AN EXAMINATION OF THE NON-ISOMORPHIC SOLUTIONS TO A PROBLEM IN COVERING DESIGNS ON FIFTEEN POINTS

J. L. Allston, R. W. Buskens, and R. G. Stanton
Department of Computer Science
University of Manitoba
Winnipeg, Canada R3T 2N2

For Joseph Mark Gani on his sixty-fifth birthday

1. Introduction.

Determination of the number of non-isomorphic Balanced Incomplete Block Designs with a given parameter set (v,b,r,k,λ) is a problem of considerable importance and even greater difficulty. Probably the case that has been most studied is that of a Steiner Triple System (15, 35, 7, 3, 1); cf. Fisher [4], Mathon and Rosa [7]. It is well known that there are eighty possible solutions for this set of design parameters, and the different designs are a fruitful source of examples and constructions. Since [4], there have been various papers devoted to similar problems in Balanced Incomplete Block Designs on a small number of varieties; cf., for example, [2], [3], [6], [14].

Stanton, Kalbfleisch, and Mullin [13] discussed the more general concept of a covering design; in such a design, every variety pair occurs at least once and we normally impose a minimality condition by demanding that the cardinality of the design be as small as possible. It is clear that, in a covering design, we may have to permit the repetition of a small number of pairs in order to ensure that all pairs do appear. The analogue of the BIBD identity $bk = rv$ is the inequality

$$k\, N(t,k,v) \geq v\, N(t-1, k-1, v-1),$$

where $N(t,k,v)$ is the minimum cardinality of a family of k-sets that cover every t-set from a given set of v elements ($t \leq k \leq v$). In this paper, we shall only be concerned with pair coverings by sets of size 4, that is, the design will contain $N(2,4,v)$ blocks. It is well known (see, for example, Mills [9], [10]) that, provided v is not contained in the set {7, 9, 10, 19}, then

$$N(2,4,v) = \lceil v \lceil (v-1)/3 \rceil /4 \rceil.$$

For various general results on covering designs, we refer to [13].

Relatively little has been done in considering the number of non-isomorphic solutions for covering designs. The case of quadruples on 9 symbols (one of the four exceptional cases) was discussed in **[12]**; see also **[1]** for a small correction. Other results for small values are given in **[10]** and **[11]**. In this paper, we wish to consider the analogue of the case discussed in Fisher's 1940 paper; we have a variety set of 15 elements, but we wish to determine the number of designs that cover all pairs (minimally) by quadruples. Since N(2,4,15) = 19, we shall have a total of 19 quadruples in the design.

In a certain sense, 15 seems to be about the break point for manageable designs. If we look at the case of triples and take v less than 15, then the discussion is relatively simple. For example, if v is equal to 4, then N =3, and the unique design may be taken as 123, 124, 134, with 3 repeated pairs. If v = 5, then N = 4, and the unique design may be taken as 123, 145, 245, 345, with 2 repeated pairs. If v = 6, then N = 6, and the unique design may be taken as 123, 145, 126, 245, 346, 356, with 3 repeated pairs. It is well known that there are unique solutions for the case v = 7 (the Fano geometry generated cyclically from the block 124) and for v = 9 (the affine geometry found by deleting points 0, 1, 3, 9, from the projective geometry on 13 points generated cyclically from the block 0139). There are two solutions for v = 13 (both are given in Marshall Hall's book or in [7]). We have already noted that there are 80 solutions for v equal to 15. For v greater than 15, the number of solutions climbs astronomically; the number for v = 19 is not known, but Stinson and Seah **[16]** have shown that the number of triple sytems S(2,3,19) that satisfy the additional very powerful constraint that they contain both a subsystem S(2,3,7) and a subsystem S(2,3,9) is 13,529 (the number containing a subsystem S(2,3,9) is 244,457). The total number of systems on 19 points is well into the millions (cf. [17], where 2,395,687 are found, and where it is estimated that the total number is of the order of 10^9).

