CYCLIC (0,1)-FACTORIZATIONS OF THE COMPLETE GRAPH

Dedicated to the memory of Edward J. Green

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Abstract. Hartman and Rosa have shown that the complete graph K_{2n} has a cyclic one-factorization if and only if n is not a power of 2 exceeding 2. Here we consider the following problem: for which $n > 0$ and $0 < k < \frac{n}{2}$ does the complete graph K_n admit a cyclic decomposition into matchings of size k? We give a complete solution to this problem and apply it to obtain a new class of perfect coverings.

1. Introduction.

Let $k > 0$ and $n > 0$ be integers with $\binom{n}{2} \equiv 0 \pmod{k}$ and $k \leq \frac{n}{2}$. It is well-known (see, for example, Folkman and Fulkerson [1, Theorem 4.2]) that the complete graph K_n admits an edge-decomposition into *matchings* of size k; that is, one can partition the edge set of K_n into $\frac{n(n-1)}{2k}$ subsets each consisting of k vertex-disjoint edges.

We are concerned here with the existence of such a decomposition having an additional property, namely, that there be a cyclic permutation σ of the vertices of K_n which preserves the matchings. That is, σ permutes the vertices of K_n in a single n-cycle and also maps matchings onto matchings. We will call a decomposition of this type *cyclic*. The case $k = \frac{n}{2}$ was considered and solved by Hartman and Rosa [2]:

Theorem 1.1. *[Hartman and Rosa] The complete graph K_n admits a cyclic one-factorization if and only if n is even and $n \neq 2^t$, $t \geq 3$.*

We will prove the following result in Section 2.

Theorem 1.2. *Let $n > 0$ and $0 < k < \frac{n}{2}$ be integers with $\binom{n}{2} \equiv 0 \pmod{k}$. Then the complete graph K_n admits a cyclic decomposition into matchings of size k.*

Having established the above result let now $m = \frac{n(n-1)}{2k}$. Each matching, together with the $n-2k$ vertices that it does not cover, constitutes a (0,1)-factor in K_n, and there are m of these factors. Now adjoin m new vertices $\{x_1, \ldots, x_m\}$ to K_n to obtain a new graph (denoted $K_n v \overline{K}_m$) where x_i is adjacent to each vertex of K_n for $i = 1, \ldots, m$, and $\{x_1, \ldots, x_m\}$ is an independent set of vertices. By making use of the original m (0,1)-factors in K_n we can now form, in the obvious manner, a decomposition of the edge set of $K_n v \overline{K}_m$ into edges (K_2)s and triangles (K_3)s in which there are, in all, $\frac{1}{2}n(2m-n+1)$ complete subgraphs. This is a *perfect covering* of $K_n v \overline{K}_m$ in the sense that, when $m \geq n-1$, there does not exist a decomposition of $K_n v \overline{K}_m$ into fewer than $\frac{1}{2}n(2m-n+1)$ complete subgraphs (see [3], [4] and [5]).

Thus we have

Corollary 1.3. *Let $n > 0$ and $m \geq n-1$ be integers with $\binom{n}{2} \equiv 0 \pmod{m}$, and suppose also that if $m = n-1$ then $n \neq 2^t, t \geq 3$. Then there is a perfect covering of $K_n v \overline{K}_m$ having the following properties:*

(i) *each vertex x in $\overline{K}_m$ is contained in $k = \frac{n(n-1)}{2m}$ triangles and $n-2k$ edges in the covering, and*

(ii) *there is an automorphism of the covering which permutes the vertices of K_n in a single n-cycle.*

Proof: Theorem 1.1 and Theorem 1.2. ∎

2. The proof of Theorem 1.2.

We will find the following terminology useful. Let d be a divisor of n. A *d-matching* in $\mathbf{Z}_n$ is a partition P of a subset $S \subseteq \mathbf{Z}_N$ into pairs where

(i) $|S| \leq d-1$ and for each x, y in S, $x - y \not\equiv 0 \pmod{d}$;

(ii) the residues $\{\pm(x-y) : \{x,y\} \in P\}$ are distinct $\pmod{n}$ and, furthermore, if d is even, include no difference $\equiv 0 \pmod{\frac{1}{2}d}$.

A d-matching in which $|S| = d-1$ (d odd) or $|S| = d-2$ (d even) will be called a *d-starter*. Note that when $d = n$ our notion of a d-starter corresponds to that of odd and even starters in $\mathbf{Z}_n$.

We will begin by proving Theorem 1.2 where n is odd.

