3-Circulant Graphs

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Abstract

A 3-regular graph G is called a 3-circulant if its adjacency matrix $A(G)$ is a circulant matrix. We show how all disconnected 3-circulants are made up of connected 3-circulants and classify all connected 3-circulants as one of two basic types. The rank of $A(G)$ is then completely determined for all 3-circulant graphs G.

1 Introduction.

Relating basic properties of an adjacency matrix $A(G)$ to the structure of its underlying graph G has a long history. The eigenvalues of $A(G)$ have been studied by many authors, including the pioneering work of Cvetkovic, Doob and Sachs [5]. Considering the algebraic multiplicity of any zero eigenvalues leads naturally to the study of the rank of $A(G)$. Much recent work has focused in this area, e.g. [1, 2, 3, 6, 8].

In this paper, we consider the class of graphs called circulant graphs. Let S be any subset of $\{1, 2, \ldots, n - 1\}$ such that $S = -S \bmod n$. A graph G with vertex set $\{0, 1, 2, \ldots, n - 1\}$ is called a *circulant graph* if two vertices i and j are adjacent if and only if $(i - j) \bmod n \in S$. The adjacency matrix $A(G)$ is a *circulant matrix*, i.e., $a_{i,j} = a_{i-1,j-1}$ with the subscript calculation done mod n. In other words, row $(i+1)$ of the matrix is a cyclic right shift one position from row (i). In the process of finding the rank of such $A(G)$, we are able to completely characterize circulant graphs where $|S| = 3$, i.e. the *3-circulant graphs.* We show how disconnected 3-circulants are made up of a collection of connected 3-circulants and how each connected 3-circulant is one of two basic types. We then return to $A(G)$ and determine its rank for each type.

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2 Elementary Properties and Examples.

All 3-circulant graphs are 3-regular. Since the sum of the degrees of the vertices must be two times the number of edges, 3-circulant graphs must have an even number of vertices. Therefore, in what follows we shall assume that n is even. For S to be a three-element subset of $\{1, 2, \ldots, n-1\}$ such that $S = -S \bmod n$, it is clear that S must have the form $S = \{a, \frac{n}{2}, n-a\}$. A result of Broere [4] gives precise conditions on the connectivity of 3-circulants.

Theorem 1 *Let G be a circulant graph with n vertices formed by $S = \{s_1, \ldots, s_k\}$. If $d = \gcd(s_1, \ldots, s_k, n)$, then G has d connected components each isomorphic to the circulant graph on $\frac{n}{d}$ vertices formed by $S' = \{\frac{s_1}{d}, \ldots, \frac{s_k}{d}\}$.*

We can apply this theorem to our specific 3-circulant case as follows.

Corollary 1 *Let n be even and $S = \{a, \frac{n}{2}, n-a\}$. If $\gcd(a, \frac{n}{2}, n) = d$, then the circulant graph with n vertices formed by S has d components each isomorphic to the circulant graph on $\frac{n}{d}$ vertices formed by $S' = \{\frac{a}{d}, \frac{n}{2d}, \frac{n-a}{d}\}$.*

A simple example illustrates this result (see Figure 1). Let $n = 8$, and $S = \{2, 4, 6\}$. The $\gcd(2, 4, 8) = 2$, indicating two connected components. Indeed the even numbered vertices form a subgraph isomorphic to K_4, as do the odd numbered vertices. The rank of this graph is 8 which is the sum of the ranks of the two connected components.

3 Classification of Connected 3-Circulants.

Any disconnected 3-circulant is made up of d isomorphic copies of connected 3-circulants. We now classify connected 3-circulants as isomorphic to one of two basic types. In what follows we assume that G is a connected 3-circulant with n vertices. The two basic types of connected 3-circulants are (i) circulants with $S = \{1, \frac{n}{2}, n-1\}$ and (ii) $P_2 \times C_{\frac{n}{2}}$.

It turns out that the key quantity in our analysis is $\gcd(n, a)$. If this gcd is odd, a connected 3-circulant is isomorphic to type (i). If this gcd is even, more conditions are required to classify G as type (i) or type (ii). Theorems 2 through 3 describe type (i).