2. Preliminary Results.

Since N(1,3,14) = 5, it is clear that any symbol i in a covering design on 15 elements has frequency at least 5. Since 19 quadruples only contain 76 elements, we immediately have

Lemma 1. The covering design with v = 15, k = 4, b = 19, has 14 elements of frequency 5 and a single element of frequency 6.

One might be tempted to assign valence 6 to node E and to concentrate on E, together with its four neighbours A, B, C, D, in the excess graph (all the repeated pairs form the edges of the excess graph). However, this leads to a rather cumbersome array of subcases, namely,

(1) EABC, EAD, EBD, EC, E, E;

(2) EABC, EA, EB, EC, ED, ED;

(3) EABC, EAD, EB, EC, ED, E;

(4)EAB, EAC, EBD, ECD, E, E;

(5) EAB, EAC, EBD, EC, ED, E;

(6) EAB, EBC, EA, EC, ED, ED;

(7) EAB, ECD, EA, EB, EC, ED;

(8) EAB, EBC, EAC, ED, ED, E.

Here we have written the blocks omitting all symbols other than A, B, C, D, E. We shall adopt an alternative approach that makes use of the repeated pairs that occur in the excess graph.

If we look at the repeated pairs, we see that every element of frequency 5 occurs in only one repeated pair; the element of frequency 6 must occur in four repeated pairs. So the number of repeated pairs is (14(1) + 1(4))/2 = 9; this is, of course, the same as 19(6) - 105. If we consider the excess graph as comprising the repeated pairs, we see that it is made up of four disjoint edges and one star with four rays.

We thus see that there are 18 pairs that occur in repetitions (two occurrences for each of nine pairs). We call these "repeating pairs". Since there is a total of 19 blocks in the design, we have

Lemma 2. There is at least one block in the design that is free of repeating pairs.

Suppose that we designate the block in Lemma 2 as 1234; then none of the pairs 12, 13, 14, 23, 24, 34, is repeated. We wish to show that we may take f(i) = 5 for all of i = 1, 2, 3, 4.

There are two possibilities; either f(i) is equal to 5 for all of i = 1, 2, 3, 4, or f(4) = 6. Suppose, if possible, that this second situation does arise. Then there are four other blocks containing 1, four other blocks containing 2, four other blocks containing 3, and five other blocks containing 4, as well as one block disjoint from 1234. Now there must be repeats 1a, 1a, 2b, 2b, 3c, 3c, 4d, 4d, 4e, 4e, 4f, 4f, 4g, 4g. This accounts for 14 of the 18 possible repeats, and so there are only four other repeating pairs. But there are seven other blocks that contain none of the 14 repeating pairs, namely, 1xxx (twice), 2xxx (twice), 3xxx (twice), and xxxx; hence there is one of these blocks that contains no repeating pair and it also does not contain the element 4 of frequency six. Thus we have shown that there is a block in the design that contains no repeating pair and does not contain the element of frequency 6; we now merely rename this block as 1234, and thus have

Lemma 3. We may take the block 1234 to have all elements of frequency 5 such that no pair of elements in the block is repeated.

Now there are two cases. We designate the points of the design as 1, 2, ... , 8, 9, a, b, c, d, e, f. We may take the first 17 blocks of the design as: 1234; four blocks of each of the forms 1xxx, 2xxx, 3xxx, 4xxx.

Case (1). The two blocks disjoint from 1234 are 5678, 59ab.

Case (2). The two blocks disjoint to 1234 are 5678, 9abc, and the element of frequency 6 appears in these blocks.

Case (3). The two blocks disjoint to 1234 are 5678, 9abc, and the element of frequency 6 does not appear in these blocks.

Note that, in Case (1), the element 5 must occur with 1, 2, 3, and 4. Hence f(5) = 6 (this incidentally shows that the two disjoint blocks can not have more than one element in common, since every common element would have to possess frequency 6).

3. Dissection of Case 1.

In Case 1, we have blocks 1234, 15xx, 25xx, 35xx, 45xx, 5678, 59ab; three blocks of each of the forms 1xxx, 2xxx, 3xxx, 4xxx.

