Partitioning sets of triples into designs II

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ABSTRACT

It is shown that the collection of all the $\binom{10}{3}$ triples chosen from a set of ten points can be partitioned into ten mutually disjoint 2-$(9,3,1)$ designs in precisely 77 non-isomorphic ways.

Introduction.

A *t-design* based on a v-set, X, is a collection of k-subsets (blocks) chosen from X in such a way that each unordered t-subset of X occurs in precisely λ of the blocks. Such a design has parameters t-(v,k,λ).

Two t-(v,k,λ) designs X are said to be *disjoint* if and only if they have no block in common. If the set of all the $\binom{v}{k}$ k-sets chosen from X can be partitioned into mutually disjoint t-(v,k,λ) designs, then these designs are said to form a *large set*.

For example, Kirkman [5] in 1850 and Bays[1] in 1917 showed that the 84 triples chosen from a 9-set can be partitioned into a large set of 2-$(9,3,1)$ designs in just two non-isomorphic ways; see also Kramer and Mesner [6] and Mathon, Phelps and Rosa [8]. Another example is provided by Kreher and Radziszowski [7] who showed that the 3432 7-subsets of a 14-set can be partitioned into a large set of two 6-$(14,7,4)$ designs.

A necessary condition for the existence of a large set of t-(v,k,λ) designs, each with b blocks, is that b divides $\binom{v}{k}$. However, there are examples where this condition is satisfied but a large set does not exist. For example, Cayley [3] showed that from the $\binom{7}{3}$ triples on seven points at most two disjoint 2-$(7,3,1)$ designs can be formed; the remaining 21 triples form a 2-$(7,3,3)$ design which cannot be decomposed into smaller designs. Similarly, Kramer and Mesner [6] showed that on a 12-set there are at most two disjoint 5-$(12,6,1)$ designs.

Whether or not a large set exists, it may be possible to pack the designs neatly by enlarging the set of points on which they are based, sometimes by adjoining just one extra point. Thus, if the set of all the $\binom{v}{k}$ k-sets chosen from X can be partitioned into v mutually disjoint t-$(v-1,k,\lambda)$ designs, each missing a different point of X, then we shall say that these designs form an *overlarge set*. In this paper, we shall take $X=\{0,1,...,9\}$, and label the designs of an overlarge set by their missing elements. Thus the designs in any overlarge set are labelled D_i, for $i=0,1,\ldots,9$, where element i is missing.

For example, from any $(t+1)$-$(v,k+1,1)$ design, D, we can form an overlarge set of t-$(v-1,k,1)$ designs by choosing, for each $i=1,...,v$, all the blocks of D containing i, and deleting i from each of them. These k-sets form design D_i, and this overlarge set is said to be *derived* from D. Such a construction has been used by Rosa [**12**]. However an overlarge set of designs need not be derived in this way; Sharry and Street [**13**] deal with the case $v = 8$, $k = 3$, $t = 2$, where only one of the 11 possible overlarge sets (partition A) is derived from a 3-$(8,4,1)$ design, and Breach and Street [**2**] with the case $v = 9$, $k = 4$, $t = 3$, where two overlarge sets of 3-$(8,4,1)$ designs exist, but no 4-$(9,5,1)$ design exists.

The designs.

The 2-$(9,3,1)$ design is unique up to isomorphism and has automorphism group of order 432; see for example [**8**]. We usually represent the design as a 3×3 array, the rows, columns, forward diagonals and back diagonals of which form the blocks of the design. For example, the design, P, given by the array

$$\begin{matrix} 1 & 2 & 9 \\ 3 & 7 & 5 \\ 8 & 4 & 6 \end{matrix}$$

has the following blocks:

$$129, 357, 468;\ 138, 247, 569;\ 167, 258, 349;\ 145, 236, 789.$$

The 3-$(10,4,1)$ design is unique up to isomorphism, and has automorphism group of order 1440; see, for example, [**4**]. We may form a derived overlarge set of 2-$(9,3,1)$ designs from a 3-$(10,4,1)$ design, as described above, and obviously this overlarge set is also unique up to isomorphism.

In distinguishing the isomorphism classes of overlarge sets, we also use the classification of the reducible 2-$(9,3,2)$ designs, given by Morgan [**11**] and by Mathon and Rosa [**9**]; these are the designs numbered 1, 2, 3, 6, 7, 14, 15, 23, 29, in [**11**]. There are precisely nine non–isomorphic reducible 2-$(9,3,2)$ designs, and an example of each is given in Table 1, where $P \cup \pi_i P = \mathcal{D}(i)$ for $i = 0, 1, \ldots, 8$.

Consider any two of the 2-$(9,3,1)$ designs contained in an overlarge set, and suppose that they are based on the sets $\{1,2,\ldots,9\}$ and $\{0,1,\ldots,8\}$ respectively. Replacing the point 0 by the point 9 in the second design gives a reducible 2-$(9,3,2)$ design, in which repeated blocks are only possible if they arise from blocks $xy9$ and $xy0$ present originally. Conversely, if we take a reducible 2-$(9,3,2)$ design, and replace any one of the points in one of its subdesigns by a tenth point in such a way that no repeated blocks remain, then we have a set of 24 blocks which may constitute two of the 2-$(9,3,1)$ designs contained in an overlarge set. Thus from the nine reducible 2-$(9,3,2)$ designs, we find all possible pairs of 2-$(9,3,1)$ designs that may occur in an overlarge set; we call such pairs *feasible*. It turns out that in fact all feasible pairs occur in overlarge sets, though we know this only through an exhaustive search.

