On (3,k) Ramsey Graphs: Theoretical and Computational Results

STANISŁAW P. RADZISZOWSKI
DONALD L. KREHER

School of Computer Science and Technology
Rochester Institute of Technology
Rochester, NY 14623

ABSTRACT

A $(3,k,n,e)$ Ramsey graph is a triangle-free graph on n vertices with e edges and no independent set of size k. Similarly, a $(3,k)$-, $(3,k,n)$- or $(3,k,n,e)$-graph is a $(3,k,n,e)$ Ramsey graph for some n and e. In the first part of the paper we derive an explicit formula for the minimum number of edges in any $(3,k,n)$-graph for $n \leq 3(k-1)$, i.e. a partial formula for the function $e(3,k,n)$ investigated in [3,5,7]. We prove some general properties of minimum $(3,k,n)$- graphs with $e(3,k,n)$ edges and present a construction of minimum $(3,k+1,3k-1,5k-5)$-graphs for $k \geq 2$ and minimum $(3,k+1,3k,5k)$-graphs for $k \geq 4$. In the second part of the paper we describe a catalogue of small Ramsey graphs: all $(3,k)$-graphs for $k \leq 6$ and some $(3,7)$-graphs including all 191 $(3,7,22)$-graphs, produced by a computer. We present for $k \leq 7$ all minimum $(3,k,n)$-graphs and all 10 maximum $(3,7,22)$-graphs with 66 edges.

1. Motivation

Research leading to the evaluation of the exact values of Ramsey numbers $R(3,k)$ has reached the point where further progress is possible only if a new innovative method is designed. We feel that the exact knowledge of small minimum Ramsey graphs can be helpful in better understanding their general properties. The original plans for this paper included only the construction of a catalogue, but while working on it we were able to derive several general theorems and constructions, which are presented in section 3. We believe that these theorems and the catalogue of small $(3,k)$ Ramsey graphs, together with the techniques introduced by Graver and Yackel [3] and refined by Grinstead and Roberts [5], can lead to the evaluation of the exact value of $R(3,8)$ and improvements of known bounds for $R(3,k)$, $k \geq 10$. The listing of known exact values and bounds for the classical two color Ramsey numbers $R(k,l)$ can be found in [2,8].

Only some of the $(3,k)$-graphs needed to achieve the above goals appear in different publications [3,5,7 and others]. The authors do not know of any previous systematic attempt to enumerate all $(3,k)$-graphs for small k and to put them together into a catalogue. The second part of this paper presents our construction of such a catalogue.

2. Definitions and Tools

The two color Ramsey number $R(k,l)$ is defined as the smallest integer n, such that any graph on n vertices contains a clique of size k or an independent set of size l. We say that a graph G on n vertices has the (k,l)-property, or, for short, G is a (k,l,n)-graph if and only if the existence of G proves that $R(k,l) > n$. A (k,l,n)-graph G with e edges will be called a (k,l,n,e)-graph. Following [3], let $e(k,l,n)$ and $E(k,l,n)$ be the minimum and the maximum number of edges in any (k,l,n)-graph. They are defined to be ∞ or $-\infty$, respectively, if no such graph exists. Knowing the values of $e(k,l,n)$ and $E(k,l,n)$ often makes computer searches for (k,l,n)-graphs feasible, in particular when the difference $E(k,l,n) - e(k,l,n)$ is small.

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Let $G=(V,E)$ be a fixed (k,l,n)-graph and choose some vertex $v \in V$. Define $H_1(v)$ to be the graph induced in G by vertices adjacent to v and $H_2(v)$ to be the graph induced by the set of vertices nonadjacent to v. The *Z-sum* of vertex v in G is $Z(v)= \Sigma\{deg_G(u):u \in H_1(v)\}$. Let $f(v) = \|H_2(v)\| - e(k,l-1,n-deg(v)-1)$, where $\|H_2(v)\|$ denotes the number of edges in $H_2(v)$. In such a situation we say that vertex v is *preferred*. Obviously $f(v) \geq 0$, and if $f(v)=0$, then vertex v is called *full* [5]. If G is a $(3,k,n,e)$-graph then by preferring any vertex v, $H_1(v)$ is an independent set of size $deg(v)$ and $H_2(v)$ is a $(3,k-1,n-deg(v)-1,e-Z(v))$-graph. Any vertex of degree i will be called an i*-vertex*.

If G is a $(3,k)$-graph, then we can evaluate the sum $\sum_{v \in V} f(v)$ using the following obvious variation of proposition 4 in [3].

Lemma 1: *In any $(3,k,n,e)$-graph $G=(V,E)$, if n_i denotes the number of i-vertices in G, then*

$$\sum_{v \in V} f(v) = \Delta(G,k,n,e) = ne - \sum_{i \geq 0}^{k-1} n_i(e(3,k-1,n-i-1)+i^2) \geq 0 \qquad (1)$$

and there are at least $n-\Delta(G,k,n,e)$ full vertices in G.

Finally, let $g_{kl}(n,e)$ be the number of nonisomorphic (k,l,n,e)-graphs, and if $e=e(k,l,n)$ then any such graph is called a *minimum* (k,l,n)-graph. A (k,l,n,e)-graph is called a *maximum* (k,l,n)-graph if $e=E(k,l,n)$. If the parameters k, l, and n are defined by the context or are irrelevant in some particular situation, then a minimum (maximum) (k,l,n)-graph will be called simply a minimum (maximum) graph.

3. Bounds for e(3,k,n) and E(3,k,n)

The goal of this section is to derive explicit partial formulas and bounds for functions $e(3,k,n)$ and $E(3,k,n)$, which will be given in theorems 1, 2 and 4 and lemma 4. We start with two technical lemmas stating properties of minimum $(3,k,n)$-graphs.

Lemma 2: *If G is a minimum $(3,k,n)$-graph, then*

(a) *If a component of G is a cycle, then it is a pentagon.*

(b) *If G has an isolated point, then all vertices in G are of degree less than 2.*

(c) *All 1-vertices in G are endpoints of isolated edges.*

Proof: Suppose that G is a minimum $(3,k,n)$-graph.
(a) Let C_i be a cycle component of G of length i, $i \neq 5$. Thus $i \geq 4$ since G is triangle-free. Let H be the triangle-free graph obtained from G by replacing cycle C_i with $i/2$ isolated edges if i is even, or with a pentagon and $(i-5)/2$ isolated edges if i is odd. One can easily check that H has less edges than G but the same maximal independent set size. Hence, H is a $(3,k,n)$-graph contradicting the fact that G is a minimum graph.

(b) Let v be an isolated vertex and w an r-vertex, $r \geq 2$. Define graph H from G by deleting all the edges adjacent to w and joining vertices v and w by an edge. Again one can check that H is a $(3,k,n)$-graph with fewer edges than G, which contradicts the minimality of G.