Suppose that we now delete the elements 1, 2, 3, 4, 5. We are left with four pairs and fourteen triples; these blocks contain a total of 4 + 14(3) = 46 pairs. Since we have deleted all the repeating pairs containing 1, 2, 3, 4, 5, we see that this derived design contains only a single repeating pair (46 - 45 = 1).

Now the packing number D(2,3,10) is the maximal number of triples on ten elements such that there is no repeated pair. It is given by (see **[15]**, for example)

$$D(2,3,10) = \lfloor 10\ D(1,2,9)/3 \rfloor = \lfloor 40/3 \rfloor = 13.$$

Hence the fourteen triples in the derived design of 14 triples and 4 pairs just obtained must contain both instances of the repeated pair (otherwise, we would have 14 triples on 10 elements, with no pair repeated, and this is not possible). We now delete one of the triples that contains the repeating pair, and are left with 13 triples that form a maximal (2,3,10) packing design.

Consequently, as a preliminary result, we shall need to construct the packing designs consisting of triples on ten elements. For future convenience, we shall designate these ten elements as 6, 7, 8, 9, a, b, c, d, e, f.

4. The Packing Designs on Ten Elements.

We first note that, since the packing design contains 13 triples and since D(1,2,9) = 4, there must be nine elements of frequency 4 and one element of frequency 3 (we designate this singular element as element 9). Each element occurs in one missing pair, except for the element 9 which occurs in three missing pairs; this gives a total of 6 missing pairs. We assign 96, 97, and 98 to be the pairs that do not occur with 9. Then there are two cases.

Design A: The elements 6, 7, 8, occur together in a block.

The design is then 9xx (thrice), 678, 6xx (thrice), 7xx (thrice), 8xx (thrice), and the missing pairs are 96, 97, 98, and F_1, where F_1 is a 1-factor on the elements a, b, c, d, e, f. We observe at once that each of 9, 6, 7, 8, must occur with a 1-factor on a, b, c, d, e, f. Since K_6 has a unique 1-factorization, we can write down Design A as follows.

9ab, 9cf, 9de; 6ac, 6bd, 6ef; 7ad, 7bf, 7ce; 8ae, 8bc, 8df; 678.

The defect graph, which consists of the missing pairs, contains 96, 97, 98, af, be, cd.

Design B: The pairs 67, 68, 78, all occur in distinct blocks. Hence there are three blocks 9xx; 2 blocks of each of the forms 6xx, 7xx, 8xx; three blocks 67x, 68x, 78x; one block abc. We first fill in the letters a, b, c, to give the design in the following form.

9ax, 9bx, 9cx; 67a, 68b, 78c; 6cx, 6xx; 7bx, 7xx; 8ax, 8xx; abc.

We next fill in the pairs de, ef, df, in the three available places and can then complete the design as

9ad, 9bf, 9ce; 67a, 68b, 78c; 6cf, 6de; 7bd, 7ef; 8ae, 8df; abc.

Again the missing pairs are 96, 97, 98, af, be, cd.

These two designs, A and B, will be used in the next section.

5. The (2,4,15) Coverings that are Extensions.

We now take the two (2,3,10) packing designs of the last section and see whether or not they can be extended back to give (2,4,15) covering designs. We first look at Design A, which is made up of the 13 triples

9ab, 9cf, 9de; 6ac, 6bd, 6ef; 7ad, 7bf, 7ce; 8ae, 8bc, 8df; 678.

The missing pairs are 96, 97, 98, af, be, cd. We first need to construct the derived design consisting of 14 triples and 4 pairs; with no loss of generality, it can be taken as the 13 triples just listed, the extra triple 967, and the pairs 98, af, be, cd.

