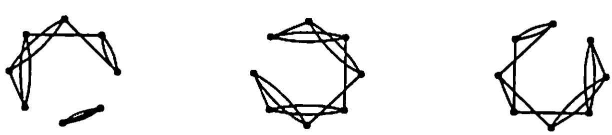


We also use f-trades on the factorizations pictured above as follows:

3-factorization in $S(4,3)$	result after f-trades
(6,7)	(10,13) ∈ $S(4,4)$
	(4,17) ∈ $S(4,5)$
	(6,17) ∈ $S(4,5)$
	(12,13) ∈ $S(4,4)$
	(5,14) ∈ $S(4,4)$
	(6,14) ∈ $S(4,4)$
	(7,14) ∈ $S(4,4)$
	(8,14) ∈ $S(4,4)$
	(5,18) ∈ $S(4,5)$
(6,9)	(9,14) ∈ $S(4,4)$
	(4,15) ∈ $S(4,4)$
	(5,15) ∈ $S(4,4)$

	$(6,15) \in S(4,4)$
	$(4,19) \in S(4,5)$
	$(4,23) \in S(4,6)$
	$(5,19) \in S(4,5)$
	$(5,23) \in S(4,6)$
(6,11)	$(8,15) \in S(4,4)$
	$(10,15) \in S(4,4)$
	$(11,16) \in S(4,4)$
	$(3,16) \in S(4,4)$
	$(5,16) \in S(4,4)$
	$(6,16) \in S(4,4)$
	$(2,15) \in S(4,4)$
	$(1,16) \in S(4,4)$
	$(1,20) \in S(4,5)$
	$(5,20) \in S(4,5)$
	$(1,24) \in S(4,6)$
	$(5,24) \in S(4,6)$
	$(3,20) \in S(4,5)$
	$(3,24) \in S(4,6)$
	$(2,19) \in S(4,5)$
	$(2,23) \in S(4,6)$
	$(2,27) \in S(4,7)$
(7,7)	$(11,13) \in S(4,4)$
	$(13,13) \in S(4,4)$
	$(10,14) \in S(4,4)$
	$(7,15) \in S(4,4)$
	$(9,15) \in S(4,4)$
	$(11,15) \in S(4,4)$
	$(7,16) \in S(4,4)$
(7,12)	$(2,16) \in S(4,4)$
	$(10,16) \in S(4,4)$
	$(9,16) \in S(4,4)$
	$(2,20) \in S(4,5)$
	$(2,24) \in S(4,6)$

Now $(4,14) \in S(4,4)$ by f-trades from the solution $(0,6) \in S(4,3)$. Consider $(10,11) \in S(4,3)$, as depicted here:

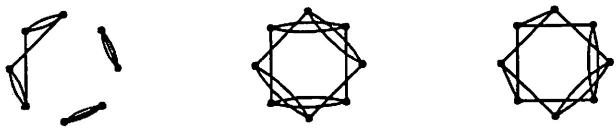


F-trades from this factorization show

$$(3,15) \in S(4,4)$$
$$(14,15) \in S(4,4)$$
$$(13,15) \in S(4,4)$$

$$(12,15) \in S(4,4)$$
$$(15,16) \in S(4,4)$$
$$(3,19) \in S(4,5)$$
$$(7,19) \in S(4,5)$$
$$(3,23) \in S(4,6)$$
$$(7,23) \in S(4,6)$$
$$(3,27) \in S(4,7)$$
$$(7,27) \in S(4,7)$$

Consider $(8,10) \in S(4,3)$ as depicted here.

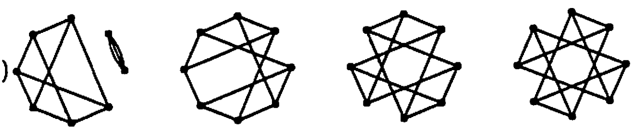


F-trades from this factorization show

$$(2,14) \in S(4,4)$$
$$(2,18) \in S(4,5)$$
$$(2,22) \in S(4,6)$$
$$(2,26) \in S(4,7)$$
$$(12,14) \in S(4,4)$$

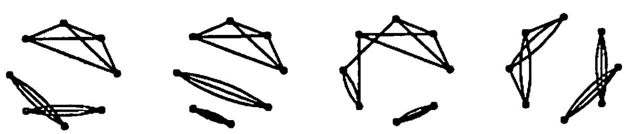
Similar f-trades to show $\{(11,14), (13,14), (12,16), (14,14), (14,16), (15,16)\} \subset S(4,4)$ are easy to find. We require one further "ad hoc" example:

$(0,15) \in S(4,4)$

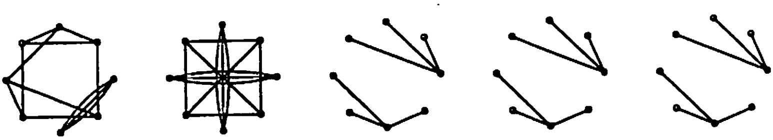


All of the solutions in $S(4,4)$ presented thus far are in fact constructions in $S(K_{4,4})$. Hence we can combine all of them with available factorizations in $S(K_4 \cup K_4)$. This is our main strategy; we supplement it only with certain small examples, given next.