(c) Let v be a 1-vertex connected to an r-vertex w, $r \geq 2$. Define graph H from G by deleting all the edges adjacent to w except the edge $\{v,w\}$. The same argument as in (a) and (b) proves the assertion of the lemma. □

Lemma 3: *If G is a minimum $(3,k,n)$-graph and F is the subgraph induced by the 2-vertices in G, then F is a disjoint union of isolated points, isolated edges and pentagons.*

Proof: Graph F has only vertices of degree 2 or less, hence each component of F must be a cycle or a path (possibly of length 0). To prove the lemma it is sufficient to show that F cannot have as a component a path $v_1, v_2, \cdots, v_r$ with $r \geq 3$, since any cycle of F is a component of G and by lemma 2(a) it must be a pentagon. Since v_1 and v_r are 2-vertices in G by lemma 2(c) v_1 and v_r must have neighbors in G of degree at least 3. There are four cases. In case (i) G has a vertex x, $deg(x) \geq 3$, adjacent to both v_1 and v_r. In remaining cases G has vertices x and y, $deg(x)$, $deg(y) \geq 3$, adjacent to v_1 and v_r respectively. In case (ii) r is odd. In case (iii) r is even and x and y are nonadjacent and in case (iv) r is even and x is adjacent to y. In each of the cases we will transform graph G into a $(3,k,n)$-graph H with a smaller number of edges, thus contradicting the minimality of G.

case (i): Transform G into H by deleting all the edges adjacent to x except those connecting x to v_1 and v_r. For each independent set S in H including vertex x we can find another independent set $\bar{S}$ in H, $|S| = |\bar{S}|$ and $x \notin \bar{S}$, by shifting one position vertices of S lying on the cycle $x, v_1, \cdots, v_r$. Thus $\bar{S}$ is an independent set in G. If S does not contain vertex x then S is an independent set in G. Therefore we can deduce that H is a $(3,k,n)$-graph.

case (ii): Transform G into H by deleting all the edges adjacent to x and y except edges $\{x, v_1\}$ and $\{y, v_r\}$, then add the edge $\{x, y\}$. Now $x, v_1, \cdots, v_r, y$ is an isolated cycle C of odd length in H. To show that H is a $(3,k,n)$-graph note that any independent set S in H can be transformed into an independent set $\bar{S}$ in G, $|S| = |\bar{S}|$, by rotating vertices of S on C to the position where $\bar{S}$ does not contain none of x and y.

case (iii): Transform G into H by deleting the edge $\{y, v_r\}$ and all the edges adjacent to x except the edge $\{x, v_1\}$, then add the edge $\{x, v_r\}$. Now the cycle $x, v_1, \cdots, v_r$ has odd length in H, so as in case (iii) we can transform any independent set S in H to one avoiding points x and v_r and conclude that H is a $(3,k,n)$-graph.

case (iv): Transform G into H by deleting the edges $\{x, v_1\}$ and $\{y, v_r\}$ and adding the edge $\{v_1, v_r\}$ yielding an even cycle $v_1, \cdots, v_r$. Each independent set S in H containing x can be transformed to one avoiding v_1, and each S containing y can be transformed to one avoiding v_r. Hence H is a $(3,k,n)$-graph. □

Initial values of $e(3,k+1,n)$ and the structure of minimum $(3,k+1,n)$-graphs with $n \leq 5k/2$ can be obtained from theorem 1 below.

Theorem 1: *For $k \geq 2$*

$$e(3,k+1,n) = \begin{cases} 0 & \text{if } n \leq k, \\ n-k & \text{if } k < n \leq 2k, \\ 3n-5k & \text{if } 2k < n \leq 5k/2. \end{cases} \tag{2}$$

Furthermore, $g_{3,k+1}(n, e(3,k+1,n)) = 1$ for $n \leq 5k/2$, i.e. the minimum graphs are unique and are given by n isolated points for $n \leq k$, $2k-n$ isolated points and $n-k$ isolated edges for $k < n \leq 2k$, and $5k-2n$ isolated edges with $n-2k$ pentagons for $2k < n \leq 5k/2$.

Proof: The statement is obvious for $n \leq k$. For the case $k < n \leq 2k$ one can easily check that the graph G formed by $i = 2k-n$ isolated points and $e = n-k$ isolated edges is a $(3,k+1,n,n-k)$-graph. Assume H is a minimum $(3,k+1,n,\bar{e})$-graph with $\bar{e} \leq e$ and let $\bar{i}$ be the number of isolated points in H. If $\bar{i} > 0$, then by lemma 2(b) H has only isolated points and isolated edges, therefore $\bar{i} + \bar{e} \leq k$ and $n = \bar{i} + 2\bar{e}$, which yields $\bar{e} \geq n-k$. Thus $\bar{e} = e$, $\bar{i} = i$ and H is isomorphic to G. If $\bar{i} = 0$, then since G has only isolated points and 1-vertices and $\bar{e} \leq e$, H has only isolated edges and $i = 0$, so H and G are isomorphic.

In the case $2k<n\leq 5k/2$ we use induction on k. Since the pentagon is the unique (3,3,5)-graph theorem 1 holds for $k=2$. Note that the graph G, formed by $i=5k-2n$ isolated edges and $j=n-2k$ pentagons with $e=3n-5k$ edges, is a $(3,k+1,n,e)$ graph since $k=i+2j$, $n=2i+5j$ and $e=i+5j$. Assume H is a minimum $(3,k+1,n,\bar{e})$-graph with $\bar{e}\leq e$. Then either H has a 2-vertex or it does not.

If H has a 2-vertex v, then by preferring v in H, $H_2(v)$ is a $(3,k,n-3,\bar{e}-Z(v))$-graph. Note that $n-3\leq 5(k-1)/2$, so by induction the number of edges $\bar{e}-Z(v)$ in $H_2(v)$ is at least $3(n-3)-5(k-1)$ for $n>2k+1$ or at least $(n-3)-(k-1)$ for $n=2k+1$. In both cases, since $\bar{e}\leq e=3n-5k$, we obtain $2\leq Z(v)\leq 4$. If $Z(v)=4$, then by induction $H_2(v)$ is a minimum $(3,k,n-3)$-graph formed by pentagons and one or more isolated edges since $2(k-1)<n-3<5(k-1)/2$ for $n>2k+1$ and $n-3=2(k-1)$ for $n=2k+1$. Each maximal independent set in $H_2(v)$ is of size $k-1$ and contains exactly one endpoint of each isolated edge and a pair of nonadjacent points from each pentagon. Since H avoids $(k+1)$-independent sets, the two neighbors of v must be connected to the endpoints of an isolated edge in $H_2(v)$. This implies that v belongs to a pentagon in H and straightforward counting of pentagons and isolated edges shows that H must be isomorphic to G. If $Z(v)\leq 3$, then the 2-vertex v must be adjacent to at least one 1-vertex, which contradicts lemma 2(c).

