THE EXACT COVERING OF PAIRS ON NINETEEN POINTS WITH BLOCK SIZES TWO, THREE, AND FOUR

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Abstract. The minimum cardinality of a pairwise balanced design on nineteen points is determined; a minimal design is exhibited containing 13 triples and 22 quadruples.

1. Introduction.

The quantity $g^{(4)}(v)$ is the minimum number of blocks required to cover exactly all pairs from a set of v elements using blocks of sizes 2, 3, and 4 (see, for example, [1] and [2]). In other words, $g^{(4)}(v)$ is the minimum cardinality of a pairwise balanced design on v points in which the block sizes are restricted to the set {2,3,4}. This quantity was determined in [3] for all values of the parameter v with the exception of 17, 18, 19. In this paper, we shall discuss the case v = 19.

It is not difficult to construct an exact covering of all 171 pairs from 19 elements by a set of 36 blocks. We take a single block 1234 and then take the seven resolutions R_i (i = 1, ... ,7) of a Kirkman triple system on 15 points. We adjoin i to each triple of R_i (i = 1,2,3,4); this gives twenty additional quadruples for a total of 21 quadruples. The remaining blocks are the 15 triples from R_5, R_6, R_7. Clearly this covering is exact; so we have

Theorem 1. $g^{(4)}(19) \le 36$.

The upper bound provided by the covering obtained in Theorem 1 will be very useful in our discussion; however, we must now investigate whether this covering is optimal.

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2. Coverings with No Blocks of Length 2.

Suppose that we have an exact covering with b_i blocks of length i (i = 2,3,4) and a total of B blocks. We know that $b_4 \leq 25$, since Stinson [4] has shown that the packing number D(2,4,19) = 25. From the relations

(1) $b_2 + b_3 + b_4 = B$,

(2) $b_2 + 3b_3 + 6b_4 = 171$,

we obtain

(3) $5b_2 + 3b_3 = 6B - 171$.

If $b_2 = 0$, there are 4 solutions of (3) with B less than 36, namely,

Case (1) $B = 35, b_3 = 13, b_4 = 22$;

Case (2) $B = 34, b_3 = 11, b_4 = 23$;

Case (3) $B = 33, b_3 = 9, b_4 = 24$;

Case (4) $B = 32, b_3 = 7, b_4 = 25$.

There are two solutions of (3) with $b_2 = 3$, namely,

Case (5) $B = 35, b_2 = 3, b_3 = 8, b_4 = 24$;

Case (6) $B = 34, b_2 = 3, b_3 = 6, b_4 = 25$.

There are no solutions of (3) with $b_2 > 3$, $b_4 < 26$, $B < 36$. In this section we shall consider the first 4 cases, which involve no blocks of length 2. We let the element i (i = 1, ... ,19) have frequency q_i in the quadruples and frequency t_i in the triples. Then we have

$$3q_i + 2t_i = 18,$$

and the only solutions for (q_i,t_i) are (6,0), (4,3), (2,6), (0,9). Suppose that there are a elements with the pattern (6,0), b elements with the pattern (4,3), c elements with the pattern (2,6), and d elements with the pattern (0,9). Then we have the equations

$$a + b + c + d = 19,$$

$$6a + 4b + 2c = 4b_4,$$

and can deduce

Lemma 1. There are no elements with pattern (0,9), that is, $d = 0$.

Proof. If $d > 0$, there is an element x that appears in 9 triples (this immediately rules out Case (4) in which there are not 9 triples). In Cases 1,2,3, the element x must appear with all 18 other elements in these 9 triples; hence $a = 0$. Also $c = 0$, since any element y that occurred in 6 triples would be forced to have a repeat xy (the number of triples is at most 13 in all three cases). Also, $d = 1$ (if $d > 1$, there would be a repeated pair in the triples), and thus we have the solution $d = 1, b = 18$. It follows that $b = b_4 = 18$, and this is a contradiction.