i	π_i	$\mid aut(\mathcal{D}(i)) \mid$	Generators of $aut(\mathcal{D}(i))$
0	(1)	432	(129)(357), (23749865)
1	(12)	24	(12), (36)(45)(78), (367458)
2	(12)(34)	32	(1625)(3847), (34)(58)(67), (56)(78)
3	(123)	6	(132)(457)(698), (12)(45)(89)
4	(1234)	4	(13)(24)(56), (24)(78)
5	(12)(46)(57)	108	(129)(365478), (134)(276)(589), (29)(36)(47)(58)
6	(1236)	4	(12)(35)(46), (12)(36)(45)(79)
7	(12369)	18	(129)(375)(468), (173642)(598)
8	(12459)	6	(136)(278)(495), (12)(38)(49)(67)

Table 1: The nine non-isomorphic reducible $2-(9,3,2)$ designs, $\mathcal{D}(i)$, and their automorphism groups, $aut(\mathcal{D}(i))$, $i = 0,1,\dots,8$. Here $\mathcal{D}(i) = P \cup \pi_i P$ for the design P given in the text.

In $\mathcal{D}(i)$ for $i = 0,1,3,5$, no single point can be replaced by 0 in such a way as to remove all repeated blocks, so these four designs do not give rise to any feasible pairs of 2-$(9,3,1)$ designs. However, from $\mathcal{D}(i)$ for $i = 2,4,7$, we obtain (up to isomorphism) one pair of feasible designs for each value of i; we denote these by $\mathcal{E}(i)$. Similarly, from $\mathcal{D}(i)$ for $i = 6,8$, we obtain two pairs of feasible designs for each i; we denote these by $\mathcal{F}(i)$ and $\mathcal{G}(i)$. This gives all seven of the feasible pairs of 2-$(9,3,1)$ designs, as shown in Table 2, where ρ_i, σ_i, τ_i respectively are the permutations which must be applied to P to give $\mathcal{E}(i) = P \cup \rho_i P$, $\mathcal{F}(i) = P \cup \sigma_i P$, $\mathcal{G}(i) = P \cup \tau_i P$ respectively.

Overlarge sets of 2-(9,3,1) designs.

The set of all $\binom{10}{3}$ triples chosen from a 10-set can be partitioned into an overlarge set of 2-$(9,3,1)$ designs in precisely 77 non–isomorphic ways. These are given in Table 3, together with their automorphism groups. As shown in Table 4, an automorphism group of order 6, 16, 20, 40, 48 or 1440 uniquely determines an overlarge set, but an automorphism group of any other order does not. (Note that partition 77 is that derived from a 3-$(10,4,1)$ design.) All but two pairs of the remaining overlarge sets may be easily distinguished using the information given in Table 5; that is, the numbers of feasible pairs of each type contained in each overlarge set.

Feasible Pair	Permutation	Group Order	Group Generators
$\mathcal{E}(2)$	$\rho_2 = (09)(12)(34)$	32	(34)(58)(67) (1625)(3847) (09)(56)(78)
$\mathcal{E}(4)$	$\rho_4 = (09)(1234)$	4	(13)(24)(56) (09)(24)(78)
$\mathcal{F}(6)$	$\sigma_6 = (07)(1236)$	2	(12)(35)(46)
$\mathcal{G}(6)$	$\tau_6 = (08)(1236)$	4	(12)(35)(46) (08)(12)(36)(45)(79)
$\mathcal{E}(7)$	$\rho_7 = (04)(12369)$	2	(04)(28)(56)(79)
$\mathcal{F}(8)$	$\sigma_8 = (03)(12459)$	1	(1)
$\mathcal{G}(8)$	$\tau_8 = (09)(12459)$ $= (012459)$	2	(09)(17)(23)(45)(68)

Table 2: The seven feasible pairs of $2-(9,3,1)$ designs, and their automorphism groups. The two designs in a feasible pair are based on distinct 9–sets of the set $\{0,1,\ldots,9\}$, Here $\mathcal{E}(i) = P \cup \rho_i P$, $\mathcal{F}(i) = P \cup \sigma_i P$, and $\mathcal{G}(i) = P \cup \tau_i P$ respectively.

This leaves just partitions 21 and 22, and partitions 41 and 42, which cannot be distinguished by the information given in Tables 4 and 5. To deal with these two cases, not only the numbers of the different types of feasible pairs are needed, but also the way in which they are arranged; see Tables 6 and 7.

In partition 41, the design D_2, with either D_1 or D_5, forms a pair of type $\mathcal{E}(7)$, whereas in partition 42, no two pairs of type $\mathcal{E}(7)$ have a design D_i in common; this is shown in Table 7.

In both partitions 21 and 22, the designs D_1 and D_2 form a pair of type $\mathcal{E}(2)$, but this is the only pair of type $\mathcal{E}(2)$ in either of these partitions; this is shown in Table 6. Thus a permutation which mapped partition 21 to partition 22 would have to belong to the group of order 32, generated by the permutations

$$(09)(58)(67), (3645)(0897), (12)(56)(78),$$

which is the automorphism group of the pair D_1, D_2. None of these 32 permutations does in fact map partition 21 to partition 22, providing easy confirmation of the fact that these overlarge sets are not isomorphic to each other. However, these are the most awkward of the partitions to identify, since the group of the unique pair of type $\mathcal{E}(2)$ must be computed.