With no loss of generality, we may assign the quadruples as 1234, 1598, 25af, 35be, 45cd. Now we note that the symbol 2 must occur in 3 further blocks and it still must occur with the four numbers 6, 7, 8, 9. Hence 2 must occur in a block that contains 2 or more of these symbols. This shows that 2 must either appear in the block 678x or in the block 967x. The same argument holds for the symbols 3 and 4. Since there are only two triples, 967 and 678, to accommodate extension by the three symbols 2, 3, 4, we have established

Theorem 1. The (2,3,10) packing design A can not be extended to a (2,4,15) covering design.

The situation with Design B is different. Here the triples are

9ad, 9bf, 9ce; 67a, 68b, 78c; 6cf, 6de; 7bd, 7ef; 8ae, 8df; abc.

with omitted pairs 96, 97, 98, af, be, cd.

Just as before, we can form a "fourteenth triple" 967 and then assign quadruples 1598, 25af, 35be, 45cd, 1234. We now try to extend the other triples to give a (2,4,15) covering design. We shall proceed by successively trying symbols 1, 2, 3, 4, 5, with the triple 967.

First we try the block 9671; then 1 must appear in two more blocks with symbols a, b, c, d, e, f, and this is clearly impossible. So we have

Lemma 4. An extension of Design B containing the block 9671 is not possible.

Next we try the block 9672. Then 2 must occur in two more blocks with the five symbols 8, b, c, d, e (there is a sixth symbol which produces the repeated pair containing 2). We successively try the four triples containing 8, namely, 68b, 78c, 8ae, and 8df. Since the remaining symbols in each case (cde, bde, bcd, and bce, respectively) are not triples, we can not have an extension of this type. We have thus established

Lemma 5. An extension of Design B that includes the block 9672 is not possible.

Next we try the block 9673. Then 3 must occur in two more blocks that contain the five symbols 8, a, c, d, f. These can indeed occur with 3 if we take the blocks 38df and 3abc (3b is then the repeating pair containing 3).

Now we must place 2 in three blocks that contain 6, 7, 8, 9, b, c, d, e. Since 2 must occur with 68b, with 78c, or with 8ae, it is easy to find that there are only two possibilities, namely,

278c, 29bf, 26de,

or 268b, 29ce, 27bd.

The second extension is impossible because it would have 2b as a repeated pair (but 3b is the repeated pair containing b). Thus we take the first possibility; it has the repeated pair 2f.

Now we must place 4 in three blocks with 6, 7, 8, 9, a, b, e, f. We simply try 468b and 48ae, and find that the only possibility is 468b, 49ad, 47ef (4d is a repeated pair).

We next must place 1 in three blocks with 6, 7, a, b, c, d, e, f. Our only possibility is to have the blocks 16cf, 17bd, 18ae; this leaves the two final blocks as 59ce and 567a. We have thus proved

Theorem 2. Design B can be extended to a (2,4,15) covering design with the following blocks (we call this design B_3).

1234	9ad4	78c2	8ae1
1598	9bf2	6cf1	8df3
25af	9ce5	6de2	abc3
35be	67a5	7bd1	9673
45cd	68b4	7ef4	

Next we consider assignment of the block 9674; then 4 must appear in two more blocks with 8, a, b, e, f. So these blocks have to be 48ae and 49bf. We then find that 2, 3, and 1 can be successively placed uniquely, just as in the last case, and we end up with

Theorem 3. Design B can be extended to a (2,4,15) covering that contains the following blocks (call this design by the name B_4).

1234	9ad3	78c3	8ae4
1598	9bf4	6cf3	8df5
25af	9ce2	6de1	abc1
35be	67a5	7bd2	9674
45cd	68b2	7ef1	

Our final trial must be with the block 9675. We first try to place 2 in three blocks with 6, 7, 8, 9, b, c, d, e, and find that there are two possibilities, namely,

268b, 29ce, 27bd,

and 278c, 29bf, 26de.

Both of these cases can be completed by successive placement of 3, 4, 1, and we end up with

Theorem 4. There are two extensions of Design B that contain the block 967d. Both designs contain the common blocks

1234, 1598, 25af, 35be, 45cd, 9675.