Lemma 2.1. *Let n be an odd integer, d a divisor of n and $D = \{x \in \mathbf{Z}_n : x \equiv 0 \pmod{d}\}$. There is a set $(S_1, P_1), \ldots, (S_{\frac{n}{d}}, P_{\frac{n}{d}})$ of d-starters in $\mathbf{Z}_n$ with the property that the sets $\{\pm(x-y) : \{x,y\} \in P_i\}$ $(i = 1, \ldots, \frac{n}{d})$ form a partition of $\mathbf{Z}_n - D$.*

Proof: For each $i = 1, \ldots, \frac{n}{d}$ take

$$P_i = \left\{ \left\{ \frac{d-1}{2}, \frac{d+1}{2} + (i-1)d \right\}, \left\{ \frac{d-3}{2}, \frac{d+3}{2} + (i-1)d \right\}, \ldots, \{1, d-1+(i-1)d\} \right\}.$$

∎

Now let $\frac{n(n-1)}{2} = m \cdot k$. Write

$$m = qn + r, \quad 0 < r \leq n. \tag{2.1}$$

Then, using the usual notation (x, y) to denote the g.c.d. of x and y we have

$$\left(\frac{n-1}{2} - qk\right) n = rk = (r, n)\lambda k = (m, n)\lambda k. \tag{2.2}$$

Clearly, the relation $\frac{n(n-1)}{2} = mk$ implies that $k \cdot (m, n) \equiv 0 \pmod{n}$, since n is odd. Thus, from Equation (2.2) we have

$$\frac{\frac{n-1}{2} - qk}{\lambda} \cdot \frac{n}{(m, n)} = k \tag{2.3}$$

where each of the two left-hand side ratios is integral. From Equation (2.2) $\lambda = \frac{r}{(m,n)} \leq \frac{n}{(m,n)}$ so that we can apply Lemma 2.1 with $d = (m, n)$ to construct a set $(S_1, P_1), \ldots, (S_\lambda, P_\lambda)$ of (m, n)-matchings where $|P_i| = \frac{1}{\lambda}\left(\frac{n-1}{2} - qk\right)$ for each $i = 1, \ldots, \lambda$ (since $k < \frac{n}{2}$ Equation (2.3) implies that $\frac{1}{\lambda}\left(\frac{n-1}{2} - qk\right) \leq \frac{(m,n)-1}{2}$).

From Equation (2.3) and the definition of a d-matching we can now construct λ matchings $M_1, \ldots, M_\lambda$, each having k edges, by setting

$$M_i = P_i \cup (P_i + (m, n)) \cup (P_i + 2(m, n)) \cup \ldots \cup \left(P_i + \left(\frac{n}{(m, n)} - 1\right)(m, n)\right)$$

where by $P_i + a$ we mean $\{\{x + a, y + a\} : \{x, y\} \in P_i\}$.

Let now (S, P) be an n-starter in $\mathbf{Z}_n$ (Lemma 2.1 with $d = n$) and remove from P all pairs $\{x, y\}$ with $\pm(x-y) \in \{\pm(s-t) : \{s, t\} \in P_1 \cup P_2 \cup \ldots \cup P_\lambda\}$. There remain $\frac{n-1}{2} - \lambda \cdot \frac{1}{\lambda}\left(\frac{n-1}{2} - qk\right) = qk$ pairs in P, which we now simply partition into q matchings $N_1, \ldots, N_q$ each with k edges.

A (cyclic) decomposition of K_n into matchings of size k is now obtained by developing each of the matchings $M_1, \ldots, M_\lambda, N_1, \ldots, N_q$ modulo n. Note that each matching $M_1, \ldots, M_\lambda$ is contained in an orbit of size (m, n) while each matching $N_1, \ldots, N_q$ is contained in an orbit of size n. We have, thus, proven

Theorem 2.2. *Let $n > 0$ be an odd integer and $0 < k < \frac{n}{2}$ with $\binom{n}{2} \equiv 0 \pmod{k}$. Then the complete graph K_n admits a cyclic decomposition into matchings of size k.*

Remark: In our construction there are $\lambda + q$ matching orbits, where (letting $m = \frac{n(n-1)}{2k}$) $m = q \cdot n + \lambda(m, n)$ and $0 < \lambda \leq \frac{n}{(m,n)}$. In particular, there will be one matching orbit when and only when $m = n$, (that is, $k = \frac{n-1}{2}$).