We now suppose that $d = 0$, $c > 0$; then there is an element x that occurs 6 times in the triples, and it must therefore occur with 12 other elements in the triples. These 12 elements can not be "a" elements with the pattern (6,0). Hence we must have

$$b + c \geq 1 + 12 = 13.$$

From the equations

$$a + b + c = 19,$$

$$6a + 4b + 2c = 4b_4,$$

we deduce that

$$b + 2c = 57 - 2b_4.$$

In Case (4), $b + 2c = 7$ (impossible, since $b + c \geq 13$); in Case (3), $b + 2c = 9$, and this is impossible; similarly, in Case (2) $b + 2c = 11$, and this is also impossible. In Case (1), $b + 2c = 13$; then $b + c = 13$, and so c must be equal to 0 (this contradicts the fact that the element x occurs in the triples). We thus have established

Lemma 2. There are no elements with pattern (2,6), that is,

Cases 1, 2, 3, 4, now satisfy the relations

$$a + b = 19, 3a + 2b = 2b_4.$$

Thus $$a = 2b_4 - 38, \; b = 57 - 2b_4.$$

In Case (4), there are 7 triples and 7 "b" elements. So the 7 triples form a Fano Geometry. But each of these 7 elements occurs 4 times in the quadruples; since all pairs from these 7 elements already appear in the triples, there are at least 28 quadruples. This contradiction shows that Case (4) does not occur and so we have established

Theorem 2. $g^{(4)}(19) > 32.$

In Case (3) there are nine "b" elements occurring in the nine triples. So the quadruples contain ten "a" elements having frequency 6 and nine "b" elements having frequency 4. We suppose there are u quadruples of the form aaaa, w quadruples of the form aaab, x quadruples of the form aabb, y quadruples of the form abbb, and z quadruples of the form bbbb. Counting of the pairs aa, ab, and bb produces the relations

$$u + w + x + y + z = 24,$$

$$6u + 3w + x = 45,$$

$$x + 3y + 6z = 36 - 27 = 9,$$

$$3w + 4x + 3y = (10)(9) = 90.$$

The third equation shows that $x = 9 - e$, where e is not negative; then the fourth equation shows that $w + y = (54 + 4e)/3$. This gives a contradiction in the first equation. So we have

Theorem 3. Case (3) does not occur, that is, $g^{(4)}(19) > 33.$

In Case (2), there are 11 "b" elements; hence there are 55 - 33 = 22 pairs of "b" elements that occur in the quadruples. Proceed as before; we find that

$$u + w + x + y + z = 23, \qquad x + 3y + 6z = 22,$$

$$6u + 3w + x = 28, \qquad 3w + 4x + 3y = 88.$$

These equations do have solutions, namely,

Case (A): $x = 22, u = 1$;

Case (B): $x = 19, y = 1, w = 3$.

In Case (A), let the elements of pattern (6,0) be 1, 2, 3, 4, 5, 6, 7, 8; then there is a block 1234. But then 5 occurs in blocks containing 51, 52, 53, 54, 56, 57, 58 (a contradiction, since 5 has frequency 6).

In Case (B), $w = 3$. If any of the "a" elements occurs twice in these three "w" blocks, then it must appear in the block of the form abbb. Hence there can be only one repeated element in these three blocks, and they must thus have the form 123x, 145x, 678x. Look at the specific block 123p. The element p must occur with five more "a" elements in three blocks; this is not possible. Thus we have

Lemma 3. Case (2) does not occur, that is, we can not have a design in 34 blocks that comprises only quadruples and triples.

We finally consider Case (1) in which there are six "a" elements (1,2,3,4,5,6) and thirteen "b" elements (p,q,r,s,t,c,d,e,f,g,h,j,k). We proceed as before to obtain the equations

$$u + w + x + y + z = 22,$$

$$6u + 3w + x = 15,$$

$$x + 3y + 6z = 39,$$

$$3w + 4x + 3y = 78.$$

Again, there are two solutions of these equations, namely,

Case (A): $x = 15, w = 0, y = 6, z = 1$;

Case (B): $x = 12, w = 1, y = 9$.