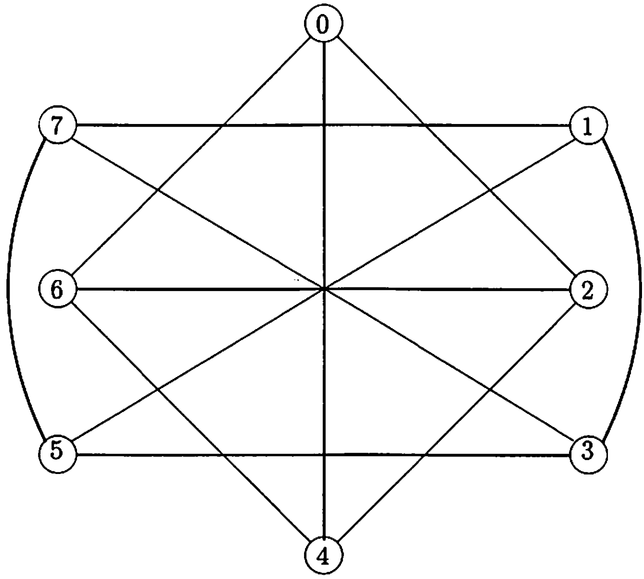

Figure 1: Circulant graph with 8 vertices formed by $S = \{2, 4, 6\}$.

Theorem 2 *Let n be even and $S = \{a, \frac{n}{2}, n-a\}$. If $gcd(n, a) = 1$, then the circulant on n vertices formed by S is isomorphic to the circulant on n vertices formed by $S' = \{1, \frac{n}{2}, n-1\}$.*

Proof: Let n be even and $\gcd(n, a) = 1$. Then a is odd. Also, the cyclic subgroup of $\mathbf{Z}_n$ generated by a is all of $\mathbf{Z}_n$. That is the elements of $\langle a \rangle = \{a, 2a, 3a, \ldots, na\}$ modulo n are distinct. Hence, the circulant on n vertices formed by the set $\{a, n-a\}$ is a cycle on n vertices. The vertex directly across the cycle from the vertex ka is $(\frac{n}{2}+k)a$. But $(\frac{n}{2}+k)a-(ka+\frac{n}{2}) = \frac{n}{2}(a-1)$ is divisible by n since a is odd. Hence, $(\frac{n}{2}+k)a \equiv (ka+\frac{n}{2}) \pmod{n}$. Therefore, the circulant on n vertices formed by S is isomorphic to the circulant on n vertices formed by $S' = \{1, \frac{n}{2}, n-1\}$. □

We illustrate Theorem 2 with an example. Consider the circulant graph with $n = 10$ vertices formed by $S = \{3, 5, 7\}$ (see Figure 2). In the proof of Theorem 2, the cyclic subgroup $\langle a \rangle = \langle 3 \rangle = \{3, 6, 9, 2, 5, 8, 1, 4, 7, 0\}$ is the entire vertex set. Relabeling the vertices as $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 0\}$ results in the circulant graph formed by $S = \{1, 5, 9\}$ (see Figure 3). We have shown the original vertex labels around the outside of the graph.

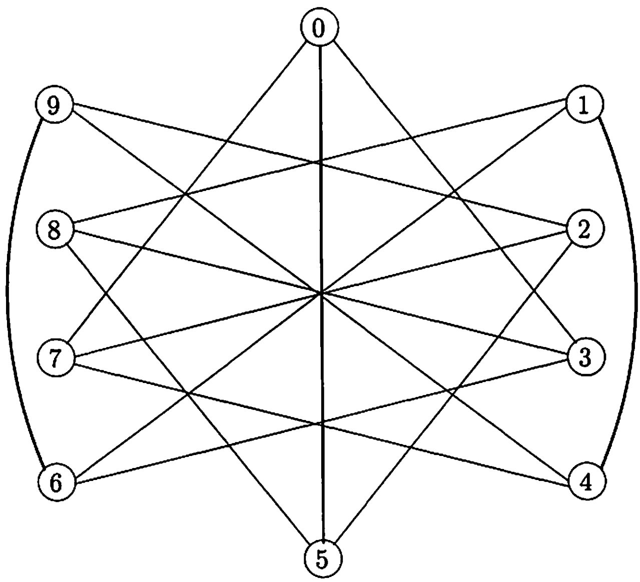

Figure 2: Circulant graph with 10 vertices formed by $S = \{3, 5, 7\}$.