Now we can assume that H has no 2-vertices. Then H must have some r-vertex v, $r\geq 3$, since otherwise H would have only 0- and 1-vertices and all such graphs are covered by the case $n\leq 2k$. Note that $n-r-1\leq 5(k-1)/2$. By preferring v in H and using the inductive assumption, $H_2(v)$ is a $(3,k,n-r-1,e_2)$-graph with $e_2\geq 3(n-r-1)-5(k-1)$ for $n-r-1>2(k-1)$ or $e_2\geq n-r-k$ for $n-r-1\leq 2(k-1)$. In either case it can be easily derived that $e_2\geq\bar{e}-7$, implying that the Z-sum of vertex v satisfies $3\leq Z(v)\leq 7$. Hence, v has as a neighbor at least one 1-vertex which contradicts lemma 2(c). □

Corollary 1:

(a) *Any graph formed by isolated points and isolated edges is a minimum graph.*

(b) *Any graph formed by isolated edges and pentagons is a minimum graph.*

Proof: By theorem 1 the graph formed by i isolated points and j isolated edges is the unique minimum $(3,i+j+1,i+2j)$-graph and the graph formed by i isolated edges and j pentagons is the unique minimum $(3,i+2j+1,2i+5j)$-graph. □

Theorem 2: *Let $k\geq 2$. Then for all $n\geq 0$*

$$e(3,k+1,n)\geq 5n-10k. \tag{3}$$

Proof: By theorem 1 and simple arithmetic we have $e(3,k+1,n)\geq 5n-10k$ for $n\leq 5k/2$. For $n>5k/2$ we use induction on k. Inequality (3) holds for $k=2$ and $k=3$ since the only relevant parameter situation is $k=3$, $n=8$, and it is a well known fact that $e(3,4,8)=10$ [7]. Suppose G is a minimum $(3,k+1,n,e)$-graph for some $k\geq 4$, $n>5k/2$ and $e<5n-10k$. Let n_i be the number of i-vertices in G. Then by applying (1) and induction to G we obtain

$$0\leq \Delta(G,k+1,n,e)=ne-\sum_{i=0}^{k}n_i(i^2+e(3,k,n-i-1))\leq$$

$$ne-\sum_{i=0}^{k}n_i(i^2-5i+5n-10k+5).$$

Recall that $\sum_{i=0}^{k}n_i=n$, hence

$$0\leq \Delta(G,k+1,n,e)\leq n(e-(5n-10k-1))-\sum_{i=0}^{k}n_i(i-2)(i-3). \tag{4}$$

The coefficient $(i-2)(i-3)$ is nonnegative for all integers i. Hence, inequality (4) and $e<5n-10k$ imply that $e=5n-10k-1$, $\Delta(G,k+1,n,e)=0$ and consequently G has only vertices of degree 2 and 3. Hence, all the vertices of G are full, and we conclude that for any vertex v in G $Z(v)=4$ if $deg(v)=2$ and $Z(v)=9$ if $deg(v)=3$. Consequently, every component of G is a cycle or a cubic graph.

We first show that $n_2=0$. If $n_2>0$, then by lemma 2(a) G is a disjoint union of a pentagon and some $(3,k-1,n-5,5n-10k-6)$-graph H. But H cannot exist by induction since $5(n-5)-10(k-2)>5n-10k-6$. Hence we may assume that G is a cubic minimum $(3,k+1,n,e)$-graph with $e=3n/2$ edges. On the other hand $e=5n-10k-1=3n/2$ implies that

$$7n=20k+2 \tag{5}$$

whence $k\equiv 2 mod 7$. Thus G is a cubic $(3,10+7p,26+20p,39+30p)$-graph for some $p\geq 0$. To complete the proof of theorem 2 it is now sufficient to show that no such graph can exist.

By preferring any vertex v in G, $H_2(v)$ is a $(3,k,\bar{n},\bar{e})$-graph with $\bar{n}=n-4$ vertices and $\bar{e}=3n/2-9$ edges. Note that $H_2(v)$ must be a minimum $(3,k,\bar{n})$-graph since by induction and by (5) we have $e(3,k,n-4)\geq 5n-10k-10=5n-(7n-2)/2-10=\bar{e}$. We first show that the graph $H_2(v)$ can have only vertices of degree 2 and 3. Since $H_2(v)$ is a subgraph of cubic graph G, $H_2(v)$ can have only vertices of degree 3 or less. If $H_2(v)$ has an isolated point, then by lemma 2(b) it would have at most $\bar{n}/2$ edges, which is a contradiction with $\bar{e}=3n/2-9$ and $n\geq 26$. If $H_2(v)$ has a 1-vertex, then by lemma 2(c) it is a union of an isolated edge and a $(3,k-1,\bar{n}-2,\bar{e}-1)$-graph. But such a graph cannot exist, for by induction $e(3,k-1,\bar{n}-2)\geq 5n-10k-10=\bar{e}$. Thus by counting edges we conclude that $H_2(v)$ has 6 2-vertices and $n-10$ 3-vertices. If u is a 2-vertex in $H_2(v)$ then $Z(u)\leq 5$ with respect to the graph $H_2(v)$, since by induction and (5) $e(3,k-1,\bar{n}-3)\geq 5(n-7)-10(k-2) = \bar{e}-5$. Thus any 2-vertex in $H_2(v)$ has at least one other 2-vertex as a neighbor. Finally consider the subgraph F of $H_2(v)$ induced by it's 2-vertices. F cannot have isolated vertices since this would imply $Z(u)=6$ in $H_2(v)$, nor can F have a pentagon, since F has 6 vertices and a pentagon would imply the existence of an isolated vertex. Hence, by lemma 3 there are exactly 3 isolated edges in F. Consequently, if any vertex v in G is preferred, then the situation has to be as in figure 1. The 6 vertices in $H_2(v)$ showed in figure 1 are the vertices of degree 2 in $H_2(v)$.

It is clear that there are exactly 3 pentagons passing through each vertex v in G. Thus the total number of pentagons in G is $3n/5$ (each pentagon was counted 5 times), which implies that n is divisible by 5. On the other hand (5) implies that $n\equiv 1 mod 5$. This contradicts the existence of G and completes the proof of theorem 2.