#	D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	|G|	Group Generators
1	123 478 596	234 579 608	134 507 689	124 560 879	129 365 708	126 374 809	125 348 907	128 364 590	127 345 096	120 378 645	16	(12)(56)(78) (39)(40)(78) (25)(48)(70)
2	123 478 596	234 579 608	134 507 689	124 560 879	129 365 708	126 374 809	125 348 907	128 345 096	127 364 590	120 378 645	32	(12)(56)(78) (39)(40)(78) (13)(27)(40)(59)(68)
3	123 478 596	234 579 608	134 507 689	124 560 879	129 365 708	126 374 809	125 348 907	128 345 096	120 379 645	127 364 580	8	(1753)(4809) (13)(26)(49)(57)(80)
4	123 478 596	234 579 608	134 507 689	124 560 879	129 365 708	126 374 809	127 384 590	128 345 096	125 346 907	120 378 645	4	(12)(37)(48)(59)(60) (13)(27)(40)(59)(68)
5	123 478 596	234 579 608	134 507 689	124 560 879	129 365 708	126 374 809	127 384 590	120 398 645	125 346 907	128 345 076	2	(15)(23)(49)(68)(70)
6	123 478 596	234 579 608	134 507 689	124 560 879	129 365 708	126 374 809	128 345 097	125 348 906	120 379 645	127 364 580	2	(12)(35)(46)(70)(89)
7	123 478 596	234 579 608	134 507 689	124 560 879	129 365 708	126 374 809	120 378 945	128 364 590	127 345 096	125 348 607	2	(25)(48)(70)
8	123 478 596	234 579 608	134 507 689	124 560 879	129 365 708	126 348 907	125 374 809	128 364 590	127 345 096	120 378 645	8	(12)(56)(78) (1329)(4506)
9	123 478 596	234 579 608	134 507 689	124 560 879	129 365 708	128 347 096	120 378 945	126 354 809	127 364 590	125 348 607	4	(12)(38)(47)(50)(69) (13)(28)(45)(70)
10	123 478 596	234 579 608	134 507 689	125 486 790	129 365 708	127 364 890	124 305 987	120 398 645	126 374 509	128 345 076	2	(12)(36)(45)(79)(80)
11	123 478 596	234 579 608	134 507 689	125 486 790	129 365 708	128 347 096	124 305 987	126 354 809	120 379 645	127 364 580	2	(12)(38)(47)(50)(69)
12	123 478 596	234 579 608	134 507 689	125 486 790	129 365 708	120 378 649	124 305 987	128 364 590	127 345 096	126 374 805	2	(12)(39)(40)(56)
13	123 478 596	234 579 608	134 507 689	125 486 790	129 365 708	120 378 649	124 305 987	128 345 096	127 364 590	126 374 805	6	(12)(39)(40)(56) (28)(30)(46)(59)

Table 3: The 77 non-isomorphic overlarge sets of $10 \times 2 - (9,3,1)$ designs, and their automorphism groups. Design D_i is based on the set $\{0,1,\ldots,9\}\backslash\{i\}$.

#	D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	|G|	Group Generators
	123	234	134	127	128	120	129	125	126	124		
14	478	579	507	456	356	364	354	348	374	387	2	(12)(35)(46)(70)(89)
	596	608	689	809	097	798	870	906	509	605		
	123	234	134	127	128	120	129	125	126	124		
15	478	579	507	486	356	378	354	348	390	360	2	(12)(35)(46)(70)(89)
	596	608	689	905	097	649	870	906	745	578		
	123	234	134	127	129	126	125	120	124	128		(39)(40)(78)
16	478	579	507	486	365	374	348	398	379	345	8	(26)(37)(89)
	596	608	689	905	708	809	907	645	560	076		(15)(37)(40)(89)
	123	234	134	127	129	126	125	120	124	128		
17	478	579	507	486	365	348	374	398	379	345	2	(39)(40)(78)
	596	608	689	905	708	907	809	645	560	076		
	123	234	134	129	120	128	127	125	126	124		(12)(56)(78)
18	478	579	507	465	356	364	345	348	374	387	4	(34)(56)(78)(90)
	596	608	689	708	897	790	098	906	509	605		
	123	234	134	129	120	128	127	125	126	124		(12)(56)(78)
19	478	579	507	465	356	347	384	346	354	387	8	(3748)(5069)
	596	608	689	708	897	096	590	908	709	605		
	123	234	134	129	120	128	127	125	126	124		(12)(56)(78)
20	478	579	507	456	365	347	384	364	345	387	8	(3748)(5069)
	596	608	689	807	798	096	590	809	907	605		
	123	234	134	120	129	127	128	125	126	124		(12)(56)(78)
21	478	579	507	465	356	346	354	348	374	387	8	(3546)(7980)
	596	608	689	798	807	098	790	906	509	605		
	123	234	134	120	129	128	127	126	125	124		(12)(56)(78)
22	478	579	507	465	356	347	384	345	364	387	8	(3748)(5069)
	596	608	689	798	807	096	590	908	709	605		
	123	234	134	120	129	126	125	128	127	124		(12)(56)(78)
23	478	579	507	456	365	374	348	364	345	387	48	(34)(56)(78)(90)
	596	608	689	897	708	809	907	590	096	605		(152)(374980)
	123	234	134	120	129	126	125	128	127	124		(12)(56)(78)
24	478	579	507	456	365	348	374	364	345	387	8	(34)(56)(78)(90)
	596	608	689	897	708	907	809	590	096	605		(39)(40)(78)
	123	234	134	120	129	127	128	125	126	124		(12)(56)(78)
25	478	579	507	456	365	364	345	348	374	387	8	(3546)(7980)
	596	608	689	897	708	890	097	906	509	605		
	123	234	134	120	125	129	128	124	126	127		
26	478	579	587	479	360	348	354	356	309	380	1	(1)
	596	608	690	865	798	076	970	809	547	645		

Table 3: (cont'd) The 77 non-isomorphic overlarge sets of $10 \times 2-(9,3,1)$ designs, and their automorphism groups. Design D_i is based on the set $\{0,1,\ldots,9\}\backslash\{i\}$.

