When is a Stable Graph not Stable or Are There Any Stable Graphs Out There? (A Correction)

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In the last issue of this journal, we published the article "Cohesion Stability under Edge Destruction" [3]. Further work in this area resulted in a reexamination of the example of a stable graph given in that paper and the subsequent discovery that the example was in error. Unfortunately the discovery did not come in time to prevent publication and thus one section of the paper contains an erroneous result. The example of a stable graph and the resultant Theorem 1 are incorrect. In this paper, we explain the error and the current status of the stable graph problem, including a new and correct example.

We proceed informally here and refer the reader to the previous paper for careful definitions and a list of references on cohesion. Recall that the cohesion of a vertex x, denoted $\mu(x)$, is the minimum number of edges whose deletion results in a subgraph for which x is a cutvertex. An isolated vertex is assigned cohesion zero, and a degree one vertex has cohesion the degree of its neighbor minus one. A stable edge is an edge whose removal changes the cohesion of no vertex. The cohesion of a graph G is given by $\mu(G) = \Sigma\, \mu(x)$, where the sum is taken over all vertices in the graph. An s-stable edge is an edge whose removal does not change the cohesion of the graph. A graph is called stable if all its edges are s-stable.

Figure 1 (a) shows the erroneous stable graph from [3]. Notice what happens when the edge uv is deleted. Figures 1 (b) and (c) indicate where the example is in error. In the previous paper, the lower three vertices in either K_6 were believed to have cohesion three as a result of a neighborhood cohesion set with the cutvertex as its center. In reality, each of these vertices has

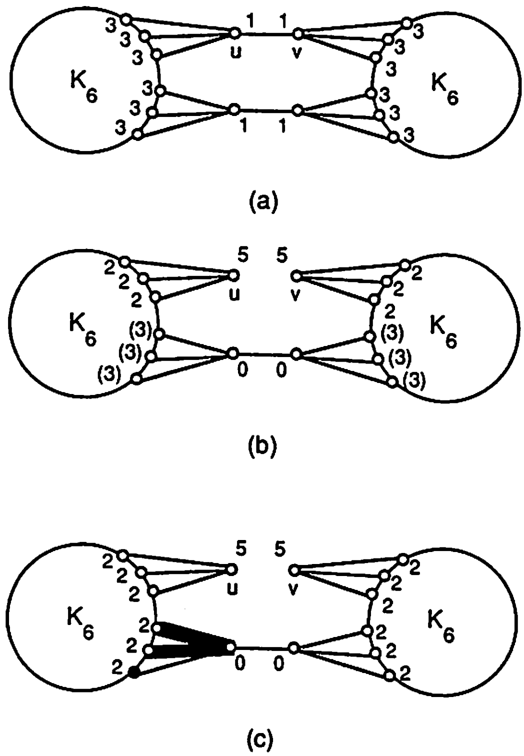


Figure 1. The erroneous stable graph.

cohesion two as illustrated by the darkened vertex whose cohesion set consists of the two darkened edges. Hence, the graph is not stable and the infinite class of stable graphs based upon it is invalid. It appears that no changes in this graph can correct the problem.

Are there any stable graphs? The following theorem from [1], stated here without proof, shows some elementary examples.

Theorem 1: K_2 and P_5 are the only stable trees.

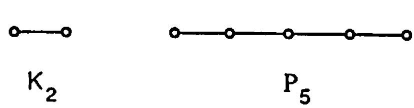


Figure 2. Two stable trees.

Here we use the fact that K_2 has vertex connectivity one and so both vertices are considered cutvertices. But are there any "nontrivial" stable graphs? A result which limits any possible search is as follows [1].

Theorem 2: If a maximal clique M of a graph G with $|M| \geq 3$ has at least two simplicial vertices, then G is not stable.

A simplicial vertex is a vertex of a maximal clique whose neighborhood is contained in the clique. The proof proceeds by removing an edge which forces $\mu(G)$ to decrease. Such is not always the case since there are graphs for which $\mu(G)$ either increases or remains the same when any edge is removed. Such graphs are called superstable and Figure 3 gives an example.

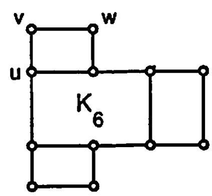
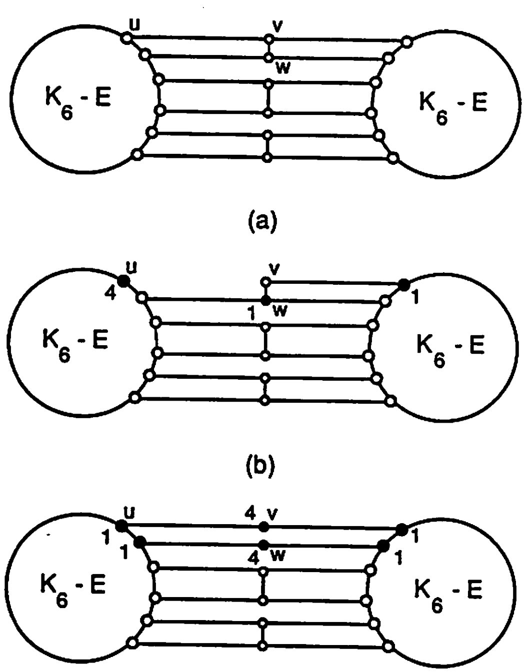


Figure 3. A superstable graph.

The edges of the K_6 are all stable; edge uv increases $\mu(G)$ by one when it is deleted; and edge vw increases $\mu(G)$ by six when it is deleted.

The above superstable graph leads one to our desired correct example of a stable graph (Figure 4). The label K_6- E means a K_6 from which a one-factor E has been removed.

All cohesions are two and $\mu(G) = 36$. Removal of edge uv results in only three vertices changing cohesion as shown in Figure 4 (b); removal of edge vw changes the cohesion of six vertices as shown in Figure 4 (c). Notice that when both K_6- E subgraphs are replaced by any 4 edge-connected, 4 regular graph with an even number of vertices and the corresponding cross edges inserted, then the resultant graph is also stable [1]. Consequently, there are many such stable graphs. Furthermore, other constructions result in different infinite classes of stable graphs [1].



(c)

Figure 4. A stable graph.

So the question of whether there are stable graphs is answered, but what are their properties? Are there methods to systematically create stable graphs? At present we do have methods to produce graphs which have a large proportion of edges which are actually stable. These are to be dealt with in a forthcoming paper [2].

References

1. V. Rice, "Cohesion Properties in Graphs", Ph.D. dissertation, Clemson University, exp. 1988.

2. V. Rice and R.D. Ringeisen, "Graphs with a Large Proportion of Stable Edges", in preparation.

3. R.D. Ringeisen and V. Rice, "Cohesion Stability under Edge Destruction", Journal of Combinatorial Mathematics and Combinatorial Computing, 4 (1988).