The proof of the nonexistence of a cubic (3,10+7p,26+20p,39+30p)-graph can also be obtained by using the result of Staton [9] on the independence ratio of triangle-free graphs with bounded degree. □

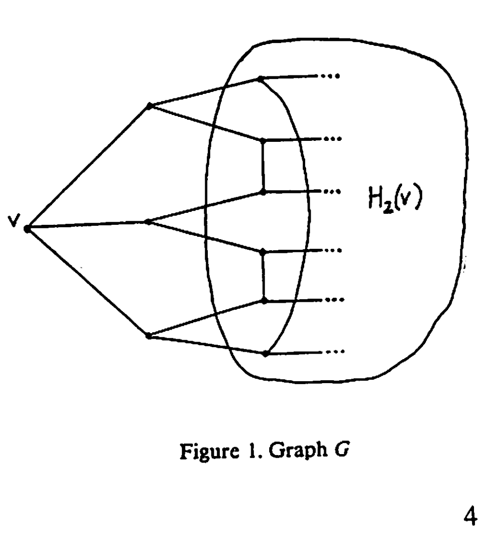

Figure 1. Graph G

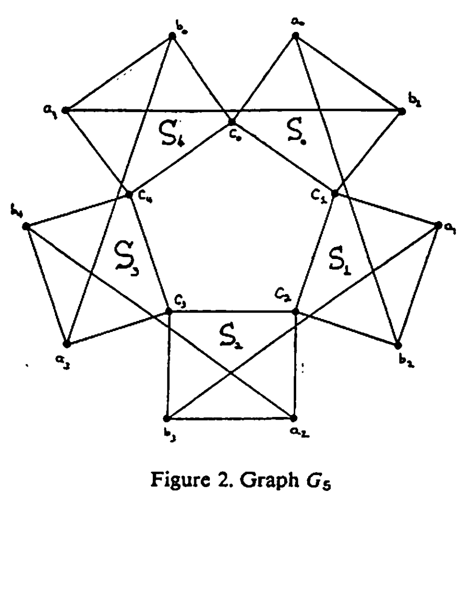

Figure 2. Graph G_5

The construction given in the next theorem defines an infinite family of minimum $(3,k+1,3k)$-graphs. We use the group theory terminology as in [10].

Theorem 3: *For all $k \geq 4$ define the graph $G_k=(V_k,E_k)$ as follows:*

$V_k = A_k \cup B_k \cup C_k$ *where*

$$A_k = \{a_x : x \in \mathbb{Z}_k\},$$
$$B_k = \{b_x : x \in \mathbb{Z}_k\} \text{ and}$$
$$C_k = \{c_x : x \in \mathbb{Z}_k\};$$

$E_k = F_k \cup \left(\bigcup_{x \in \mathbb{Z}_k} S_x\right)$ *where*

$$F_k = \{\{a_x, b_{x+2}\} : x \in \mathbb{Z}_k\} \text{ and}$$
$$S_x = \{\{c_x, c_{x+1}\}, \{c_{x+1}, b_{x+1}\}, \{b_{x+1}, a_x\}, \{a_x, c_x\}\},\ x \in \mathbb{Z}_k.$$

Then G_k is a minimum $(3,k+1,3k,5k)$-graph and it's full automorphism group is isomorphic to the dihedral group D_k on k symbols.

Proof: First note that each S_x is a 4-cycle and we refer to them as squares. Thus among the squares there are $4k$ edges. Hence, G_k has $4k + |F_k| = 5k$ edges. Furthermore, if $k > 3$ it is easy to see that G_k contains no triangles. The graph G_k is drawn in figure 2 for $k=5$.

Suppose that I is an independent set in G_k and consider how I intersects vertices of the cyclic sequence $S_0, S_1, \ldots, S_{k-1}$ of squares. Let $V(S_x)$ denote the set of vertices of S_x. Observe that the edges in F_k imply that if $I \cap V(S_x) = \{a_x, c_{x+1}\}$ then $I \cap V(S_{x+1}) = \{c_{x+1}\}$. Let α count the number of such pairs S_x, S_{x+1}. This accounts for 2α elements of I. Similarly, if $I \cap V(S_x) = \{b_{x+1}, c_x\}$, then $I \cap V(S_{x-1}) = \{c_x\}$ and we let β count the number of such pairs S_{x-1}, S_x. They account for 2β elements of I. All of the other squares intersect I in at most one vertex and there are $k-(2\alpha+2\beta)$ of them. Thus it follows that $|I| \leq 2\alpha+2\beta+k-2\alpha-2\beta=k$. Whence G_k has no independent sets of size $k+1$. By theorem 2 G_k is a minimum graph.

We now show that the full automorphism group Γ of the graph G_k is isomorphic to the dihedral group D_k on k symbols. Note that the following permutations are in Γ

$$\alpha = \begin{pmatrix} a_x & b_x & c_x \\ a_{x+1} & b_{x+1} & c_{x+1} \end{pmatrix}_{x \in \mathbb{Z}_k},$$
$$\beta = \begin{pmatrix} a_x & b_x & c_x \\ b_{-x} & a_{-x} & c_{-x} \end{pmatrix}_{x \in \mathbb{Z}_k}.$$

Thus D_k is a subgroup of Γ. Note that C_k is a block of imprimitivity, since C_k has all vertices of degree 4 in G_k. Furthermore, it is relatively easy to see that the pointwise stabilizer in Γ of C_k is the identity group. Hence, the restriction of Γ to the the subgraph induced by C_k is an isomorphism. Thus, since this induced subgraph is a cycle of length k it follows that Γ is isomorphic to D_k, as claimed. □

Corollary 2: *There exist minimum $(3,k+1,3k-1,5k-5)$-graphs for $k \geq 2$.*

Proof: A pentagon is the desired graph for $k=2$. For $k \geq 3$ observe that any 3-vertex in the graph G_{k+1} defined in theorem 3 has Z-sum equal to 10. Therefore by preferring any 3-vertex v in G_{k+1}, the graph $H_2(v)$ is a $(3,k+1,3k-1,5k-5)$-graph. By theorem 2 $e(3,k+1,3k-1) \geq 5k-5$, so $H_2(v)$ is a minimum graph. We remark that for all $k \geq 3$ this $H_2(v)$-graph has the full automorphism group of order 8. □

The graph G_k is the unique minimum $(3,k+1,3k)$-graph for $4 \le k \le 7$. The uniqueness of (3,5,12,20)- and (3,6,15,25)-graphs was shown in [7] and [5], and these graphs are isomorphic to G_4 and G_5, respectively. The uniqueness of G_6 and G_7 was established by the computer programs described in section 4. For $k \ge 8$ a minimum $(3,k+1,3k)$-graph nonisomorphic to G_k can be obtained as a disjoint union of G_i and G_{k-i} for any i, $4 \le i \le k-4$. For $3 \le k \le 5$, corollary 2 establishes minimum graphs for the parameter situations (3,4,8), (3,5,11) and (3,6,14). These minimum graphs are unique and were described in the literature [3,5,7].

Corollary 3:

(a) *There exist* $(3,k+1,3k+1,5k+7)$*-graphs for* $5 \le k \le 7$.

(b) *There exist* $(3,k+1,3k+1,5k+6)$*-graphs for* $k \ge 8$.

