Realizability of p-Vertex, q-Edge Graphs with Prescribed Vertex Connectivity and Minimum Degree

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ABSTRACT. It is an established fact that some graph-theoretic extremal questions play an important part in the investigation of communication network vulnerability. Questions concerning the realizability of graph invariants are generalizations of the extremal problems. We define a (p, q, κ, δ) graph as a graph having p vertices, q edges, vertex connectivity κ and minimum degree δ. An arbitrary quadruple of integers (a, b, c, d) is called (p, q, κ, δ) realizable if there is a (p, q, κ, δ) graph with $p = a, q = b, \kappa = c$ and $\delta = d$. Necessary and sufficient conditions for a quadruple to be (p, q, κ, δ) realizable are derived. In earlier papers, Boesch and Suffel gave necessary and sufficient conditions for $(p, q, \kappa), (p, q, \lambda), (p, q, \delta), (p, \Delta, \delta, \lambda)$ and $(p, \Delta, \delta, \kappa)$ realizability, where Δ denotes the maximum degree for all vertices in a graph and λ denotes the edge connectivity of a graph.

1 Introduction

Here we consider an undirected graph $G = (V, X)$ with a finite vertex set V and a set X whose elements, called edges, are two vertex subsets of V. The number of vertices is denoted by p, and the number of edges $|X|$ is denoted by $q(G)$ or q. This paper uses the notation and terminology of Harary [14]; however a few basic concepts are now reproduced.

The edge connectivity of a graph G (denoted by $\lambda(G)$ or λ) is the minimum number of edges whose removal results in a disconnected graph. A graph is called trivial if it has just one vertex. The vertex connectivity (denoted by $\kappa(G)$ or κ) is the minimum number of vertices whose removal results in a disconnected or trivial graph. The number of edges connected to a vertex v of G is the degree of that vertex, denoted by $d_v(G)$ or d_v. The minimum degree is denoted by δ or $\delta(G)$ and the maximum degree is denoted by Δ. If $\delta = \Delta$, the graph is called regular. A p vertex graph with

$\delta = p-1$ is called complete and is denoted by K_p. A set of κ vertices whose removal disconnects G, or makes G trivial, is called a minimum vertex disconnecting set. The graph obtained from C_p (the cycle on p vertices) by adding edges between all pairs of vertices that are distance at least two but not greater than A apart is denoted by C_p^A.

It is an established fact that some graph-theoretic extremal questions play an important part in the investigation of communication network vulnerability [1-13]. Harary [15] found the maximum vertex connectivity among all graphs with a given number of vertices and a given number of edges. Questions concerning the realizability of graph invariants are generalizations of these extremal problems. We define a (p, q, κ, δ) graph as a graph having p vertices, q edges, vertex connectivity κ and minimum degree δ. An arbitrary quadruple of integers (a, b, c, d) is called (p, q, κ, δ) realizable if there is a (p, q, κ, δ) graph with $p = a, q = b, \kappa = c$ and $\delta = d$. Necessary and sufficient conditions for a quadruple to be (p, q, κ, δ) realizable (or, more briefly, realizable) are derived. Boesch and Suffel derived necessary and sufficient conditions for $(p, q, \kappa), (p, q, \lambda), (p, q, \delta), (p, \Delta, \delta, \lambda)$ and $(p, \Delta, \delta, \kappa)$ realizability in earlier papers [6-8].

2 Preliminaries

We start by reviewing some known results that are pertinent to the realizability question.

Lemma 1. [7] *If a graph is not complete, and $p \geq 2$, then $p \geq 2\delta + 2 - \kappa$.*

Lemma 2. [15] *If $2 \leq \delta \leq p-1$, then there is a graph on p vertices with $q(G) = \left\lceil \frac{p\delta}{2} \right\rceil$ and $\lambda = \delta = \kappa$. (This graph is a power of cycle and is usually called the Harary graph on p vertices).*

We now give some new results.