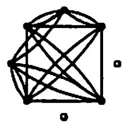
$(3,9) \in S(4,4)$



$\{(0,17), (1,17), (2,17), (3,17)\} \subset S(4,5)$: Taking

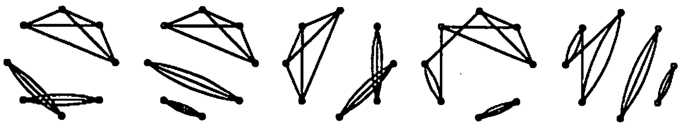


as partial 3-factors leaves



to be 2-factorized, which can be done with 0,1,2, or 3 repeated edges.

$(3,11) \in \mathcal{S}(4,5)$



Combining all of the small examples here, and the observations made, we do not determine $\mathcal{S}(4,7)$ completely. However, we have established that $\mathcal{A}(4,7) \backslash \mathcal{L} \subseteq \mathcal{S}(4,7)$ where $\mathcal{L}$ is given in the table here.

t	s
0	9,10,11,13,14,25
1	12,13,14,15,18,19,21,23,25,26,27
2	11,12,13,19,25
3	12,13,14,18,23,25
4	9,11,13,17,25
5	11,12,13,17,25
6	8,10
7	8,9,10,11,25
8	13
9	13
11	12

While many of these could be settled affirmatively, we suspect that $\mathcal{S}(4,7) \neq \mathcal{A}(4,7)$, and hence have determined only those cases most useful in our recursion.

6. Small Cases III. $n = 5$.

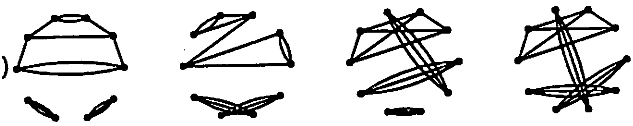
Since there are over five hundred cases for $\mathcal{S}(5,9)$, we describe only the basic methods used. Primarily, constructions D and E are applied to the 3-factorizations of K_6. F-trades are applied in a few cases as follows:

Input factorization	After f-trade
$(0,22) \in \mathcal{S}(5,5)$	$(0,27) \in \mathcal{S}(5,6)$, $(0,32) \in \mathcal{S}(5,7)$
$(0,23) \in \mathcal{S}(5,5)$	$(0,28) \in \mathcal{S}(5,6)$
$(0,24) \in \mathcal{S}(5,5)$	$(0,29) \in \mathcal{S}(5,6)$
$(1,21) \in \mathcal{S}(5,5)$	$(1,26) \in \mathcal{S}(5,6)$, $(1,31) \in \mathcal{S}(5,7)$
$(1,22) \in \mathcal{S}(5,5)$	$(1,27) \in \mathcal{S}(5,6)$, $(1,32) \in \mathcal{S}(5,7)$
$(1,23) \in \mathcal{S}(5,5)$	$(1,28) \in \mathcal{S}(5,6)$
$(1,24) \in \mathcal{S}(5,5)$	$(1,29) \in \mathcal{S}(5,6)$
$(2,21) \in \mathcal{S}(5,5)$	$(2,26) \in \mathcal{S}(5,6)$, $(2,31) \in \mathcal{S}(5,7)$
$(2,22) \in \mathcal{S}(5,5)$	$(2,27) \in \mathcal{S}(5,6)$

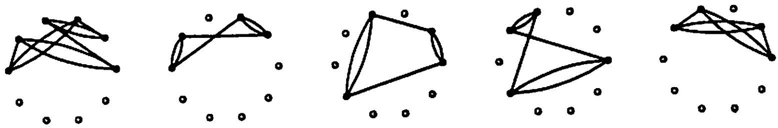
$(2,23) \in S(5,5)$ $(2,28) \in S(5,6)$
$(2,24) \in S(5,5)$ $(2,29) \in S(5,6)$
$(3,22) \in S(5,5)$ $(3,27) \in S(5,6)$, $(3,32) \in S(5,7)$

We also construct $(3,9) \in S(5,4)$ and $(3,11) \in S(5,5)$ similarly to these cases for $n = 4$. We employ two additional factorizations.