Proof: For (a) apply corollary 6 of Graver and Yackel in [3] to the graph G_k. To establish (b) take a disjoint union of the unique (3,5,13,26)-graph [7] and G_{k-4}. □

Corollary 3(a) gives the construction of (3,6,16,32)- (3,7,19,37)- and (3,8,22,42)-graphs. They are minimum graphs since $e(3,6,16)=32$, $e(3,7,19)=37$ [5,7] and we have found by computer that $e(3,8,22) \ge 42$. We do not know whether all the graphs given by corollary 3(b) are minimum graphs, but we consider it possible. Corollary 3(b) and theorem 2 imply that for all $k \ge 8$ we have $5k+5 \le e(3,k+1,3k+1) \le 5k+6$.

As a consequence of theorems 2 and 3 we obtain the following:

Theorem 4: *For* $k \ge 4$ *and* $5k/2 < n \le 3k$

$$e(3,k+1,n)=5n-10k.$$

Proof: By theorem 2 it is sufficient to construct $(3,k+1,n,5n-10k)$-graphs for $k \ge 4$ and $5k/2 < n \le 3k$. For $k=4$ and $k=5$ the parameters of graphs of consideration are: (3,5,11,15), (3,5,12,20), (3,6,13,15), (3,6,14,20) and (3,6,15,25). Note that a (3,6,13,15)-graph can be obtained as a union of a pentagon and a (3,4,8,10)-graph, for the remaining four parameter situations the graphs exist as mentioned after the proof of corollary 2. The general construction for $k \ge 6$ is by induction on k. For $5k/2 < n < 3k$ let G be the graph obtained by a disjoint union of a pentagon and any minimum $(3,k-1,n-5)$-graph H. By induction H has $5n-10k-5$ edges since $5(k-2)/2 < n-5 \le 3(k-2)$. So it is obvious that G is a $(3,k+1,n,5n-10k)$-graph. For $n=3k$ the construction of $(3,k+1,n,5n-10k)$-graphs is given by theorem 3. □

The values of $e(3,7,n)$ for $19 \le n \le 22$ were given in [5]; all the values for $n<19$ can be obtained from theorems 1 and 4. Thus the calculation of $e(3,7,n)$ is completed for all n. The smallest parameter situation in which minimum $(3,k+1,n)$-graph is not unique for $5k/2 < n \le 3k$ is (3,7,16). The two minimum (3,7,16,20)-graph are obtained by two disjoint copies of the unique minimum (3,4,8,10)-graph and by a disjoint union of a pentagon and the unique (3,5,11,15)-graph.

Corollary 4: *Any graph which is a disjoint union of pentagons, (3,4,8,10)-graphs and minimum* $(3,q+1,r)$*-graphs with* $5q/2 < r \le 3q$ *is also a minimum graph.*

Proof: Let $C_1,\ldots,C_p$ be components of some graph G of the form required by corollary 3. By theorem 1 and 4 each C_i is a $(3,k_i+1,n_i,e_i)$-graph with $5k_i/2 \le n_i \le 3k_i$ and $e_i=5n_i-10k_i$ for $1 \le i \le p$. Furthermore, $5k_i/2=n_i$ only if C_i is a pentagon. If all C_i's are pentagons, then G is a minimum graph by corollary 1(b). In all cases G is a $(3,k+1,n,e)$-graph with

$$k=\sum_{i=1}^{p} k_i, \quad n=\sum_{i=1}^{p} n_i \quad \text{and} \quad e=\sum_{i=1}^{p} e_i.$$

Note that $e=5n-10k$, and if at least one C_i is not a pentagon then $5k/2 < n \le 3k$. Hence G is a minimum $(3,k+1,n)$-graph by theorem 4. □

It is an interesting problem to decide whether corollary 4 can be extended to give a compact and complete specification of all minimum $(3,k+1,n)$ graphs with $5k/2<n\leq 3k$, using constructions as in theorems 3 and 4 and in corollary 2. These constructions provide such a characterization for $k\leq 6$ (see the list of minimum graphs in section 4).

Finally, observe that theorems 1 and 2 say that $e(3,k+1,n)$ is minorized simultaneously by a set of linear functions. That is for $k\geq 2$

$$e(3,k+1,n)\geq max\{a_i n-b_i k : 1\leq i\leq 4\} \qquad (6)$$

where a_i=0,1,3,5 and b_i=0,1,5,10, respectively. It seems that this set can be extended to $i\leq 5$ by defining a_5=6 and b_5=13, however we were unable to prove it nor to find a counterexample. Note also that (6) becomes an equality for $n\leq 3k$ by theorems 1 and 4, with the exception of two parameter situations $k=2, n=6$ and $k=3, n=9$, when $R(3,k+1)\leq 3k$ and thus $e(3,k+1,3k)=\infty$.

Relatively little work has been done with respect to the function $E(3,k,n)$, probably because of the following two reasons: the situation seems to be simpler than for function $e(3,k,n)$ and so far there are no substantial applications of $E(3,k,n)$ in obtaining information about $(3,k)$-graphs. The basic properties of $E(3,k,n)$ are stated in lemma 4.

Lemma 4: *For* $k\geq 2$

$$E(3,k+1,n)=\lfloor n^2/4\rfloor \text{ for } n\leq 2k \text{ and} \qquad (7)$$

$$E(3,k+1,n)\leq nk/2 \text{ for } n>2k. \qquad (8)$$

Proof: The well known theorem of Turán, see Harary [6, p.17], says that the number of edges in a triangle-free graph on n vertices is bounded by $\lfloor n^2/4\rfloor$. Furthermore it is easy to see that a complete bipartite graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ is a $(3,k+1,n,\lfloor n^2/4\rfloor)$-graph for $n\leq 2k$, so (7) follows. To show (8) note that the maximal degree of any vertex in a $(3,k+1,n)$-graph is k. □

We note that the bound given by (8) is tight for $2\leq k\leq 5$ and $2k<n<R(3,k+1)$ with the exception of parameter situations (3,5,9), (3,6,11) and (3,6,12) (see the tables in section 4).

4. Construction of the Catalogue

Using computer techniques we have constructed the following graphs:

(A) all $(3,k)$-graphs for $3\leq k\leq 6$,

(B) all minimum (3,7)-graphs,

(C) all (3,7,22)-graphs.

The basic approach to accomplish (A) was different from the one used in (B) and (C), since there are simply too many (3,7)-graphs and the exhaustive construction of all $(3,k)$-graphs is still feasible for $k=6$. Despite the difference in approaches common algorithms and data structures were used in order to maintain a uniform database of graphs.

4.1. General Graph Algorithms

The elementary block of data for all implemented algorithms is a file $G(k,n,e,d)$ formed by the set of $(3,k,n,e)$-graphs with fixed k,n,e and given minimum degree d. We will call it a *graph file*. Each line of a graph file represents one graph encoded similarly as in [1]. We explain this encoding in detail since the graphs listed in the appendix are presented this way. Given a 0-1 incidence matrix A of a graph G on n vertices, first write G as the 0-1 string formed by the $n(n-1)/2$ entries $A[i,j]$ of the incidence matrix A for $i=1,...,n$ and $j=i+1,...,n$.