Lemma 3. *For all graphs,*

$$q \leq \frac{1}{2}\delta(\delta+1) + \frac{1}{2}(p-\delta-1)(p-\delta-2) + \kappa(p-\delta-1).$$

Proof: If $\delta = p-1$,the result is obvious. Let G be a graph with $\delta < p-1$ and S be a minimum vertex disconnecting set of G. The vertex set of $G-S$ may be partitioned into two sets T and U such that no edges of $G-S$ join T and U. Let a be a vertex in T. Since $d_a(G) \geq \delta$, we have $|T|+|S| \geq \delta+1$, and similarly $|U|+|S| \geq \delta+1$. Therefore $|T| \geq \delta+1-\kappa$ and $|U| \geq \delta+1-\kappa$. Let $|T| = N$, thus $|U| = p - N - \kappa$. Noting that T union S has $N+\kappa$ vertices, it follows that

$$q(G) \leq \frac{1}{2}(N+\kappa)(N+\kappa-1) + \frac{1}{2}(p-N-\kappa)(p-N-\kappa-1) + \kappa(p-N-\kappa).$$

We wish to maximize the right side of the inequality on

$$\delta + 1 - \kappa \le N \le p - \delta - 1.$$

Since this quantity is a quadratic in N with a leading term of N^2, the maximum must take place at one of the bounds of the interval. It is easily verified that the value of the right side of the inequality is the same at each bound and the result follows.

Lemma 4. *If $\kappa = 1$ and $p = 2\delta + 2$, then $q > \left\lceil \frac{p\delta}{2} \right\rceil = \frac{p\delta}{2}$.*

Proof: Let G be a graph with $\kappa = 1$ and $p = 2\delta + 2$. Since G is not complete, there exists a vertex (denote it by a) whose removal disconnects G. Thus the vertex set of $G - a$ may be partitioned into two sets T and U such that no edges of $G - a$ join T and U. We note that $|T| \ge \delta$ and $|U| \ge \delta$. Thus, either $|T| = \delta$ or $|U| = \delta$. Suppose without loss of generality that $|T| = \delta$. Therefore, a must be adjacent to every vertex in T. Since G is connected, it follows that $d_a(G) \ge \delta + 1$. Thus $q(G) > \frac{p\delta}{2}$.

Lemma 5. *If $\kappa = 1$ and $\delta = 2$, then $q > p$.*

Proof: Since the only connected regular graph of degree two is a cycle, there is no graph with $\kappa = 1, \delta = 2$ and $q = \left\lceil \frac{p\delta}{2} \right\rceil = p$. The result follows.

Lemma 6. *For all graphs,*

$$q \ge \left\lceil \frac{p\delta}{2} + \frac{1}{2} \max(0, (p - \delta - 1)(2\delta + 2 - p) - \kappa(\kappa - 1)) \right\rceil.$$

Proof: We note that if $(p - \delta - 1)(2\delta + 2 - p) - \kappa(\kappa - 1) \le 0$ the result is obviously true. Henceforth, we will assume $(p - \delta - 1)(2\delta + 2 - p) - \kappa(\kappa - 1) > 0$. Since $p - \delta - 1 \ge 0$ and $\kappa(\kappa - 1) \ge 0$, then $p \le 2\delta + 1$. It also follows that $p \ge 3$ and $0 < \delta < p - 1$. Let G be a graph with $(p - \delta - 1)(2\delta + 2 - p) - \kappa(\kappa - 1) > 0$. Let S, T and U be defined as they were in the proof of Lemma 3 (we again have $|T| \ge \delta + 1 - \kappa$ and $|U| \ge \delta + 1 - \kappa$).

We wish to show that

$$q \ge \frac{1}{2}[p\delta + (p - \delta - 1)(2\delta + 2 - p) - \kappa(\kappa - 1)],$$

or equivalently,

$$\sum_{j \in G} d_j \ge p\delta + (p - \delta - 1)(2\delta + 2 - p) - \kappa(\kappa - 1).$$

Let V denote the union of T and U. Since $\Sigma_{j\in G}d_j = \Sigma_{j\in S}d_j + \Sigma_{j\in V}d_j$, we wish to find lower bounds for $\Sigma_{j\in S}d_j$ and $\Sigma_{j\in V}d_j$. Since $|V| = p - \kappa$, we have $\Sigma_{j\in V}d_j \geq (p-\kappa)\delta$. Note that

$$\sum_{j\in V} d_j(G-S) + \sum_{j\in S} d_j(G) \;\geq\; \sum_{j\in V} d_j(G) \;\geq\; (p-\kappa)\delta,$$

thus $\Sigma_{j\in S}d_j(G) \geq (p-\kappa)\delta - \Sigma_{j\in V}d_j(G-S)$. An argument similar to that used to prove Lemma 3 shows that

$$\sum_{j\in V} d_j(G-S) \leq (\delta-\kappa+1)(\delta-\kappa) + (p-\delta-1)(p-\delta-2),$$

and we have

$$\sum_{j\in S} d_j(G) \geq (p-\kappa)\delta - (\delta-\kappa+1)(\delta-\kappa) - (p-\delta-1)(p-\delta-2).$$