Design B_{5a} contains the additional blocks

9ad3	68b2	7bd2	abc1
9bf4	78c3	7ef1	
9ce2	6cf3	8ae4	
67a4	6de1	8df5	

Design B_{5b} contains the additional blocks

9ad4	68b4	7bd1	abc5
9bf2	78c2	7ef4	
9ce3	6cf1	8ae1	
67a3	6de2	8df3	

We now examine the four designs obtained so far and use the "Groups & Graphs" package described by Kocay [5] which works by converting each design to a bipartite graph with 15 "point vertices" and 19 "block vertices". Designs B_3 and B_{5b} are isomorphic, both having an automorphism group of degree 4. Also, Designs B_4 and B_{5a} are isomorphic, with an automorphism group of degree 2. In both cases, the isomorphism is achieved by use of the permutation (12)(67)(be)(8f)(9a). Thus we may state

Theorem 5. There are exactly two (2,4,15) covering designs that are extensions of a (2,3,10) packing design.

6. The Non-extension (2,4,15) Coverings.

We now consider the second case from Section 2. For convenience we alter the notation slightly and write the design on elements 1, 2, 3, 4, 5, 6, 7, 8, a, b, c, d, e, f, g. We take the two blocks disjoint to 1234 as 5678 and abcd; the three additional elements are e, f, g. We designate 5 as the element of frequency 6.

Clearly e, f, and g will have frequencies of 5, since they will be the repeated elements with three of the elements 1, 2, 3, 4; the fourth element (say 4) will be repeated with the element 5).

We now split our discussion into five subcases by looking at the repeated pairs (other than the pair 54) that involve elements from the last two blocks 5678 and abcd..

(a) The repeated pairs are 5a, 5b, 5c, 67, 8d.

(b) The repeated pairs are 56, 5a, 5b, 7c, 8d.

(c) The repeated pairs are 56, 5a, 5b, 78, cd.

(d) The repeated pairs are 56, 57, 5a, bc, 8d.

(e) The repeated pairs are 56, 57, 58, ab, cd.

Lemma 6. Case (a) is not possible.

Proof. We must have element 5 occurring in blocks 15xx, 25xx, 35xx, 45xx (twice), 5678. This gives us only 5 blocks in which we may place the 7 pairs 5a, 5a, 5b, 5b, 5c, 5c, 5d, none of which can occur together. Clearly this can not be done.

Lemma 7. Case (b) is not possible.

Proof. Again we consider the blocks that contain 5, namely, 15xx, 25xx, 35xx, 45xx (twice), 5678. There are only five blocks available to contain the six pairs 5a, 5a, 5b, 5b, 5c, 5d, none of which can occur together. So Case (b) is also impossible.

Now we consider Case (c) in which the repeated pairs are 5a, 5b, 56, 78, cd. Since we need 5a, 5a, 5b, 5b, 5c, 5d, we see that the "triples" that occur with the elements 1, 2, 3, 4, must contain 5cd. We write down these triples as the following array:

axx	axx	axx	axx
bxx	bxx	bxx	bxx
cxx	cxx	cxx	cd5
dxx	dxx	dxx	xxx

Now 6, 7, 8, must occur in these 4 sets and each must occur with a, b, c, d. So the last set can be taken as a6x, b78, cd5, xxx. Since 5 must occur twice with a and b, this shows that the last four blocks are a56, b78, cd5, efg. We now fill in 6, 7, and 8 with c and d to give the array

axx	axx	axx	a56
bxx	bxx	bxx	b78
c8x	c7x	c6x	cd5
d6x	d8x	d7x	efg

This determines the array

a7x	a5x	a8x	a56
b5x	b6x	b5x	b78
c8x	c7x	c6x	cd5
d6x	d8x	d7x	efg

We now must place e, e, f, g, in the first column. We try e and e in all 6 possible positions and find that only two positions are possible; both of these immediately lead to completions and so we have

Theorem 6. In Case (c), there are two solutions. We call these solutions Design C_1 and Design C_2. They are set out as follows.