Then break the string into 32 bit words and convert them into hexadecimal notation. We recognize that intuition can be lost in such representation, but an arbitrary graph on 22 vertices can be defined just in 1 line! For example the hexadecimal numbers 265, 312 and 0f2 are three different representations of a pentagon. Although an arbitrary graph usually has a large number of different such representations depending on the incidence matrix used, we will give just one. A simple conversion routine from the hexadecimal encoding to the incidence matrix of a graph is given in the appendix.

The major time consuming computational problem is the deletion of copies of isomorphic graphs from each graph file. In order to solve it we have basically implemented the techniques described in [1]. Our "early isomorphism rejection" strategy was to calculate the number of 3-independent sets passing through each vertex and degree sequence of the graph. The obvious sufficient condition for two graphs to be nonisomorphic was that their corresponding combined degree- and 3-independent set sequences were not identical. Otherwise, a full isomorphism algorithm was run.

The general graph algorithms we implemented also included conversions between different representations, graphics editor of graphs, finding cliques and independent sets, calculation of automorphism groups, and a manager of the graph data base.

4.2. Building (3,k,n)-graphs for k ≤ 6

The following steps were executed for k=3,4,5 and 6:

a) Create graph files $G(k,n,e,d)$ containing all $(3,k,n)$-graphs with $n<k$ by generating all triangle free graphs on n vertices with copies of isomorphic graphs deleted. The number of graphs obtained is relatively small, so we were able to check completely the correctness of this step by hand.

b) For each $n=k,k+1,\dots,R(3,k)-1$,
for each $e=e(3,k,n),e(3,k,n)+1,\dots,\lfloor n(k-1)/2\rfloor$,
for each $d=0,1,\dots,min(k-1,\lfloor n(k-1)/2\rfloor)$:
Expand all the graphs G, if any, in the graph files $G(k,n-1,e-d,\bar{d})$, $d-1\leq\bar{d}$, by adding new vertex of degree d to G and connecting it to all d-independent sets in G. Store the result in $G(k,n,e,d)$ if the obtained graph is a $(3,k)$-graph and its minimal degree is d. After completion delete copies of isomorphic graphs from $G(k,n,e,d)$.

It is straightforward to see that the above steps produce all $(3,k,n)$-graphs with $3\leq k\leq 6$ and consequently

$$g_{3k}(n,e)=\sum_{d\geq 0}|G(k,n,e,d)|$$

where $|G(k,n,e,d)|$ is the number of graphs in file $G(k,n,e,d)$. The correctness of the results was checked with all available data in [3,5,7] and with the lemmas and theorems in this paper. The number of $(3,k,n,e)$-graphs for $3\leq k\leq 6$ and for all possible n and e is reported in the tables I, II, III and IV. A blank entry in all tables denotes 0. We note that the values of $e(3,k,n)$ and $E(3,k,n)$ can be easily read in the table of g_{3k} by finding the location of the first and last nonzero entry in column n.

edges	number of vertices n					total
e	1	2	3	4	5	
0	1	1				2
1		1	1			2
2			1	1		2
3				1		1
4				1		1
5					1	1
total	1	2	2	3	1	9

Table I. Number of $(3,3,n,e)$-graphs, g_{33}

edges e	number of vertices n 1	2	3	4	5	6	7	8	total
0	1	1	1						3
1		1	1	1					3
2			1	2	1				4
3				2	2	1			5
4				1	3	1			5
5					2	4			6
6					1	4	1		6
7						3	2		5
8						1	3		4
9						1	2		3
10							1	1	2
11								1	1
12								1	1
total	1	2	3	6	9	15	9	3	48

Table II. Number of $(3,4,n,e)$-graphs, g_{34}

edges e	number of vertices n 1	2	3	4	5	6	7	8	9	10	11	12	13	total
0	1	1	1	1										4
1		1	1	1	1									4
2			1	2	2	1								6
3				2	3	3	1							9
4				1	4	6	2	1						14
5					2	8	7	1						18
6					1	7	13	5						26
7						4	17	13	1					35
8						2	15	27	3					47
9						1	10	39	11					61
10							4	41	28	1				74
11							1	27	59	2				89
12							1	15	73	10				99
13								6	62	32				100
14								2	33	69				104
15								1	14	86	1			102
16								1	4	65	6			76
17									2	32	19			53
18										12	31			43
19										3	30			33
20										1	13	1		15
21											4	2		6
22											1	5		6
23												2		2
24												2		2
25														0
26													1	1
total	1	2	3	7	13	32	71	179	290	313	105	12	1	1029

Table III. Number of $(3,5,n,e)$-graphs, g_{35}

Some of the initial values of $e(3,k,n)$ for $k \leq 5$ appeared indirectly in the early work of Greenwood and Gleason [4]. Kalbfleisch [7] systematically constructed all $(3,k,n)$-graphs and thus calculated $g_{3k}(n,e)$ for the following parameter situations: (3,3,5), (3,4,7), (3,4,8), (3,5,12), (3,5,13) and (3,6,17). He also partially analyzed the situation (3,6,16,32). Graver and Yackel [3] showed the uniqueness of minimum graph for (3,5,11). They also calculated the values of $e(3,6,n)$ for $n \geq 12$. Grinstead and Roberts in [5] have constructed all $(3,k,n,e)$-graphs and thus calculated $g_{3k}(n,e)$ for the parameter situations (3,5,11,16), (3,6,15,25), (3,6,15,26) and (3,6,16,32). We remark that each of the previous results known to the authors was obtained by considering only a particular parameter situation.

edges e	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	total
	number of vertices n																	
0	1	1	1	1	1													5
1		1	1	1	1	1												5
2			1	2	2	2	1											8
3				2	3	4	3	1										13
4				1	4	7	7	3	1									23
5					2	9	13	10	2	1								37
6					1	7	20	23	8	1								60
7						4	20	44	26	5								99
8						2	18	63	70	16	1							170
9						1	11	73	142	60	3							290
10							5	63	234	175	16							493
11							1	40	284	451	64	1						841
12							1	21	267	864	265	4						1422
13								9	185	1255	900	20						2369
14								3	106	1344	2353	119						3925
15								2	47	1114	4444	644	1					6252
16								1	22	707	6134	2693	4					9561
17									8	377	6239	7968	45					14637
18									3	167	4823	16445	375					21813
19									1	71	2885	23986	2402					29345
20									1	28	1405	25267	10176	1				36878
21										13	565	19704	27975	16				48273
22										4	206	11672	51188	177				63247
23										2	64	5404	64221	1588				71279
24										1	20	2016	56809	8494				67340
25										1	6	630	36312	27013	1			63963
26											2	169	17208	53157	7			70543
27												41	6189	67224	101			73555
28												8	1729	56478	822			59037
29												1	377	32235	3998			36611
30													66	12784	10910			23760
31													8	3550	17552			21110
32													1	699	16896	5		17601
33														94	9957	39		10090
34														9	3587	200		3796
35														1	794	547		1342
36															100	803		903
37															7	634		641
38																275		275
39																62		62
40																11	2	13
41																	3	3
42																	2	2
total	1	2	3	7	14	37	100	356	1407	6657	30395	116792	275086	263520	64732	2576	7	761692