Therefore,

$$\begin{aligned}\sum_{j\in G} d_j(G) &= \sum_{j\in S} d_j(G) + \sum_{j\in V} d_j(G)\\ &\geq (p-\kappa)\delta - (\delta-\kappa+1)(\delta-\kappa)\\ &\quad - (p-\delta-1)(p-\delta-2) + (p-\kappa)\delta.\end{aligned}$$

Consequently, if we can show

$$\begin{aligned}2(p-\kappa)\delta - (\delta-\kappa+1)(\delta-\kappa) - (p-\delta-1)(p-\delta-2)\\ \geq p\delta + (p-\delta-1)(2\delta+2-p) - \kappa(\kappa-1)\end{aligned}$$

the proof will be finished. In fact, simplifying both sides shows they are equal and we are done.

The alert reader may wonder if Lemmas 2 and 6 are contradictory. However, since $\delta \geq 2$ and $\kappa = \delta$ imply

$$(p-\delta-1)(2\delta+2-p) - \kappa(\kappa-1) = (p-\delta-1)(2\delta+2-p) - \delta(\delta-1) \leq 0,$$

there is no contradiction. [To prove the above inequality consider each of the following four cases (details are left to the reader):

(1) $p \geq 2\delta + 2$,

(2) $p = 2\delta + 1$,

(3) $\delta + 1 < p < 2\delta + 1$ – for this case recall Lemma 1,

(4) $p = \delta + 1$.]

3 The (p, q, κ, δ) realizability theorem

Theorem . *A quadruple of non-negative integers (p, q, κ, δ) is realizable if and only if exactly one of the following conditions holds:*

(I) $\delta \le \lfloor \frac{1}{2}(p+\kappa-2) \rfloor$, *and if* $\kappa > 0$ *then* $q \ge p-1$.

(A) $1 = \kappa \le \delta$,

$$\lceil \frac{p\delta}{2} + \frac{1}{2}\max(0, (p-\delta-1)(2\delta+2-p)) \rceil \le q \le \frac{1}{2}\delta(\delta+1) + \frac{1}{2}(p-\delta)(p-\delta-1),$$

and if $p = 2\delta+2$ *or* $\delta = 2$, *then* $q > \frac{p\delta}{2}$.

(B) $1 \ne \kappa \le \delta$ *and*

$$\lceil \frac{p\delta}{2} + \frac{1}{2}\max(0, (p-\delta-1)(2\delta+2-p) - \kappa(\kappa-1)) \rceil \le q \le \frac{1}{2}\delta(\delta+1) + \frac{1}{2}(p-\delta-1)(p-\delta-2) + \kappa(p-\delta-1).$$

(II) $\delta = \kappa = p-1$ *and* $q = \frac{1}{2}p(p-1)$.

Proof: If a graph is not complete, the necessity of $\delta \le \lfloor \frac{1}{2}(p+\kappa-2) \rfloor$ in (I) follows from Lemma 1. The other conditions in (I) are a consequence of Lemmas 3, 4, 5 and 6, and some obvious facts about graphs. If a graph is complete the conditions in (II) are obvious.

We now provide constructions to prove sufficiency.

Case 1. Suppose that $p \ge 2\delta+2$, κ is even, $2 \le \kappa < \delta$, and

$$\left\lceil \frac{p\delta}{2} \right\rceil \le q \le \frac{1}{2}\delta(\delta+1) + \frac{1}{2}(p-\delta-1)(p-\delta-2) + \kappa(p-\delta-1).$$

Let H denote the Harary graph on $p-\delta-1$ vertices with $\lceil \frac{1}{2}(p-\delta-1)(\delta-1) \rceil$ edges and $\kappa(H) = \delta(H) = \delta-1$. Take the union of H and $K_{\delta+1}$ to form a single graph. We pause to note that H is not complete, and if both p and δ are even, then H has one vertex of degree δ. Let N denote the number of vertices of degree $\delta-1$ in H. Denote the vertices in $K_{\delta+1}$ by A and denote the vertices in H by B. Recall that $\kappa \ge 2$ and H is a power of cycle. Therefore if p and δ are not both odd it is possible to partition $N-\kappa$ of the vertices of degree $\delta-1$ in B into pairs, such that in each pair the two vertices are not adjacent. We now add an edge for each pair, joining the vertices that belong to the pair. If both p and δ are odd it is possible to partition $N-\kappa-3$ of the vertices of degree $\delta-1$ in B into nonadjacent pairs, with the following being true. We can find three vertices of degree