$(4,10) \in S(5,4)$



To show $(11,11) \in S(5,5)$, we depict only the portions; their completion to 3-factors is straightforward:

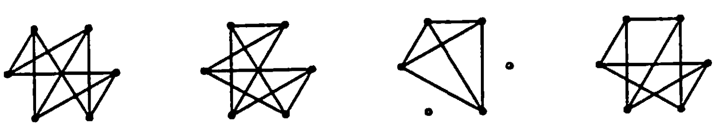


With these examples and those constructed by f-trades, we have $\mathcal{A}(5,9) \setminus \mathcal{M} = S(5,9)$, where $\mathcal{M}$ is given in the table below.

t	s
0	11,12,13,14,17
1	12-17
2	11-16
3	12-17
4	11,12,16
5	11-16
6	11,13
7	10-13,16
8	11,12
9	11,12,13
10	11

7. Intermediate cases: $6 \le n \le 11$.

For $n = 6$, constructions B and C settle the bulk of the cases. We explicitly construct $(3,9) \in S(6,4)$ and $(3,11) \in S(6,5)$ by modifying the 3-factorizations for $n = 5$ given earlier. For $(0,11) \in S(6,4)$, we form a factorization by completing the following portions:



For the remaining values, we observe that addition of two factorizations of K_6 produces a factorization of $K_6 \cup K_6$. Since $(4,4) \in S(K_{6,6})$, we obtain the following:

$(t,s) \in S(K_6 \cup K_6)$	$(t',s') \in S(6,11)$
(0,13)	(4,17)
(1,13)	(5,17)
(1,14)	(5,18)
(1,15)	(5,19)
(3,12)	(7,16)

A similar construction is to form a 1-factorization $\{F_1,\ldots,F_6\}$ of $K_{6,6}$ so that some factor F of the 3-factorization of $K_6 \cup K_6$ has two 3-times repeated edges $\{e,f\}$ so that $F_1 \cup F_2 \cup \{e,f\}$ contains a K_4. Then triplicating each factor, and replacing the factorization of the resulting $3K_4$ by a K_4 in each of F_1, F_2 and F accomplishes the following

$(t,s) \in S(K_6 \cup K_6)$	$(t',s') \in S(6,11)$
(0,10)	(0,16)
(1,12)	(1,18)
(1,13)	(1,19)
(1,14)	(1,20)
(2,11)	(2,17)
(2,12)	(2,18)
(2,13)	(2,19)
(3,12)	(3,18)
(3,13)	(3,19)
(3,14)	(3,20)

The same strategy applies when e is chosen to be a 2-times repeated edge, giving

$(t,s) \in S(K_6 \cup K_6)$	$(t',s') \in S(6,11)$
(1,12)	(0,17)
(2,12)	(1,17)
(4,12)	(3,17)

These cases, together with constructions A-E,G establish that $A(6,11) = S(6,11)$.

Now we turn to $n = 7$. Although $S(4,5)$ is far from being determined, addition of values in $S(4,5)$ with those in $S(3,5)$, together with the example $(0,11) \in S(7,4)$, establishes that $A(7,5)\backslash\{(1,22),(1,23),(2,22)\} \subseteq S(7,5)$. All of these values are realized in $S(K_6 \cup (K_8-f-f'))$ where f,f' are 1-factors of K_8. It is an easy matter to 1-factorize the complement in such a way that two of the 1-factors contain a C_4 in their union. Appropriate placement of the C_4 so that it meets a two 3-times repeated edges in a 3-factor of $3(K_6 \cup (K_8-f-f'))$, or in one two-times and one three-times repeated edge, yields the following:

$(t,s) \in S(K_6 \cup (K_8-f-f'))$	$(t',s') \in S(7,13)$
(2,17)	(1,22)
(1,17)	(1,23)
(3,17)	(2,22)

Applying constructions A-E,G together with these gives $A(7,13) = S(7,13)$.

For $8 \leq n \leq 11$, addition alone suffices to show that $A(n,5) = S(n,5)$, with one exception which is easily constructed: $(0,11) \in S(8,4)$. The basic construction then establishes that $A(n,2n-1) = S(n,2n-1)$ for $8 \leq n \leq 11$.

8. Conclusion.

In sections 4-7, we established that $A(n,2n-1) = S(n,2n-1)$ for $6 \leq n \leq 11$. Hence the Main theorem follows directly from Lemma 3.1. Remarkably, our results also prove the strong statement that $A(n,5) = S(n,5)$ for $n \geq 8$; the Main Theorem for $n \geq 8$ follows easily from this.

A simpler problem than we have solved is to determine the *support size* of the factorization, which is the sum over all factors of the number of distinct edges in the factor. Naturally the support size of a 3-factorization of K_{2n} is at least $n(2n-1)$ and at most $3n(2n-1)$. A Corollary of our Main Theorem is a complete determination of support sizes:

Theorem 2. Every s satisfying $n(2n-1) \leq s \leq 3n(2n-1)$ is the support size of a 3-factorization of K_{2n} with the exception for $n > 3$ of $s = n(2n-1) + t$ for $t \in \{1,2,3,5\}$, and the exception of $s = n(2n-1) + t$ for $t \in \{1,2,3,4,5,7\}$ for $n = 3$. □

The complete determination of $S(n,2n-1)$ for $n \geq 6$ is useful primarily as a stepping stone in the classification of triple systems with index 3 according to numbers of doubly and triply repeated blocks; we explore this problem in a forthcoming paper with Mathon, Rosa and Shalaby.

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