#	D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	|G|	Group Generators
	123	234	134	126	127	128	129	120	125	124		
27	478	579	506	490	390	306	347	346	397	356	2	(16)(27)(30)(49)(58)
	596	608	789	857	685	947	580	859	046	708		
	123	234	134	126	127	128	129	120	125	124		
28	478	579	506	490	390	397	347	346	306	356	2	(16)(27)(30)(49)(58)
	596	608	789	857	685	046	580	859	947	708		
	123	234	134	124	129	126	127	120	125	128		(26)(38)(40)(79)
29	478	579	560	507	365	374	384	398	346	345	4	(15)(23)(49)(68)(70)
	596	608	879	689	708	809	590	645	907	076		
	123	234	134	124	129	126	125	128	127	120		
30	478	579	560	507	365	348	374	364	345	378	2	(19)(23)(45)(60)(78)
	596	608	879	689	708	907	809	590	096	645		
	123	234	134	124	129	128	120	126	127	125		(2670)(3954)
31	478	579	560	507	365	347	378	354	364	348	8	(18)(23)(40)(57)(69)
	596	608	879	689	708	096	945	809	590	607		
	123	234	134	129	125	128	120	126	127	124		(2076)(3459)
32	478	579	560	408	398	397	378	354	309	305	8	(18)(23)(46)(57)(90)
	596	608	879	675	067	640	945	809	546	768		
	123	234	134	129	128	127	120	124	125	126		
33	478	579	578	470	309	308	379	306	346	348	2	(13)(28)(47)(59)(60)
	596	608	960	658	576	649	845	958	790	057		
	123	234	134	125	129	127	120	128	124	126		(28)(30)(46)(59)
34	478	579	576	486	350	398	354	356	376	380	4	(17)(24)(35)(68)(90)
	596	608	089	790	876	640	798	049	905	547		
	123	234	134	125	129	126	124	128	127	120		(34)(59)(60)
35	478	579	598	460	306	398	378	350	309	376	4	(17)(28)(59)(60)
	596	608	076	798	758	047	950	649	546	845		
	123	234	134	129	125	126	124	128	127	120		(34)(59)(60)
36	478	579	598	406	360	398	378	350	309	376	4	(17)(28)(59)(60)
	596	608	076	758	798	047	950	649	546	845		
	123	234	135	128	129	120	127	125	126	124		
37	478	579	470	496	378	346	359	398	350	356	2	(18)(29)(30)(46)(57)
	596	608	698	057	605	987	840	046	479	708		
	123	234	135	126	128	124	127	120	129	125		
38	478	579	407	475	356	360	395	398	354	364	2	(12)(37)(49)(58)(60)
	596	608	698	908	907	879	480	645	076	708		
	123	234	135	128	129	126	120	125	124	127		
39	478	579	409	457	365	308	378	364	390	350	2	(18)(25)(34)(69)(70)
	596	608	768	609	780	479	945	098	657	846		

Table 3: (cont'd) The 77 non-isomorphic overlarge sets of $10 \times 2-(9,3,1)$ designs, and their automorphism groups. Design D_i is based on the set $\{0,1,\ldots,9\} \backslash \{i\}$.

#	D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	|G|	Group Generators
	123	234	135	125	129	127	124	126	120	128		
40	478	579	486	469	308	346	350	359	354	356	4	(1756)(4809)
	596	608	790	870	675	980	879	408	796	047		
	123	234	135	125	129	127	120	128	126	124		
41	478	579	486	469	308	346	354	356	359	350	4	(15)(40)(67)(89) (16)(23)(48)(57)(90)
	596	608	790	870	675	980	798	049	407	876		
	123	234	135	125	129	120	127	128	126	124		
42	478	579	486	469	308	364	345	356	359	350	4	(12)(37)(58)(60) (13)(27)(49)(50)(68)
	596	608	790	870	675	798	980	049	407	876		
	123	234	135	125	129	127	120	128	126	124		
43	478	579	486	469	350	346	354	356	359	308	4	(17)(23)(49)(56)(80) (14)(26)(35)(79)(80)
	596	608	790	870	876	980	798	049	407	675		
	123	234	135	120	129	126	128	124	125	127		
44	478	579	486	497	365	374	359	305	379	364	2	(26)(37)(89)
	596	608	790	685	708	809	407	986	640	085		
	123	234	135	120	129	126	128	124	127	125		
45	478	579	486	497	365	374	359	305	364	378	2	(26)(37)(89)
	596	608	790	685	708	809	407	986	095	640		
	123	234	135	124	120	128	127	125	129	126		
46	478	579	480	507	365	367	345	348	340	374	2	(18)(20)(34)(57)(69)
	596	608	976	689	789	490	098	906	657	805		
	123	234	135	128	120	127	129	125	126	124		
47	478	579	480	495	397	304	347	389	345	305	2	(16)(37)(40)(59)
	596	608	976	067	685	869	058	460	907	786		
	123	234	135	128	127	124	129	125	120	126		
48	478	579	408	495	305	390	308	346	356	345	2	(12)(37)(40)(58)
	596	608	967	760	986	687	475	890	749	078		
	123	234	135	127	129	126	128	124	125	120		
49	478	579	407	465	376	348	309	398	304	347	8	(2705)(3468) (19)(2603)(4785)
	596	608	986	098	850	907	475	056	769	658		
	123	234	135	128	129	126	125	124	127	120		
50	478	579	407	457	365	348	374	398	395	378	8	(2705)(3468) (19)(23)(45)(60)(78)
	596	608	986	609	708	907	809	056	460	645		
	123	234	135	126	128	127	124	129	120	125		
51	478	579	496	475	356	390	305	346	397	376	2	(13)(28)(47)(56)(90)
	596	608	087	908	907	486	789	058	645	840		
	123	234	136	125	120	129	127	126	124	128		
52	478	579	459	408	367	340	390	345	395	365	10	(12)(38)(45)(67)(90) (14)(26)(39)(58)(70)
	596	608	708	967	589	786	845	908	067	470		

Table 3: (cont'd) The 77 non-isomorphic overlarge sets of $10\times2-(9,3,1)$ designs, and their automorphism groups. Design D_i is based on the set $\{0,1,\ldots,9\}\setminus\{i\}$.