$\delta - 1$ in B, none of which are members of the $\frac{1}{2}(N - \kappa - 3)$ pairs mentioned above, such that at least one of the three vertices is not adjacent to either of the other two. Now add for each pair an edge joining the vertices that belong to that pair and add two edges which are incident to the three vertices mentioned above. Next add κ edges joining κ distinct vertices in A to the κ vertices of degree $\delta - 1$ in B. We now partition the vertices of degree $\delta + 1$ in A into pairs, and for each pair delete the edge joining the vertices that belong to that pair. Let C denote the set of vertices in A that were not previously partitioned into pairs. Adding $q - \left\lceil \frac{p\delta}{2} \right\rceil$ edges which do not join any vertices in C to any vertices in B yields the desired graph. Since $\kappa < \delta$ and we deleted an independent set of edges from $K_{\delta+1}$ our graph has the desired vertex connectivity. It is easily verified that our graph has the other desired properties. Thus any quadruple satisfying this case is realizable.

Case 2. Suppose that $p \geq 2\delta + 2, \kappa$ is odd, $3 \leq \kappa < \delta$, and

$$\left\lceil \frac{p\delta}{2} \right\rceil \leq q \leq \frac{1}{2}\delta(\delta+1) + \frac{1}{2}(p-\delta-1)(p-\delta-2) + \kappa(p-\delta-1).$$

This construction is similar to the one used for Case 1. Here we note only how the two cases differ. In Case 2 we will partition $N - (\kappa - 1)$ of the vertices of degree $\delta - 1$ in B into pairs. In addition, instead of adding κ edges joining vertices in A to vertices in B (as in Case 1), we wish to add $\kappa + 1$ edges joining $\kappa + 1$ distinct vertices in A to κ distinct vertices in B (denote these κ vertices by D) with the following being true. The vertex set D has the following properties: (1) D contains exactly one of the vertices in B that was partitioned into pairs (denote this vertex by v), (2) v is adjacent to exactly one vertex in A, and (3) the vertex in D which is adjacent to two vertices in A (denote this vertex in D by v_1) is adjacent to v. We pause to note that it is possible for D to have these properties because $\delta - 1 \geq 3$ and H (which is a power of cycle) has at most two vertices which cannot be included in a partitioning of vertices into nonadjacent pairs. Next we delete the edge $\{v, v_1\}$ and note that this edge can be replaced by a path containing vertices in A – giving us the appropriate vertex connectivity. To achieve the upper bound for q we delete one edge joining a vertex in A to v_1 and then add edges as we did in Case 1.

Case 3. Suppose that $p \geq 2\delta + 2$, $\kappa = 1$, $\delta \geq 3$,

$$\left\lceil \frac{p\delta}{2} \right\rceil \leq q \leq \frac{1}{2}\delta(\delta+1) + \frac{1}{2}(p-\delta)(p-\delta-1),$$

and if $p = 2\delta+2$, then $q > \frac{p\delta}{2}$. Let H_1 denote the Harary graph on $p-\delta-1$ vertices with $\lceil \frac{1}{2}(p-\delta-1)\delta \rceil$ edges and $\kappa(H_1) = \delta(H_1) = \delta$. Take the union of H_1 and $K_{\delta+1}$ to form a single graph. Denote the vertices in $K_{\delta+1}$