Design C_1

a7e	a5f	a8g	a56
b5e	b6f	b5g	b78
c8f	c7g	c6e	cd5
d6g	d8e	d7f	efg

Design C_2

a7f	a5g	a8e	a56
b5e	b6g	b5f	b78
c8g	c7e	c6f	cd5
d6e	d8f	d7g	efg

In Case (d), the repeats are 56, 57, 5a, bc, 8d. The discussion is along the same lines as that used in Theorem 6. One first establishes the composition of the fourth set of triples; then one places 5, 6, 7, 8 (two ways); then one completes each case (two ways).

Theorem 7. There are 4 solutions in Case (d), as listed under the headings D_i $(i = 1, 2, 3, 4)$.

Design D_1

a8g	a7f	a5e	a56
b7e	b5g	b6f	bc8
c5f	c6e	c7g	d75
d6g	d8e	d8f	efg

Design D_2

a8g	a7f	a5e	a56
b7e	b5g	b6f	bc8
c5f	c6e	c7g	d75
d6g	d8f	d8e	efg

Design D_3

a8g	a5f	a7e	a56
b5e	b7g	b6f	bc8
c7f	c6e	c5g	d75
d6g	d8f	d8e	efg

Design D_4

a8g	a5f	a7e	a56
b5e	b7g	b6f	bc8
c7f	c6e	c5g	d75
d6g	d8e	d8f	efg

The final case is Case (e) in which the repeated pairs are 56, 57, 58, ab, cd. We first try the array

axx, bxx, cxx, dxx (three times each); aby, cdx, xxx, xxx.

Then element y must appear with three more of {a,b,c,d}, and this is not possible.

In Case (e_1), we put ab and cd in the last two columns of the array, and suppose that 5 occurs with ab . Then we may fill in the other occurrences of 5. If we have cde, then we would get a repeat on 5e and so we must take cd6; this position the pair 56 and so we may fill in the element 6.

Next we fill in symbols 7 and 8; the remaining elements in the fourth column have to be e, f, and g. This allows us to fill in the pairs fg, eg, ef. The array now has the form

a6x	a8x	a7x	ab5
b7x	b6x	b8x	c7f
c58	ceg	cd6	d8g
def	d57	5fg	56e

The choice a6f leads to a contradiction; hence we must have a6g and this determines b6f. Completion of the design is immediate.

Theorem 8. There is one design E_1 as given below.

a6g	a8f	a7e	ab5
b7g	b6f	b8e	c7f
c58	ceg	cd6	d8g
def	d57	5fg	56e

We now pass on to Case (e_2) in which e occurs with ab. We can immediately fill in the occurrences of e in the array. Since cdf would give a repeat on ef, we may take cd6 and fill in 6. This permits us to fill in 7 and 8, and the array takes the form

a6	a8	a7	abe
b7	b6	b8	c7
ce8	c	cd6	d8
d	de7	e	e6

In Case (e_{2a}), we put e65 in the last column; then f and g go in the last column; this forces the block efg. We now place 5 and the array is uniquely completable. In Case (e_{2b}), we put e6g in column 4 and likewise obtain a unique completion. This gives us

Theorem 9. There are two designs possible in Case (e_2).

Design E_{2a}

a6g	a8f	a75	abe
b7g	b6f	b85	c7f
ce8	c5g	cd6	d8g
d5f	de7	efg	e65

Design E_{2b}

a6f	a85	a7g	abe
b7f	b65	b8g	c75
ce8	cfg	cd6	d8f
dg5	de7	ef5	e6g

In Case (e_3), we start with ab6 in column 4, cd7 in column 3 (note that we have already considered the case when either pair occurs with 5 or with e). It is fairly easy to reach a contradiction. Consequently, we have

Theorem 10. We obtain a total of three designs in Case (e).

So far, we have constructed nine designs from Case (2); we now have to consider Case (3).

7.The Third Case.