#	D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	|G|	Group Generators
53	123 478 596	234 579 608	136 459 708	129 470 865	128 365 709	126 374 908	127 390 485	120 389 546	124 395 067	125 376 840	8	(26)(37)(89) (19)(24)(37)(58)(60)
54	123 478 596	234 579 608	136 497 805	129 407 658	128 365 709	126 374 908	127 384 095	124 389 506	125 367 490	120 346 857	4	(26)(37)(89) (15)(37)(40)
55	123 478 596	234 579 608	136 470 985	125 460 798	120 379 568	129 304 786	124 395 087	128 364 905	126 350 479	127 356 840	2	(17)(28)(36)(49)(50)
56	123 478 596	234 579 608	136 470 985	127 456 890	126 350 879	120 374 968	124 395 087	125 398 460	129 304 576	128 364 705	2	(19)(26)(38)(47)(50)
57	123 478 596	234 579 608	136 407 985	120 496 785	128 305 976	127 304 869	129 347 508	125 389 460	124 395 067	126 384 705	1	(1)
58	123 478 596	234 579 608	138 469 705	125 486 097	128 305 976	120 346 789	129 304 587	126 309 458	124 359 670	127 384 065	2	(13)(27)(48)(50)(69)
59	123 478 596	234 579 608	139 406 758	127 486 905	129 365 708	126 374 809	125 380 497	120 398 645	124 379 560	128 345 076	32	(27486903) (2863)(4709) (15)(23)(49)(68)(70)
60	123 478 596	234 579 608	139 460 875	127 458 609	129 365 708	126 374 809	125 384 097	124 389 605	120 397 456	128 364 570	24	(15)(37)(40)(89) (15)(38)(79) (146502)(387)
61	123 478 596	234 579 608	139 460 875	128 459 607	129 365 708	126 374 809	125 384 097	124 389 560	120 397 645	127 305 486	24	(124)(389)(560) (15)(37)(40)(89) (15)(38)(79)
62	123 478 596	234 579 608	130 457 896	129 407 658	128 365 709	126 374 908	125 379 840	124 389 506	127 390 465	120 348 657	8	(15)(37)(40) (1652)(4809)
63	123 478 596	234 579 608	130 475 968	127 458 609	129 365 708	126 374 809	125 308 947	124 389 605	120 397 456	128 364 570	8	(39)(40)(78) (26)(37)(89) (15)(38)(79)
64	123 478 596	234 579 608	130 475 968	128 459 607	129 365 708	126 374 809	125 308 947	124 389 560	120 397 645	127 305 486	8	(39)(40)(78) (26)(37)(89) (15)(37)(40)(89)
65	123 478 596	234 570 698	134 508 976	124 506 789	120 376 859	129 304 768	125 308 947	128 359 640	127 390 546	126 387 045	4	(18)(30)(45)(69) (14)(27)(39)(58)(60)

Table 3: (cont'd) The 77 non-isomorphic overlarge sets of $10\times 2-(9,3,1)$ designs, and their automorphism groups. Design D_i is based on the set $\{0,1,\ldots,9\}\backslash\{i\}$.

#	D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	|G|	Group Generators
66	123 478 596	234 570 698	134 508 976	129 405 768	126 350 798	124 378 069	125 308 947	128 359 640	127 390 546	120 376 845	8	(34)(50)(69) (1679)(2580)
67	123 478 596	234 570 698	134 568 079	127 468 905	126 350 798	120 378 649	129 354 807	124 398 065	125 374 960	128 307 546	8	(13)(25)(46)(78)(90) (14)(25)(38)(69)(70)
68	123 478 596	234 570 698	134 568 079	128 490 765	127 389 506	120 378 649	129 354 807	126 384 095	125 374 960	124 305 768	4	(1806)(3497)
69	123 478 596	234 570 698	136 450 798	126 475 908	129 378 065	128 367 490	125 308 947	120 356 849	127 390 546	124 305 768	8	(2679)(3054) (18)(23)(46)(57)(90)
70	123 478 596	234 570 698	136 450 798	126 475 908	120 376 859	129 304 768	125 308 947	124 398 065	127 390 546	128 365 470	10	(12)(38)(45)(67)(90) (10573)(28649)
71	123 478 596	234 570 698	136 457 890	127 468 905	126 350 798	120 367 489	129 354 807	124 398 065	125 374 960	128 307 546	8	(13)(25)(46)(78)(90) (14)(25)(38)(69)(70)
72	123 478 596	234 570 698	137 490 856	124 590 867	125 378 960	120 348 679	128 350 497	129 354 806	127 396 045	126 304 578	20	(12)(35)(40)(67)(89) (24)(58)(60)(79)
73	123 478 596	234 570 698	138 457 690	125 460 789	129 378 065	127 346 908	120 384 597	126 398 450	124 350 679	128 354 706	8	(1042)(5869) (16)(29)(37)(45)(80)
74	123 478 596	234 570 698	138 457 690	128 490 765	129 378 065	120 348 679	125 308 947	126 398 450	127 390 546	124 305 768	40	(1623)(49)(5780) (27)(35)(40)(69) (1346287905)
75	123 478 596	234 570 698	138 457 690	129 405 768	127 389 506	120 364 789	125 374 908	126 384 095	124 309 657	128 365 470	8	(1903)(4876) (1407)(25)(3698)
76	123 478 596	234 507 689	134 579 608	120 456 897	129 365 708	126 348 907	125 374 809	128 364 590	127 345 096	124 387 605	32	(12)(56)(78) (34)(56)(78)(90) (1329)(4506)
77	123 478 596	230 469 758	130 498 765	120 467 895	127 309 865	129 307 684	128 347 509	124 350 968	126 359 407	125 340 786	1440	(23)(69)(78) (132)(467589) (12)(34)(58)(69)(70)