by E and denote the vertices in H_1 by F. If p and δ are both odd we pick a vertex in F which is adjacent to the vertex of degree $\delta+1$ in F and denote it by a. If p and δ are not both odd then let a denote an arbitrary vertex in F. Let e and f denote two vertices in E. Next we delete the edge $\{e, f\}$ and add the edges $\{a, f\}$ and $\{a, e\}$. If p and δ are both odd, then delete the edge joining a to the vertex of degree $\delta+1$ in F. But if $p = 2\delta+2$ then $q > \frac{p\delta}{2}$. Thus, if $p = 2\delta + 2$ our construction is finished except for adding edges to realize $q > \frac{p\delta}{2} + 1$. However if p and δ are not both odd, and if $p > 2\delta+2$, then we do the following. Recalling that H_1 is a power of cycle let c denote the vertex in F which follows a in the clockwise direction. Noting that H_1 is not complete, we let G denote the set of vertices in H_1 which are not adjacent to c. Let b denote the vertex in G which is closest to c in the counterclockwise direction. The vertices b and c are both adjacent to a. Next we delete the edges $\{a, c\}$ and $\{a, b\}$, and add the edge $\{b, c\}$. To complete our construction for Case 3 we must be able to realize the upper bound for q. To accomplish this delete the edge $\{a, f\}$ and add the desired number of edges, none of which join vertices of $E - \{e\}$ to vertices of F. Since $\lambda(H_1) = \delta \geq 3$ and we removed only two edges from H_1, we see that our graph has the desired vertex connectivity. Our construction fulfills the requirements of this case.

Case 4. Suppose that $p \geq 2\delta + 2$, $\delta \geq 2$, $\kappa < 2$,

$$\left\lceil \frac{p\delta}{2} \right\rceil \leq q \leq \frac{1}{2}\delta(\delta+1) + \frac{1}{2}(p-\delta-1)(p-\delta-2) + \kappa(p-\delta-1),$$

and if $\kappa = 1$ then $\delta = 2$ and $q > \frac{p\delta}{2}$. Let H_2 denote the Harary graph on $p-\delta-1$ vertices with $\lceil \frac{1}{2}(p-\delta-1)\delta \rceil$ edges and $\kappa(H_2) = \delta(H_2) = \delta$. We now take the union of $K_{\delta+1}$ and H_2. If $\kappa = 0$ then adding $q - \left\lceil \frac{p\delta}{2} \right\rceil$ edges to H_2 yields the desired graph. If $\kappa = 1$ then add an edge joining a vertex in $K_{\delta+1}$ (denote this vertex by a) to a vertex in H_2. Next we add $q - \frac{p\delta}{2} - 1$ edges, none of which join a vertex in $K_{\delta+1} - \{a\}$ to a vertex in H_2.

Case 5. Suppose that $2\delta + 2 - \kappa \leq p \leq 2\delta + 1, \kappa < \delta$, and

$$\begin{aligned}&\left\lceil \frac{p\delta}{2} + \frac{1}{2}\max(0, (p-\delta-1)(2\delta+2-p) - \kappa(\kappa-1)) \right\rceil \\ &\qquad \leq q \leq \frac{1}{2}\delta(\delta+1) + \frac{1}{2}(p-\delta-1)(p-\delta-2) + \kappa(p-\delta-1).\end{aligned}$$

Here we note that $\kappa \geq 1$. Take a set containing p vertices and partition the set into three mutually disjoint subsets A, B and C, where $|A| = \delta + 1 - \kappa$, $|B| = p - \delta - 1$ and $|C| = \kappa$. If $(p-\delta-1)(2\delta+2-p) \geq \kappa(\kappa-1)$ then construct complete graphs on A and B and join each vertex of A to each vertex of C. Note that at present the vertices of A have degree δ, the vertices of B have degree $p - \delta - 2$ and the vertices of C have degree

$\delta+1-\kappa$. We wish to show that $p \geq \kappa+\delta$. First, assume $p < \kappa+\delta$ which implies $0 \leq p-\delta-1 < \kappa-1$. But the latter inequality together with $0 < 2\delta+2-p \leq \kappa$ (which is a consequence of $2\delta+2-\kappa \leq p \leq 2\delta+1$) contradicts $(p-\delta-1)(2\delta+2-p) \geq \kappa(\kappa-1)$. Thus we have $p \geq \kappa+\delta$ and $|B| = p-\delta-1 \geq \kappa-1$. If $|B| > \kappa-1$ then join κ vertices of B in a 1-1 fashion to the vertices of C. If $|B| = \kappa-1$ then join the vertices of B in a $1-1$ fashion to $\kappa-1$ vertices of C. Since $\delta-(p-\delta-2) = 2\delta+2-p \leq \kappa$ we may add edges joining vertices in B to vertices in C so that every vertex in B will have degree δ. We do this in such a way that the degrees of the vertices in C differ by at most one. We also have $p-\delta-2 \leq \delta-1$, therefore each vertex in B is adjacent to at least one vertex in C. Denote this graph by G_1. Recall that before edges were added joining vertices in C to vertices in B every vertex in C had degree $\delta-(\kappa-1)$. There are $(p-\delta-1)(2\delta+2-p)$ edges joining vertices in C to vertices in B, and since $(p-\delta-1)(2\delta+2-p) \geq \kappa(\kappa-1)$ and $|C| = \kappa$, we have $\delta(G_1) = \delta$. It is also true that $q(G_1) = \frac{1}{2}p\delta + \frac{1}{2}(p-\delta-1)(2\delta+2-p) - \frac{1}{2}\kappa(\kappa-1)$. Adding $q - q(G_1)$ edges each of which joins vertices in C, or joins a vertex in B to a vertex in C finishes the construction. It can easily be shown that our final graph has the desired properties.