In Case (A), there are 15 blocks containing two "a" elements and 6 blocks containing a single "a" element. Let the single "b" block be pqrs. Then the quadruples containing 1, 2, 3, 4, 5, 6, p, q, r, s, can be written as:

1, 2, 3, 4, 5, 6, 12p, 34p, 56p, 13q, 26q, 45q, 14r, 25r, 36r, 15s, 23s, 46s, 16, 24, 35, where we omit occurrences of the other letters in the quadruples.

The remaining nine "b" elements are t, c, d, e, f, g, h, j, k, and each must occur in four quadruples. Hence each must occur in two quadruples of the form aabb and in two quadruples of the form abbb, and there are six quadruples of this latter type.

Look at these last six quadruples as regards the b triples. The triples take the form tcd, txx, cxx, dxx, xxx, xxx. There are only two possible completions for these triples. These are:

(a) tcd, tef, cgh, djk, egj, fhk;

(b) tcd, tef, ceg, dhj, khf, kjg.

In Case (a), these six blocks form two resolutions. Since the block containing 16 (a similar argument holds for 24 and 35) contains two elements not in 1xxx and 6xxx, we see that the blocks 1xxx and 6xxx must contain triples from different resolution classes. Hence the triples may be taken as:

(7) 1tcd, 6tef, 2egj, 4cgh, 3fhk, 5djk.

The quadruples that contain 16, 24, and 35 may now be considered. In order to avoid repeated pairs, we must the following selections:

16hj or 16gk; 24tk or 24df; 35ce or 35tg.

This results in eight possible assignments for the blocks that contain 16, 24, and 35. Of these assignments, seven immediately produce contradictions and can not be completed. The exception is the asssignment 16hj, 24df, 35ce. It is somewhat astonishing that this case can be completed, but this is easily done in a unique fashion to produce a pairwise balanced design The easiest procedure is to draw six intersecting lines in the plane and label these lines as 1, 2, 3, 4, 5, 6. Their fifteen points of intersection are then represented by the pairs 12, 13, ... , 56. Each line is assigned a triple as in the list (7) previously given; when we assign hj to 16, df to 24, and ce to 35, we find that the other letters can be strung out along the six lines in only one way (of course, all nine letters must appear once and only once along each line). The resulting set of quadruples is then

pqrs; 1tcd, 2egj, 3fhk, 4cgh, 5djk, 6tef; 12pk, 13qg, 14re, 15sf, 16hj, 23st, 24df, 25rh, 26qc, 34pj, 35ce, 36rd, 45qt, 46sk, 56pg.

If one lists the pairs not covered by these quadruples, they fall into the following set of thirteen triples.

pth, pcf, pde, qfj, qhd, qke, rtj, rck, rfg, seh, sgd, sjc, tgk.

The block tgk plays a special role, since it is the only triple that does not meet the quadruple pqrs.

At this time,we are not attempting to determine all non-isomorphic solutions of the problem; hence the fact that we now have a design in 35 blocks relieves us of the need to discuss Case (5) at all. Also, we do not need to complete the discussion of Case (1). We may thus state

Theorem 4. $g^{(4)}(19) \leq 35$.

In order to establish that $g^{(4)}(19) = 35$, we shall have to exclude Case 6 in which there are 3 pairs, 6 triples, and 25 quadruples.

3. Discussion of Case 6.

We let q_i, t_i, and p_i be the frequencies with which a particular element occurs in the quadruples, triples, and pairs, respectively. Then we have

$$3q_i + 2t_i + p_i = 18.$$

The solutions for (q_i,t_i,p_i) in which $t_i \leq 6$, $p_i \leq 3$, are (5,0,3), (3,3,3), (1,6,3), (4,2,2), (2,5,2), (5,1,1), (3,4,1), (6,0,0), (4,3,0), (2,6,0). Several of these possible patterns can be excluded at once.

Lemma 4. Patterns (2,6,0) and (1,6,3) are not possible.