Table 3: (cont'd) The 77 non-isomorphic overlarge sets of $10\times 2-(9,3,1)$ designs, and their automorphism groups. Design D_i is based on the set $\{0,1,\ldots,9\}\backslash\{i\}$.

Group Order	Number of Overlarge Sets
1	2
2	25
4	14
6	1
8	23
10	2
16	1
20	1
24	2
32	3
40	1
48	1
1440	1
	77

Table 4: The possible orders of the automorphism groups of overlarge sets of $2-(9,3,1)$ designs, and the number of non-isomorphic overlarge sets with groups of each order.

Computation.

Backtrack searches and the *nauty* software package [**10**] were used in computation, which was carried out on the Pyramid 9810 in the Department of Mathematics, University of Queensland. The number of large sets in each isomorphism class, as found by the backtrack search, was cross-checked against the number expected from the order of the automorphism group, as explained in [**14**].

Acknowledgements.

The work was sponsored by the Australian Research Grants Scheme, and by a University of Queensland Special Project Grant.

#	$\mathcal{E}(2)$	$\mathcal{E}(4)$	$\mathcal{F}(6)$	$\mathcal{G}(6)$	$\mathcal{E}(7)$	$\mathcal{F}(8)$	$\mathcal{G}(8)$
1	17	4	24	0	0	0	0
2	29	0	16	0	0	0	0
3	17	0	24	0	0	0	4
4	17	0	24	2	2	0	0
5	9	0	24	4	4	0	4
6	9	0	24	2	2	0	8
7	9	4	24	4	4	0	0
8	9	8	24	0	0	0	4
9	9	0	24	6	6	0	0
10	5	0	16	8	8	0	8
11	5	0	16	6	6	0	12
12	5	4	16	10	10	0	0
13	9	0	24	6	6	0	0
14	1	4	12	8	2	12	6
15	1	0	12	4	6	16	6
16	9	4	24	0	0	0	8
17	5	4	16	4	4	0	12
18	1	8	12	8	4	12	0
19	1	12	12	8	8	0	4
20	1	4	12	16	0	8	4
21	1	12	8	8	0	0	16
22	1	12	8	8	0	0	16
23	9	12	24	0	0	0	0
24	5	16	16	0	0	0	8
25	5	8	16	8	8	0	0
26	0	7	5	11	3	9	10
27	1	6	4	4	12	12	6
28	1	6	4	4	4	20	6
29	5	0	16	10	10	0	4
30	5	8	16	6	6	0	4
31	5	0	16	12	12	0	0
32	1	0	8	4	12	0	20
33	0	2	4	3	6	20	10
34	1	0	4	4	4	20	12
35	3	14	4	0	12	8	4
36	3	14	4	0	4	16	4
37	0	4	10	5	8	10	8
38	0	2	8	3	4	8	20
39	0	0	12	2	6	14	11

Table 5: The number of feasible pairs of $2-(9,3,1)$ designs of each type contained in the overlarge sets of $2-(9,3,1)$ designs.

#	$\mathcal{E}(2)$	$\mathcal{E}(4)$	$\mathcal{F}(6)$	$\mathcal{G}(6)$	$\mathcal{E}(7)$	$\mathcal{F}(8)$	$\mathcal{G}(8)$
40	1	0	20	4	4	12	4
41	5	0	12	4	4	12	8
42	5	0	12	4	4	12	8
43	5	0	8	6	6	8	12
44	3	10	8	0	14	8	2
45	3	10	8	0	6	16	2
46	0	4	12	6	2	10	11
47	1	2	4	4	4	24	6
48	1	6	4	4	8	16	6
49	1	0	8	4	4	8	20
50	5	0	16	4	4	0	16
51	0	2	8	4	4	10	17
52	0	0	10	10	10	0	15
53	1	4	4	16	8	0	12
54	1	8	4	8	4	12	8
55	1	2	2	6	10	14	10
56	0	4	4	4	10	14	9
57	1	4	2	10	2	14	12
58	1	4	4	8	2	12	14
59	5	8	16	0	0	0	16
60	3	6	12	0	24	0	0
61	3	6	12	0	0	24	0
62	1	12	4	8	0	8	12
63	7	10	4	0	16	8	0
64	7	10	4	0	8	16	0
65	5	8	0	14	14	0	4
66	5	8	0	8	8	0	16
67	1	8	0	0	12	8	16
68	1	4	0	8	8	24	0
69	5	0	0	12	12	0	16
70	5	0	0	10	10	0	20
71	1	0	0	0	4	24	16
72	5	0	0	10	10	0	20
73	1	0	0	4	20	8	12
74	5	0	0	20	20	0	0
75	1	0	0	4	12	16	12
76	5	24	0	0	0	0	16
77	45	0	0	0	0	0	0

Table 5: (cont'd) The number of feasible pairs of $2-(9,3,1)$ designs of each type contained in the overlarge sets of $2-(9,3,1)$ designs.