If $(p-\delta-1)(2\delta+2-p) < \kappa(\kappa-1)$ and $p > \kappa+\delta$ then repeat the construction that previously gave us graph G_1. However, now there is at least one vertex in C with degree $\delta-1$ or less, and there are no vertices with degree exceeding δ. To realize $q = \left\lceil \frac{p\delta}{2} \right\rceil$ add the appropriate Harary graph to the vertices of C and/or join pairs of vertices in C. If C has two vertices of degree $\delta+1$ then we add the Harary graph so that these two vertices are adjacent and delete the edge joining the two vertices of degree $\delta+1$. Next add edges as we did when $(p-\delta-1)(2\delta+2-p) \geq \kappa(\kappa-1)$ to realize a graph with q edges. If $(p-\delta-1)(2\delta+2-p) < \kappa(\kappa-1)$ and $p \leq \kappa+\delta$ we again let A, B and C be sets of vertices with $|A| = \delta+1-\kappa$, $|B| = p-\delta-1$ and $|C| = \kappa$. Now construct a complete graph on A and join each vertex in C to each vertex in A and B. At this stage of the construction every vertex in A has degree δ, every vertex in B has degree κ and every vertex in C has degree $p-\kappa$. To make all vertices in B have degree δ (except possibly one vertex, which may have degree $\delta+1$), we add the appropriate Harary graph to the vertices of B. We now do the same to C. If both B and C have a vertex of degree $\delta+1$, delete the edge joining the two vertices of degree $\delta+1$. Adding $q - \left\lceil \frac{p\delta}{2} \right\rceil$ edges each of which joins vertices in C, or joins vertices in B yields the desired graph.

Case 6. Suppose that $\delta \geq 2$, $\kappa = \delta \leq p-1$, and

$$\left\lceil \frac{p\delta}{2} \right\rceil \leq q \leq \frac{1}{2}\delta(\delta+1) + \frac{1}{2}(p-\delta-1)(p-\delta-2) + \kappa(p-\delta-1).$$

Let H_3 denote the Harary graph on p vertices with $\left\lceil \frac{p\delta}{2} \right\rceil$ edges and $\kappa(H_3) = \delta(H_3) = \delta$, and let v denote one of the vertices in H_3. Adding $q - \left\lceil \frac{p\delta}{2} \right\rceil$ edges to H_3 in such a way that none of the added edges are incident to v yields the desired graph.

Case 7. Suppose that $\delta = 0$. Note we then have $\kappa = 0, p \geq 1$ and $q \leq \frac{1}{2}(p-1)(p-2)$. Let G_2 be the graph on p vertices with no edges and let v_1 be one of the vertices in G_2. We construct our graph by adding q edges to G_2 with no edges incident to v_1.

Case 8. Suppose that $\delta = 1$. First we assume that $\kappa = 0$. Thus we have $p \geq 4$ and

$$\left\lceil \frac{p}{2} \right\rceil \leq q \leq 1 + \frac{1}{2}(p-2)(p-3).$$

If p is even then let G_3 be the graph composed of $\frac{1}{2}p$ copies of K_2. On the other hand, if p is odd then let G_3 be the graph composed of $K_{1,2}$ and $\frac{1}{2}(p-3)$ copies of K_2. Take one copy of K_2 in G_3 and denote it by A. Adding $q - \left\lceil \frac{p}{2} \right\rceil$ edges, none of which are incident to either vertex in A, finishes our construction.