Proof. If element 1 has pattern (2,6,0), then there are six blocks of the form 1xx. The twelve elements that occur with 1 in these blocks must all have pattern (5,1,1). This means that they must all occur in the pairs, and there is room for at most six elements in the pairs.

If element 1 has pattern (1,6,3), there are again six triples 1xx; however, in this case, the three pairs have the form 1x as well. Any element that occurs with 1 in the triples must again have pattern (5,1,1), and this gives an immediate repeat in the pairs.

Lemma 5. Pattern (2,5,2) is not possible.

Proof. In this case, we have pairs 1x, 1x, xx, and triples 1xx (five times) and xxx. Consider the triple 1rs. The element r can not have the pattern (4,2,2) or we would have a repeat of 1r in the pairs. Hence the element r has pattern

(5,1,1). Thus we must have the pairs 1x, 1x, rx. Now the element s must also have the pattern (5,1,1), and this leads to a repeat in the pairs (either on 1s or on rs).

Lemma 6. Pattern (3,4,1) is not possible.

Proof. In this case, we have blocks 1x, xx, xx, 1xx (4 times), xxx (twice). Consider a block 1rs.

If r has pattern (4,2,2), then the blocks can be written as 1x, rx, rx, 1rs, 1xx (three times), rxx, xxx; then s can not appear in the pairs and so s must have pattern (4,3,0); but this gives an immediate repeat of 1s in the triples. So the only possible patterns for r are (5,1,1) and (4,3,0).

If r has pattern (4,3,0), then the triples may be taken as 1rs, 1tu, 1vw, 1xy, rxx, rxx, and the pairs are 1x, sx, xx. Clearly s, t, u, v, w, x, y, all have pattern (5,1,1) or there would be a repeat in the triples. But then there is no room in the pairs for all of these elements.

We conclude that all the eight elements that occur with 1 in the triples must have pattern (5,1,1). This is again impossible, since at most six distinct elements can appear in the pairs. This completes the proof of Lemma 6.

Lemma 7. The pattern (3,3,3) is not possible.

Proof. Suppose there is an element 1 with pattern (3,3,3). Let the triples be 1rs, 1xx, 1xx, xxx, xxx, xxx. Clearly both r and s must have pattern (4,3,0). This forces another occurrence of rs in the triples, and so we can reject the pattern (3,3,3).

We thus have to consider only the patterns (5,0,3), (4,2,2), (5,1,1), (6,0,0), and (4,3,0). We suppose that the frequencies of these patterns are u, w, x, y, and z, respectively. Then we have

$$u + w + x + y + z = 19,$$

$$2w + x + 3z = 18,$$

$$3u + 2w + x = 6.$$

In order to avoid a repeated pair, we must have u = 1 or 0. If u = 1, we must have w = 0 (or there would be a repeated pair). So we have the possible solution u = 1, w = 0, x = 3, z = 5, y = 10.

However, if $z = 5$, we have 5 elements that appear in the six triples three times each. This is not possible without having a repeated pair. So we have

Lemma 8. Pattern (5,0,3) can not occur.

The other possibility is when $u = 0$. Then $2w + x = 6$, $z = 4$. If $z = 4$, the six triples must take the form 12x, 13x, 14x, 23x, 24x, 34x; consequently, the quadruples have the form 1xxx, 2xxx, 3xxx, 4xxx (six times each), and one other block that does not contain 1, 2, 3, or 4. If any element in the triples has pattern (4,2,2), it must be an element r that occurs in two blocks of the form (say) 12r, 34r; hence it is not able to appear the necessary 4 times in the quadruples. Consequently, there are six elements in the triples that have pattern (5,1,1). But then, if we take a triple 12r, we find that r can appear in only three quadruples (with 3, with 4, and with none of 1, 2, 3, 4). Hence we must reject this possibility also, and thus have

Theorem 5. Case (6) can not occur, that is, $g^{(4)}(19) > 34$.

We may now combine Theorems 4 and 5 to obtain the final result, namely,

Theorem 6. $g^{(4)}(19) = 35$.

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