Table IV. Number of $(3,6,n,e)$-graphs, g_{36}

4.3. Building (3,7)-graphs

$R(3,7)=23$ [3], thus the maximal number of vertices in a (3,7)-graph is 22. Since the exhaustive construction of all $(3,7,n)$-graphs is not feasible, we restricted our search to all minimum $(3,7,n)$-graphs for $n>15$ (all minimum graphs with $n \leq 15$ are described by theorem 1), and all (3,7,22)-graphs. This task becomes tractable since we have already constructed the complete catalogue of all (3,6)-graphs.

In addition to algorithms described in section 4.1, we have implemented algorithms *DELTA* and *EXPAND* performing the following functions:

$DELTA(k+1,n,e)$ finds all degree sequences n_i, $i \leq k$, of graphs satisfying (1) and the two equations below, counting vertices and edges of the desired graph

$$\sum_{i=0}^{k} n_i = n \quad \text{and} \quad \sum_{i=0}^{k} i \cdot n_i = 2e.$$

$EXPAND(H,k,n,e,d,Z(v))$ constructs from a $(3,k,n-d-1,e-Z(v))$-graph H all $(3,k+1,n,e)$-graphs G such that G has some d-vertex v, and by preferring v in G, $H_2(v)$ is isomorphic to the graph H.

The algorithm *DELTA* is a simple case of integer linear programming. In the implementation of *EXPAND* we have basically generalized the algorithms described by Grinstead and Roberts [5]. Similarly as in [5] we use the concept of a *good* pair of independent sets, but without any restrictions concerning sizes of involved sets. To construct all $(3,k+1,n,e)$-graphs we proceed as follows. If algorithm *DELTA* finds no solution then no such graph exists. If *DELTA* produces some solutions, then define a small (if possible) set S of parameters for algorithm *EXPAND* exhausting all possible degree sequences obtained by *DELTA*. Then running *EXPAND* for all elements in S produces the desired graphs. We remark that the above steps give an automatic method to produce results similar to many of the lemmas appearing in [5]. In this section we report results concerning only $(3,k)$ graphs with $k=7$, but are optimistic about applying this technique with some refinements also for $k\geq 8$.

Let us illustrate the method by an example of the search for minimum (3,7,16,20)-graphs. The set of degree sequences obtained by algorithm *DELTA* is given in table V.

n_5	n_4	n_3	n_2	Δ
1	0	5	10	0
0	2	4	10	2
0	1	6	9	5
0	0	8	8	8

Table V. Results of *DELTA*(7,16,20)

Hence any (3,7,16,20)-graph must have a full 3-vertex or a full 2-vertex. The minimum (3,6,12)- and (3,6,13)-graphs are unique, thus running *EXPAND* twice with the corresponding parameters will yield two minimum (3,7,16)-graphs mentioned after the proof of theorem 4. They have degree sequences as in two last rows of table V.

The results of search for (3,7)-graphs is summarized in table VI, which gives the values of $e(3,7,n)$ and the number of minimum $(3,7,n)$-graphs for $n\geq 16$, and in table VII, which gives the number of $(3,7,22,e)$-graphs for all possible e.

n	16	17	18	19	20	21	22
$e=e(3,7,n)$	20	25	30	37	44	51	60
$g_{37}(n,e)$	2	2	1	11	15	4	1

Table VI. $g_{37}(n,e(3,7,n))$ for $n\geq 16$

e	60	61	62	63	64	65	66	total
$g_{37}(22,e)$	1	6	30	60	59	25	10	191

Table VII. $g_{37}(22,e)$

Kalbfleisch [7] constructed a (3,7,22,64)- and a (3,7,22,65)-graph. He also derived some general properties of (3,7,21)- and (3,7,22)-graphs. Graver and Yackel in [3] obtained $e(3,7,20)=44$ and gave a construction of a regular (3,7,22,66)-graph with automorphism group of order 10. The values of $g_{37}(n,e(3,7,n))$ and the corresponding graphs for $n\geq 20$, including the unique (3,7,22,60)-graph, were described by Grinstead and Roberts in [5]. They also established that $e(3,7,19)=37$. The unique (3,7,18,30)-graph is that obtained by the general construction of theorem 3, corollary 2 gives one of the (3,7,17,25)-graphs. The second of (3,7,17,25)-graphs is a union of a pentagon and the unique (3,5,12,20)-graph. Also, several bounds for some values of $e(3,k,n)$ with $k\geq 8$ appeared in [3,5,7].

4.4. Maximum (3,7,22,66)-graph

We conclude this paper by presenting an interesting maximum (3,7,22,66)-graph with full automorphism group of order 22, the largest group of symmetries among automorphism groups for all 191 (3,7,22)-graphs.

Define the graph $H=(V,E)$ by $V=V_1 \cup V_2$, where $V_1=\mathbb{Z}_{11}$ and $V_2=\{\bar{x}:x\in\mathbb{Z}_{11}\}$, and for each $i,j\in\mathbb{Z}_{11}$

$$\{i,j\}\in E \ \textit{iff}\ i-j\equiv\pm1,\pm3 \ \textit{mod}\ 11,$$

$$\{i,\bar{j}\}\in E \ \textit{iff}\ i-j\equiv\pm2 \ \textit{mod}\ 11,$$

$$\{\bar{i},\bar{j}\}\in E \ \textit{iff}\ i-j\equiv\pm2,\pm3 \ \textit{mod}\ 11.$$

The full automorphism group of the graph H is isomorphic to the dihedral group D_{11} and is generated by permutations $(i\rightarrow i+1,\bar{i}\rightarrow\overline{i+1})$ and $(i\rightarrow -i,\bar{i}\rightarrow\overline{-i})$. One easily notes that H is triangle-free. To prove that H is a (3,7)-graph, first observe that the subgraphs induced in H by V_1 and V_2 are both isomorphic to the unique cyclic (3,5,11,22)-graph (see table III in section 4). Thus any independent set of size 7 in H must have 4 points in V_1 and 3 points in V_2 or 3 points in V_1 and 4 points in V_2. The (3,5,11,22)-graph has unique up to symmetry 4-independent set, given by the difference sequence 2225 in the case of V_1 and 1415 in the case of V_2. The remaining details needed to show that H has no independent set of size 7 are left for the reader. The hexadecimal encoding of the graph H appears in the first row of the table IX in the appendix. We would like to point out that we had previously constructed the above graph by the algorithm described in [8]. This algorithm searches for Ramsey graphs with a given automorphism group.