	D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9
D_0	*	$\mathcal{F}(6)$	$\mathcal{F}(6)$	$\mathcal{E}(4)$	$\mathcal{E}(4)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{G}(6)$	$\mathcal{G}(6)$	$\mathcal{E}(4)$
D_1	$\mathcal{F}(6)$	*	$\mathcal{E}(2)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{F}(6)$	$\mathcal{F}(6)$	$\mathcal{F}(6)$
D_2	$\mathcal{F}(6)$	$\mathcal{E}(2)$	*	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{F}(6)$	$\mathcal{F}(6)$	$\mathcal{F}(6)$
D_3	$\mathcal{E}(4)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	*	$\mathcal{E}(4)$	$\mathcal{G}(6)$	$\mathcal{G}(6)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{E}(4)$
D_4	$\mathcal{E}(4)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{E}(4)$	*	$\mathcal{G}(6)$	$\mathcal{G}(6)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{E}(4)$
D_5	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{G}(6)$	$\mathcal{G}(6)$	*	$\mathcal{E}(4)$	$\mathcal{E}(4)$	$\mathcal{E}(4)$	$\mathcal{G}(8)$
D_6	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{G}(6)$	$\mathcal{G}(6)$	$\mathcal{E}(4)$	*	$\mathcal{E}(4)$	$\mathcal{E}(4)$	$\mathcal{G}(8)$
D_7	$\mathcal{G}(6)$	$\mathcal{F}(6)$	$\mathcal{F}(6)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{E}(4)$	$\mathcal{E}(4)$	*	$\mathcal{E}(4)$	$\mathcal{G}(6)$
D_8	$\mathcal{G}(6)$	$\mathcal{F}(6)$	$\mathcal{F}(6)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{E}(4)$	$\mathcal{E}(4)$	$\mathcal{E}(4)$	*	$\mathcal{G}(6)$
D_9	$\mathcal{E}(4)$	$\mathcal{F}(6)$	$\mathcal{F}(6)$	$\mathcal{E}(4)$	$\mathcal{E}(4)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{G}(6)$	$\mathcal{G}(6)$	*

(a) Partition 21

	D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9
D_0	*	$\mathcal{F}(6)$	$\mathcal{F}(6)$	$\mathcal{E}(4)$	$\mathcal{E}(4)$	$\mathcal{G}(6)$	$\mathcal{G}(6)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{E}(4)$
D_1	$\mathcal{F}(6)$	*	$\mathcal{E}(2)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{F}(6)$	$\mathcal{F}(6)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{F}(6)$
D_2	$\mathcal{F}(6)$	$\mathcal{E}(2)$	*	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{F}(6)$	$\mathcal{F}(6)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{F}(6)$
D_3	$\mathcal{E}(4)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	*	$\mathcal{E}(4)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{G}(6)$	$\mathcal{G}(6)$	$\mathcal{E}(4)$
D_4	$\mathcal{E}(4)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{E}(4)$	*	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{G}(6)$	$\mathcal{G}(6)$	$\mathcal{E}(4)$
D_5	$\mathcal{G}(6)$	$\mathcal{F}(6)$	$\mathcal{F}(6)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	*	$\mathcal{E}(4)$	$\mathcal{E}(4)$	$\mathcal{E}(4)$	$\mathcal{G}(6)$
D_6	$\mathcal{G}(6)$	$\mathcal{F}(6)$	$\mathcal{F}(6)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{E}(4)$	*	$\mathcal{E}(4)$	$\mathcal{E}(4)$	$\mathcal{G}(6)$
D_7	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{G}(6)$	$\mathcal{G}(6)$	$\mathcal{E}(4)$	$\mathcal{E}(4)$	*	$\mathcal{E}(4)$	$\mathcal{G}(8)$
D_8	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{G}(6)$	$\mathcal{G}(6)$	$\mathcal{E}(4)$	$\mathcal{E}(4)$	$\mathcal{E}(4)$	*	$\mathcal{G}(8)$
D_9	$\mathcal{E}(4)$	$\mathcal{F}(6)$	$\mathcal{F}(6)$	$\mathcal{E}(4)$	$\mathcal{E}(4)$	$\mathcal{G}(6)$	$\mathcal{G}(6)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	*

(b) Partition 22

Table 6: Relationships between pairs of $2-(9,3,1)$ designs contained in partitions 21 and 22.