We turn our attention now to the case where n is even. We begin with the appropriate analogue to Lemma 2.1.

Lemma 2.3. *Let n be an even integer, d a divisor of n with $d \equiv 0 \pmod 2$ and $D = \{x \in \mathbf{Z}_n : x \equiv 0 \pmod{\frac{d}{2}}\}$. Then there is a set $(S_1, P_1), \ldots, (S_{\frac{n}{d}}, P_{\frac{n}{d}})$ of d-starters in $\mathbf{Z}_n$ with the property that the sets $\{\pm(x - y) : \{x, y\} \in P_i\}$ $(i = 1, \ldots, \frac{n}{d})$ form a partition of $\mathbf{Z}_n - D$.*

Proof: The lemma is vacuously true if $d = 2$. If $d = 4$ then for each $i = 1, \ldots, \frac{n}{4}$ take $P_i = \{\{2, 3 + (i-1) \cdot 4\}\}$. For $d \geq 6$ let $s = \frac{\lfloor d \rfloor}{4}$ and $t = \frac{\lfloor 3d \rfloor}{4}$; for each $i = 1, \ldots, \frac{n}{d}$ take $P_i = \{\{1, 2s-1+(i-1)d\}, \{2, 2s-2+(i-1)d\}, \ldots, \{s-1, s+1+(i-1)d\}, \{2s, d-1+(i-1)d\}, \{2s+1, d-2+(i-1)d\}, \ldots, \{t-1, t+(i-1)d\}\}$. ∎

Now let $\frac{n(n-1)}{2} = m \cdot k$ and let $\frac{n(n-1)}{2} - k(m,n) = (m - (m,n))\, k = m'k$. Write

$$m' = q \cdot n + r', \quad 0 < r' \leq n. \tag{2.4}$$

Then we have

$$\left(\frac{n}{2} - qk - \frac{k(m,n) + \frac{n}{2}}{n}\right) \cdot n = r'k = (r', n)\lambda k = (m', n)\lambda k. \tag{2.5}$$

Now $(m', n) = (m - (m,n), n)$, so that (m', n) is a multiple of (m, n). Let 2^k be the highest power of 2 dividing m. Since $\frac{n(n-1)}{2} = mk$, 2^k is the highest power of 2 dividing (m, n) and so 2^{k+1} divides $m - (m,n) = m'$. But 2^{k+1} also divides n, whence 2^{k+1} divides (m', n). Thus, (m', n) is a multiple of $2(m, n)$. Now the relation $\frac{n(n-1)}{2} = mk$ implies $(m,n) \cdot k \equiv 0 \pmod{\frac{n}{2}}$, so that by the foregoing we have $(m', n) \cdot k \equiv 0 \pmod n$. From Equation (2.5) we can, therefore, write

$$\frac{\frac{n}{2} - qk - \frac{k(m,n) + \frac{n}{2}}{n}}{\lambda} \cdot \frac{n}{(m', n)} = k \tag{2.6}$$

where each of the two left-hand side ratios is integral.

From Equation (2.5) $\lambda = \frac{r'}{(m',n)} \leq \frac{n}{(m',n)}$; we consider two cases.

Case 1: $\lambda = \frac{n}{(m',n)}$.

Here Equation (2.5) and Equation (2.4) imply that $(m', n) = n$, that is, $\lambda = 1$. Let (S', P') be an (m, n)-starter on the symbols $\{0, 1, \ldots, (m,n) - 1\}$

(Lemma 2.1 and Lemma 2.3) and choose a subset $P'' \subseteq P'$ of size $\frac{1}{n}(k(m,n) - \frac{n}{2})$ (this can be done since $k < \frac{n}{2}$, whence $\frac{1}{n}(k(m,n) - \frac{n}{2}) < \frac{(m,n)}{2}$). Construct a matching L in $\mathbf{Z}_n$ as follows:

$$L = \left(P'' \cup \{\{0, \frac{n}{2}\}\} \right) \cup \left(P'' \cup \{\{0, \frac{n}{2}\}\}\right) + (m,n)$$
$$\cup \left(P'' \cup \{\{0, \frac{n}{2}\}\}\right) + 2(m,n)$$
$$\cup \ldots \cup \left(P'' \cup \{\{0, \frac{n}{2}\}\}\right) + \left(\frac{n}{(m,n)} - 1\right)(m,n).$$

Recalling that $(m', n) = n$ is a multiple of $2(m,n)$, we have $|L| = \frac{1}{n}(k(m,n) - \frac{n}{2}) \cdot \frac{n}{(m,n)} + \frac{n}{2(m,n)} = k$.