In the final case, we take e as the element of frequency 6. Then e is repeated with two of the elements from {1,2,3,4}, whereas each of f and g is repeated with only one element from {1,2,3,4}.

Case (3a). Element e repeats with 5 and 6; the other repeats are 7c, 8d, ab. Then we must have blocks containing 7c, 7c, 7d, 7ab, as well as blocks containing 8d, 8d, 8c, 8ab. Thus 78ab is a block, and this gives an unacceptable repeat on 78. Thus we have

Lemma 8. Case (3a) is not possible.

Case (3b). Element e repeats with 5 and 6; the other repeats are 78, ab, cd. Then we look at the four sets of "triples" that appear with 1, 2, 3, 4, and write them as

5xx	5xx	5xx	5xx
6xx	6xx	6xx	6xx
7xx	7xx	7xx	78x
8xx	8xx	8xx	xxx

If 78f is a triple, then f occurs 3 more times in the first 3 columns; this gives a repeat on f5 or f6. A similar argument rules out 78a. So we may take triple 78e; then e must occur twice with 5 and twice with 6, and so we must have a triple exx that contains none of {5,6,7,8}. So the array takes the form

5ex	5ex	5xx	5xx
6xx	6ex	6ex	6xx
7xx	7xx	7xx	78e
8xx	8xx	8xx	exx

There must be a pair from {a,b,c,d} appearing in the last column. If eab is a triple, then a can not appear with all of 5,6,7,8. Thus we may take the triple 5ab, and the last column can be taken as comprising 5ab, 6cf, 78e, edg. Then f has to appear 4 times in the first three columns and this gives a repeat on one of f5, f6, f7, f8. We have thus obtained

Lemma 9. Case (3b) is impossible.

Case (3c). Element e repeats with 5 and with a; the other repeated pairs are 6b, 7c, 8d. But then we need distinct blocks containing 6b, 6b, 6a, 6d, and 6c, and this is not possible. Hence we have

Lemma 10. Case (3c) is not possible.

Case 3(d). Element e repeats with 5 and a. The other repeated pairs are 6b, 78, and cd.

As before, we place elements 5,6,7,8; the same argument as in Case (3b) shows that 78a and 78f are not possible; hence 78b or 78e must be a triple. In either case, we can place the element a in the array as follows.

5xx	5xx	5xx	5ax
6xx	6xx	6ax	6xx
7xx	7ax	7xx	78x
8ax	8xx	8xx	xxx

If 78e is a triple, then we get 5ae and 6ae. But then we can not find three more positions for e.

If 78b is a triple, we get the array in the form

5xx	5xx	5bx	5ax
6bx	6bx	6ax	6xx
7xx	7ax	7xx	78b
8ax	8xx	8xx	xxx

Then the last column must contain 6cd and so we get the array

5cx	5dx	5bx	5ax
6bx	6bx	6ax	6cd
7dx	7ax	7cx	78b
8ax	8cx	8dx	xxx

We now place efg in the fourth column. If we take 5af, we immediately get a contradiction (e must appear in 5 positions in the first 3 columns and appear with elements 5, 5, 6, 7, 8, a, a, b, c, d). Hence we take 5ae and get the array

5cx	5dx	5bx	5ae
6bx	6bx	6ax	6cd
7dx	7ax	7cx	78b
8ax	8cx	8dx	efg

The first 2 columns are equivalent under (78)(cd). So there are two possibilities for placing e; the first is to take 5be and then get

Design F

5cg	5df	5be	5ae
6bg	6bf	6ae	6cd
7de	7ag	7cf	78b
8af	8ce	8dg	efg

The second possibility is to take 5ce, 5df, 5bg, and get

Design G

5ce	5df	5bg	5ae
6be	6bf	6ag	6cd
7dg	7ae	7cf	78b
8af	8cg	8de	efg

Thus we have

Theorem 11. We obtain a total of two designs in Case 3.

7. Isomorphisms of the Non-extension Designs.

We have thus constructed eleven designs that are not obtained as extensions of a (2,3,10) packing design. It remains to investigate how many of these designs are not isomorphic.