	D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9
D_0	*	$\mathcal{F}(6)$	$\mathcal{F}(6)$	$\mathcal{F}(8)$	$\mathcal{G}(6)$	$\mathcal{F}(6)$	$\mathcal{F}(8)$	$\mathcal{F}(8)$	$\mathcal{E}(2)$	$\mathcal{G}(8)$
D_1	$\mathcal{F}(6)$	*	$\mathcal{E}(7)$	$\mathcal{G}(8)$	$\mathcal{F}(6)$	$\mathcal{G}(6)$	$\mathcal{G}(8)$	$\mathcal{E}(2)$	$\mathcal{F}(8)$	$\mathcal{F}(8)$
D_2	$\mathcal{F}(6)$	$\mathcal{E}(7)$	*	$\mathcal{E}(2)$	$\mathcal{F}(6)$	$\mathcal{E}(7)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{F}(8)$	$\mathcal{F}(8)$
D_3	$\mathcal{F}(8)$	$\mathcal{G}(8)$	$\mathcal{E}(2)$	*	$\mathcal{F}(8)$	$\mathcal{G}(8)$	$\mathcal{E}(7)$	$\mathcal{E}(7)$	$\mathcal{F}(6)$	$\mathcal{F}(6)$
D_4	$\mathcal{G}(6)$	$\mathcal{F}(6)$	$\mathcal{F}(6)$	$\mathcal{F}(8)$	*	$\mathcal{F}(6)$	$\mathcal{F}(8)$	$\mathcal{F}(8)$	$\mathcal{G}(8)$	$\mathcal{E}(2)$
D_5	$\mathcal{F}(6)$	$\mathcal{G}(6)$	$\mathcal{E}(7)$	$\mathcal{G}(8)$	$\mathcal{F}(6)$	*	$\mathcal{E}(2)$	$\mathcal{G}(8)$	$\mathcal{F}(8)$	$\mathcal{F}(8)$
D_6	$\mathcal{F}(8)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$	$\mathcal{E}(7)$	$\mathcal{F}(8)$	$\mathcal{E}(2)$	*	$\mathcal{G}(6)$	$\mathcal{F}(6)$	$\mathcal{F}(6)$
D_7	$\mathcal{F}(8)$	$\mathcal{E}(2)$	$\mathcal{G}(8)$	$\mathcal{E}(7)$	$\mathcal{F}(8)$	$\mathcal{G}(8)$	$\mathcal{G}(6)$	*	$\mathcal{F}(6)$	$\mathcal{F}(6)$
D_8	$\mathcal{E}(2)$	$\mathcal{F}(8)$	$\mathcal{F}(8)$	$\mathcal{F}(6)$	$\mathcal{G}(8)$	$\mathcal{F}(8)$	$\mathcal{F}(6)$	$\mathcal{F}(6)$	*	$\mathcal{G}(6)$
D_9	$\mathcal{G}(8)$	$\mathcal{F}(8)$	$\mathcal{F}(8)$	$\mathcal{F}(6)$	$\mathcal{E}(2)$	$\mathcal{F}(8)$	$\mathcal{F}(6)$	$\mathcal{F}(6)$	$\mathcal{G}(6)$	*

(a) Partition 41

	D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9
D_0	*	$\mathcal{F}(6)$	$\mathcal{F}(6)$	$\mathcal{F}(8)$	$\mathcal{G}(6)$	$\mathcal{G}(8)$	$\mathcal{E}(7)$	$\mathcal{F}(8)$	$\mathcal{E}(2)$	$\mathcal{G}(8)$
D_1	$\mathcal{F}(6)$	*	$\mathcal{E}(7)$	$\mathcal{G}(8)$	$\mathcal{F}(6)$	$\mathcal{F}(8)$	$\mathcal{F}(6)$	$\mathcal{E}(2)$	$\mathcal{F}(8)$	$\mathcal{F}(8)$
D_2	$\mathcal{F}(6)$	$\mathcal{E}(7)$	*	$\mathcal{E}(2)$	$\mathcal{F}(6)$	$\mathcal{F}(8)$	$\mathcal{F}(6)$	$\mathcal{G}(8)$	$\mathcal{F}(8)$	$\mathcal{F}(8)$
D_3	$\mathcal{F}(8)$	$\mathcal{G}(8)$	$\mathcal{E}(2)$	*	$\mathcal{F}(8)$	$\mathcal{F}(6)$	$\mathcal{F}(8)$	$\mathcal{E}(7)$	$\mathcal{F}(6)$	$\mathcal{F}(6)$
D_4	$\mathcal{G}(6)$	$\mathcal{F}(6)$	$\mathcal{F}(6)$	$\mathcal{F}(8)$	*	$\mathcal{G}(8)$	$\mathcal{G}(6)$	$\mathcal{F}(8)$	$\mathcal{G}(8)$	$\mathcal{E}(2)$
D_5	$\mathcal{G}(8)$	$\mathcal{F}(8)$	$\mathcal{F}(8)$	$\mathcal{F}(6)$	$\mathcal{G}(8)$	*	$\mathcal{E}(2)$	$\mathcal{F}(6)$	$\mathcal{E}(7)$	$\mathcal{G}(6)$
D_6	$\mathcal{E}(7)$	$\mathcal{F}(6)$	$\mathcal{F}(6)$	$\mathcal{F}(8)$	$\mathcal{G}(6)$	$\mathcal{E}(2)$	*	$\mathcal{F}(8)$	$\mathcal{G}(8)$	$\mathcal{G}(8)$
D_7	$\mathcal{F}(8)$	$\mathcal{E}(2)$	$\mathcal{G}(8)$	$\mathcal{E}(7)$	$\mathcal{F}(8)$	$\mathcal{F}(6)$	$\mathcal{F}(8)$	*	$\mathcal{F}(6)$	$\mathcal{F}(6)$
D_8	$\mathcal{E}(2)$	$\mathcal{F}(8)$	$\mathcal{F}(8)$	$\mathcal{F}(6)$	$\mathcal{G}(8)$	$\mathcal{E}(7)$	$\mathcal{G}(8)$	$\mathcal{F}(6)$	*	$\mathcal{G}(6)$
D_9	$\mathcal{G}(8)$	$\mathcal{F}(8)$	$\mathcal{F}(8)$	$\mathcal{F}(6)$	$\mathcal{E}(2)$	$\mathcal{G}(6)$	$\mathcal{G}(8)$	$\mathcal{F}(6)$	$\mathcal{G}(6)$	*

(b) Partition 42

Table 7: Relationships between pairs of $2-(9,3,1)$ designs contained in partitions 41 and 42.

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