Now let (S, P) be an n-starter in $\mathbf{Z}_n$ (Lemma 2.3) and remove from P all pairs $\{x,y\}$ with $\pm(x-y) \in \{\pm(s-t) : \{s,t\} \in L\}$. There remain $\frac{n}{2} - 1 - \frac{1}{n}(k(m,n) - \frac{n}{2}) = (q+1)k$ pairs in P (Equation (2.6)) which we now partition into $q+1$ matchings $N_1, \ldots, N_{q+1}$, each with k edges. Develop the matchings $L, N_1, \ldots, N_{q+1}$ modulo n.

Case 2: $\lambda < \frac{n}{(m',n)}$.

Begin by constructing a matching L on $\mathbf{Z}_n$ exactly as in Case 1. Again $|L| = k$ since (m',n) is a multiple of $2(m,n)$ (and, hence, so is n). Now apply Lemma 2.3 with $d = (m',n)$ to construct a set $(\overline{S}_2, \overline{P}_2), \ldots, (\overline{S}_{\lambda+1}, \overline{P}_{\lambda+1})$ of (m',n)-matchings where $\overline{P}_i \subseteq P_i$ and $|\overline{P}_i| = \frac{1}{\lambda}\left(\frac{n}{2} - qk - \frac{k(m,n)+\frac{n}{2}}{n}\right)$ for each $i = 2, \ldots, \lambda+1$ (since $k < \frac{n}{2}$ Equation (2.6) implies $\frac{1}{\lambda}\left(\frac{n}{2} - qk - \frac{k(m,n)+\frac{n}{2}}{n}\right) < \frac{(m',n)}{2}$). Construct λ matchings $M_2, \ldots, M_{\lambda+1}$, each having k edges, by setting

$$M_i = \overline{P}_i \cup (\overline{P}_i + (m',n)) \cup (\overline{P}_i + 2(m',n)) \cup \ldots \cup \left(\overline{P}_i + \left(\frac{n}{(m',n)} - 1\right)(m',n)\right).$$

Note that, by the way L was constructed (see Case 1) the set $D = \{\pm(x-y) \pmod{n} : \{x,y\} \in L\}$ is contained in the set $\{\frac{n}{2}\} \cup \{\pm(s-t) : \{s,t\} \in P_1\}$ where (S_1, P_1) is the first (m',n)-starter in $\mathbf{Z}_n$ referred to in Lemma 2.3; in particular, D is disjoint from the set $\{\pm(s-t) : \{s,t\} \in \overline{P}_2 \cup \ldots \cup \overline{P}_{\lambda+1}\}$.

Now let (S, P) be an n-starter in $\mathbf{Z}_n$ (Lemma 2.3) and remove from P all pairs $\{x,y\}$ with $\pm(x-y) \in \{\pm(s-t) : \{s,t\} \in L \cup \overline{P}_2 \cup \overline{P}_3 \cup \ldots \cup \overline{P}_{\lambda+1}\}$. By the foregoing there remain $\frac{n}{2} - 1 - \lambda \cdot \frac{1}{\lambda}\left(\frac{n}{2} - qk - \frac{k(m,n)+\frac{n}{2}}{n}\right) - \frac{1}{n}(k(m,n) - \frac{n}{2}) = qk$ pairs in P, which we can now partition into q matchings $N_1, \ldots, N_q$, each with k edges. Develop the matchings $L, M_2, \ldots, M_{\lambda+1}, N_1, \ldots, N_q$ modulo n.

Note that (in both cases) the matching L is contained in an orbit of size (m,n), while each matching M_i is contained in an orbit of length (m',n) and each matching N_j is contained in an orbit of length n. We now have

Theorem 2.4. *Let $n > 0$ be an even integer and $0 < k < \frac{n}{2}$ with $\binom{n}{2} \equiv 0 \pmod{k}$. Then the complete graph K_n admits a cyclic decomposition into matchings of size k.*

Remark: In the above construction there are $\lambda + q + 1$ matching orbits, where (letting $m = \frac{n(n-1)}{2k}$ and $m' = m - (m,n)$) $m' = q \cdot n + \lambda(m',n)$ and $0 < \lambda \leq \frac{n}{(m',n)}$.

Theorem 1.2 now follows from Theorem 2.2 and Theorem 2.4.

Acknowledgement.

The author wishes to thank J. A. Bondy for his helpful assistance.

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