As before, we apply the "Groups & Graphs" package [5]; the results are perhaps not what we might have expected. All of the designs have groups of orders 2 and 4, and the following isomorphisms hold.

Design C_1 is isomorphic to Design B_3.

Design C_2 is also isomorphic to Design B_3.

Designs E_1 and G are isomorphic to Design B_4.

Designs D_1, D_4, F, and E_{2b} all have a group of order 2 and are isomorphic to one another.

Designs D_2, D_3, and E_{2a} all have a group of order 4 and are isomorphic to one another.

We can sum up all of these results in

Theorem 12. There are four distinct (2,4,15) covering designs. Two of these (B_3 and B_4) can be obtained by extension from a D(2,3,10) packing design. Two of them (D_1 and D_2) can not be thus obtained.

It is worth noting that the designs D_1 and D_2 are identical except for two blocks.

REFERENCES

[1] J. A. Bate and G. H. J. van Rees, *Some Results on N(4,6,10), N(4,6,11), and Related Coverings*, Congressus Numerantium **48** (1985), 25-45.

[2] E. J. (Morgan) Billington, *Isomorphism Classes of Some Small Block Designs*, Ars Combinatoria **4** (1977), 25-35.

[3] E. J. (Morgan) Billington, *Some Small Quasi-multiple Designs*, Ars Combinatoria **3** (1977), 233-250.

[4] R. A. Fisher, *An Examination of the Different Possible Solutions of a Problem in Incomplete Blocks*, Annals of Eugenics **10** (1940), 52-75.

[5] W. L. Kocay, *Groups & Graphs, a Macintosh Application for Graph Theory*, J. Combinatorial Maths and Combinatorial Computing **3** (1988), 195-206.

[6] P. B. Gibbons, R. A. Mathon, and D. G. Corneil, *Computing Techniques for the Construction and Analysis of Block Designs*, Utilitas Mathematica **11** (1977), 161-192.

[7] R. A. Mathon, K. T. Phelps, and A. Rosa, *Small Steiner Triple Systems and Their Properties*, Ars Combinatoria **15** (1983), 3-110.

[8] W. H. Mills, *On the Covering of Pairs by Quadruples I*, J. Combinbatorial Theory A **13** (1972), 55-78.

[9] W. H. Mills, *On the Covering of Pairs by Quadruples II*, J. Combinbatorial Theory A **14** (1973), 138-166.

[10] R. C. Mullin, E. Nemeth, R. G. Stanton, and W. D. Wallis, *Isomorphism Properties of Some Small Covers*, Utilitas Mathematica **29** (1986), 269-283.

[11] R. G. Stanton, *Isomorphism Classes of Small Covering Designs with Block Size Five*, Annals of Discrete Maths **34**, 441-448.

[12] R.G. Stanton, J. L. Allston, W. D. Wallis, and D. D. Cowan, *The Number of Non-isomorphic Solutions to a Problem in Covering Designs*, Utilitas Mathematica **21A** (1982), 119-136.

[13] R. G. Stanton, J. G. Kalbfleisch, and R. C. Mullin, *Covering and Packing Designs*, Proc. Second Chapel Hill Conf. on Combinatorial Maths and its Applications, Univ. of North Carolina, Chapel Hill (1970), 428-450.

[14] R. G. Stanton, R.C. Mullin, and J. A. Bate, *Isomorphism Classes of a Set of Prime BIBD Parameters*, Ars Combinatoria **2** (1976), 251-264.

[15] R. G. Stanton and M. J. Rogers, *Packings and Coverings by Triples*, Ars Combinatoria **13** (1982), 61-69.

[16] D. R. Stinson and E. Seah, *284457 Steiner Triple Systems of Order 19 Contain a Subsystem of)rder 9*, Mathematics of Computation **46** (1986), 717-729.

[17] D. R. Stinson and H. J. Ferch, *2000000 Steiner Triple Systems of Order 19*, Mathematics of Computation **44** (1985), 533-535.