Appendix

All the graphs given in the appendix are in the hexadecimal encoding explained in section 4.1. The following procedure *HEX−INC* converts graphs from hexadecimal notation into standard incidence matrix representation.

Algorithm *HEX−INC(n,HEX,A)*

Input: n - number of vertices in graph G,
$HEX[i]$, $i=1,...,words$ - array of integers representing a graph G on n points.

Output: $A[i,j]$, $1\le i,j\le n$ - 0-1 incidence matrix of graph G.

```
begin
  bit := n(n-1)/2-1;
  word := 1;
  for i:=1 step 1 to n do
  begin
     A[i,i] := 0;
     for j:=i+1 step 1 to n do
     begin
        A[i,j] := A[j,i] := mask(HEX[word],bit mod 32);
        if (bit mod 32 = 0) then word := word+1;
        bit := bit-1
     end
  end
end
```

where $mask(x,y)$ is 1 if x has the y-th bit on and 0 otherwise. The bits of each integer are numbered from 0 to 31 starting from the least significant position.

We present all minimum $(3,k,n)$-graphs for $3 \le k \le 7$ and $5(k-1)/2 < n$ (those with $n \le 5(k-1)/2$ are described by theorem 1), and all maximum (3,7,22)-graphs. Each row of the following tables represents one $(3,k,n,e)$-graph. The last column $S = |Aut(G)|$ gives the order of the full automorphism group of the graph. Potentially those with larger automorphism group deserve more interest.

k	*n*	*e*	*HEX*[1]	*HEX*[2]	*HEX*[3]	*HEX*[4]	*HEX*[5]	*HEX*[6]	*S*
4	8	10	71a0861						8
5	11	15	361c00	80980861					8
	12	20	2	a85050c2	1c044113				8
	13	26	3141	82450823	8a85184b				52
6	13	15	1a03	80080208	100313				80
	14	20	56020c3	81800	80980861				8
	15	25	192	144220a	6043208	8044113			10
	16	32	b20130	4140221	18250824	2286184b			96
			b20130	2240a40	14250424	218a184b			12
			b05120	a4508a06	c0481201	48c0861			4
			5c41c0	a0509e00	44030412	12541044			2
			c88881	c2282212	80405885	205184a			2
	17	40	c3	8612106	b010c85	98390840	205a284c		2
			70	48ca0814	a088e12	a0950621	824a4412		2
7	16	20	608184	80042	84405040	18004021			128
			6c01c0	400203	80208	100313			80
	17	25	ac	a1010a	882800	80480008	4100313		80
			e8	230023	4801002	8001800	80894421		8
	18	30	1850048	40440884	44088047	101002	8081213		12
7	19	37	660	20c80884	a0048981	2010408	8149090c	20c021	4
			622	c40088	80440141	4041081	8111048	809e088b	4
			7a8	231008	c1141109	3002210	84a04	21024203	2
			758	431012	448240c0	4121000	41102288	2800c121	2
			2d4	f08014	44128221	440880	4842008a	450023	2
			684	40a09014	44128560	4402800	4042008a	24c025	2
			668	ca0081	8010920	408102a0	8a08442a	2180441	2
			370	16c0003	408c2084	10481020	42002288	40380205	1
			390	21628004	48620481	2082020	254050c	40208205	1
			570	10610d0	80041141	30040a	209102a	320701	1
			521	81210258	920011	d0412	404c0900	810c0445	1
7	20	44	34504283	9301204	61a80102	3002400	10101440	88404b0b	8
			30d0024a	1806040	90321130	2028414a	40081080	4800485e	4
			31820310	41448003	10a18140	20a40c0a	84460016	10090a1	4
			1e210662	8029008	48700518	80205123	2208	21181121	2
			2a420068	71a80113	a80501	80900281	44248028	80860701	2
			344c0284	c1241010	ca80300	54112042	1401802	8284d001	2
			1d4206a4	10049001	248d0041	844a0310	81280120	208928	2
			2d2404b0	81012802	14043209	34880443	40280000	61c032	2
			1a248714	112020	48a30013	6c01006	28014040	24822241	1
			31508221	91850040	24b08118	10a0490	41500010	22090528	1
			318c0298	41825000	ec80802	52007021	3802	2282e080	1
			2c830443	14a0004	18870180	11842032	18280040	1900a844	1
			351100a0	304d00da	b006	3210508	c20018	4040d830	1
			390500a0	30801647	a11042	2c1110a	1008483	2302003	1
			34410068	21228018	50a01921	80500280	c4489080	40208854	1

Table VIII. Minimum $(3,k,n)$-graphs, $n > 5(k-1)/2$

Table VIII continued:

c5c0	bc4018b	80e02080	18100400	91110191	88044c80	90a0a101	6
d1c0	3940100b	88801810	24b10103	4241101	88044c80	9020a105	1
39004	88990013	91840828	4b10103	5200142	1600c40	5020e111	1
c90c	1c0a0385	28210	258006a	50ba00	18003080	448c2561	1

Minimum (3,7,21,51)-graphs, *HEX* has 7 words

The unique (3,7,22,60)-graph with automorphism group of order 2 is given by *HEX* on 8 words

24 c2424c12 20310491 4a4414a 1a45008 a241140a 10009020 1998068c

Finally, table IX presents ten 6-regular graphs, which are all maximum (3,7,22,66)-graphs. The first one was described in section 4.3; the second one was constructed by Graver and Yackel in [3].

HEX								*S*
1	2	3	4	5	6	7	8	
1a	91038148	e108205	11081410	42035184	6248b0c1	2401c265	4a40844b	22
27	8101b808	30488221	11044402	22222844	5181b109	1207058a	aa80844b	10
70	89008708	1a00c190	288142a8	440450a1	11448d40	45118862	ce00844b	5
4b	810240c8	116086a2	10022484	14301460	942188c0	620d04ac	6700844b	4
2e	8102b088	a88638	10011442	22110d06	23028a18	980c8584	a700844b	4
51	29008c48	1f008708	10008612	441244a0	63120248	4a0652a5	4d00844b	2
59	9024828	12a0830a	10880c18	6602508	52110980	584c41cb	4940844b	2
55	3000e88	19a0850a	10426082	24302540	a0428312	2404c1ad	4540844b	2
54	31021448	15208505	101806d0	1402c104	52910920	98154320	ea80844b	2
4a	a1030148	a908311	11460420	4222604b	42a1110	b034aa0	ea80844b	2

Table IX. All (3,7,22,66)-graphs

The electronic copies of the entire catalogue (or hardcopy of a portion of it) together with basic manipulation procedures written in *C* under *UNIX*[1] can be requested from the authors.

1 - UNIX is a trademark of Bell Laboratories.

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