Next we consider $\kappa = 1$. Here we have $p \geq 2$ and

$$p - 1 \ \leq \ q \ \leq \ 1 + \frac{1}{2}(p-1)(p-2).$$

Let G_4 be the graph composed of a path on p vertices and let v_2 be one of the end-vertices of the path. Adding $q - (p-1)$ edges so that no added edges are incident to v_2 yields the desired graph.

We are done with our constructions and will now show sufficiency. Cases 1, 2, 3 and 4 show the conditions of the theorem are sufficient if we also have $\delta \geq 2$, $p \geq 2\delta + 2$ and $\kappa < \delta$. If we assume $p \leq 2\delta + 1$ and $\kappa < \delta$ then Case 5 shows the sufficiency of the conditions of the theorem. (Recall that in Case 5 we have $\kappa \geq 1$). Similarly, if we assume $\delta \geq 2$ and $\kappa = \delta$ then Case 6 is adequate. Cases 7 and 8 show the sufficiency of the conditions of the theorem for $\delta < 2$ and our proof is finished.

4 Conclusion

The (p, q, κ, δ) realizability theorem in this paper solves several extremal problems. If any three of the parameters p, q, κ and δ are given we can find the range of values for the unknown parameter. Here we will look at the problem of finding both the maximum value of κ among all (p, q, δ) graphs [denoted by max(κ, given p, q, δ)] and the minimum value of κ among all (p, q, δ) graphs [denoted by min(κ, given p, q, δ)]. The solution is given below (the proof is straightforward):

$$\max(\kappa, \text{ given } p, q, \delta) = \begin{cases} 0, & \text{if } q < p-1 \\ \delta, & \text{if } q \geq p-1 \end{cases}$$

$$\min(\kappa, \text{ given } p, q, \delta) = \begin{cases} 2\delta + 2 - p, & \text{if } \delta + 1 < p < 2\delta + 2 \\ \delta, & \text{if } p = \delta + 1 \\ 0, & \text{otherwise.} \end{cases}$$

References

[1] A.A. Ali and W. Staton, On extremal graphs with no long paths, *Electron. J. Combinat.* Vol. 3, R20 (1996), 4 pp.

[2] E.T. Baskoro and M. Miller, On the construction of networks with minimum diameter, Proceedings of the 16th Australian Computer Science Conference, (January 1993), 739–744.

[3] E.T. Baskoro and M. Miller, A procedure for constructing a minimum diameter digraph, *Bulletin of ICA* **17** (1996), 8–14.

[4] E.T. Baskoro, M. Miller, J. Plesnik and S.Znam, Regular digraphs of diameter 2 and maximum order, *Australasian J. of Combinatorics* **9** (1994), 291–306.

[5] B. Berseford-Smith, O. Diessel and H. ElGindy, Optimal algorithms for constrained reconfigurable meshes, *Australian Computer Science Communications* **17** (1995), 32–41.

[6] F.T. Boesch and C.L. Suffel, Realizability of p-point graphs with prescribed minimum degree, maximum degree, and line-connectivity, *J. Graph Theory* 4 (1980), 363–370.

[7] F.T. Boesch and C.L. Suffel, Realizability of p-point graphs with prescribed minimum degree, maximum degree, and point-connectivity, *Discrete Appl. Math.* **3** (1981), 9–18.

[8] F.T. Boesch and C.L. Suffel, Realizability of p-point, q-line graphs with prescribed point connectivity, line connectivity, or minimum degree, *Networks* **12** (1982), 341–350.

[9] G. Chartrand, A graph-theoretic approach to the communications problem, *SIAM J. Appl. Math.* **14** (1966), 778–781.

[10] G. Exoo, F. Harary and C-D. Xu, Vulnerability in graphs of diameter four, *Math. Comput. Modelling* **17** (1993), 65–68.

[11] F. Harary and J.P. Hayes, Survey of fault tolerance for graphs, *Graph Theory, combinatorics, and Applications* (Eds. Y. Alavi and A. Schwenk), Vol. 1 (1995), 477–483.

[12] F. Harary and J.P. Hayes, Node fault tolerance in graphs, *Networks* **27** (1996), 19–23.

[13] F. Harary and M. Khurrum, One node fault tolerance for caterpillars and starlike trees, *Internat. J. Computer Math.* **56** (1995), 135–143.

[14] F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA (1969).

[15] F. Harary, The maximum connectivity of a graph, *Proc. Natl. Acad. Sci. USA* **48** (1962), 1142–1146.