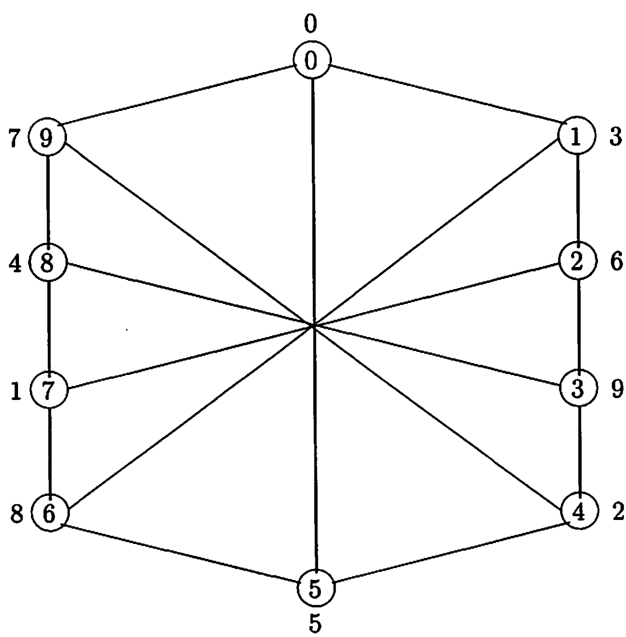

Figure 3: Circulant graph with 10 vertices formed by $S = \{1, 5, 9\}$.

Theorem 3 *Let n be even and $S = \{a, \frac{n}{2}, n-a\}$. If $gcd(n,a) = 2k+1$ where k is a nonnegative integer, then the circulant graph on n vertices formed by S has $2k+1$ components each isomorphic to the circulant on $\frac{n}{2k+1}$ vertices formed by $S' = \{1, \frac{n}{2(2k+1)}, \frac{n}{2k+1} - 1\}$.*

Proof: Let n be even and $\gcd(n,a) = 2k+1$ where k is a nonnegative integer, then $\gcd(a, \frac{n}{2}, n) = 2k+1$. Therefore, by Corollary 1, the circulant graph on n vertices formed by S has $2k+1$ components each isomorphic to the circulant on $\frac{n}{2k+1}$ vertices formed by $S'' = \{\frac{a}{2k+1}, \frac{n}{2(2k+1)}, \frac{n-a}{2k+1}\}$. Since $\gcd(n,a) = 2k+1$, then $\gcd(\frac{n}{2k+1}, \frac{a}{2k+1}) = 1$, and by Theorem 2 our graph is isomorphic to the graph with $2k+1$ components each isomorphic to the circulant on $\frac{n}{2k+1}$ vertices formed by $S' = \{1, \frac{n}{2(2k+1)}, \frac{n}{2k+1} - 1\}$. □

This completely characterizes all 3-circulant graphs formed by $S = \{a, \frac{n}{2}, n-a\}$ where $\gcd(n,a)$ is odd as having components of type (i). The next theorem describes when the $\gcd(n,a)$ is even and the 3-circulant's components are of type (i). The proof to this theorem follows from Corollary 1 and Theorem 2.

Theorem 4 *Let n be even and $S = \{a, \frac{n}{2}, n-a\}$. If $gcd(n,a) = 2k$ where k is a positive integer and 4 divides $\frac{n}{k}$, then the circulant graph on n vertices formed by S has $2k$ components each isomorphic to the circulant on $\frac{n}{2k}$ vertices formed by $S' = \{1, \frac{n}{4k}, \frac{n}{2k} - 1\}$.*

The next two theorems classify the remaining cases as type (ii).

Theorem 5 *Let $n \equiv 2 (mod\ 4)$ and $S = \{a, \frac{n}{2}, n-a\}$. If $gcd(n,a) = 2$, then the circulant on n vertices formed by S is isomorphic to $P_2 \times C_{\frac{n}{2}}$.*

Proof: Let $n \equiv 2 (\mathrm{mod}\ 4)$ and $\gcd(n,a) = 2$. Then the circulant on n vertices formed by the set $\{a, n-a\}$ has two components both isomorphic to cycles on $\frac{n}{2}$ vertices, one containing all the even vertices ($a, 2a, \ldots, \frac{n}{2}a$ modulo n) and the other containing all the odd vertices. Also, $\frac{n}{2}$ is odd and $\gcd(a, \frac{n}{2}, n) = 1$. Hence, the circulant on n vertices formed by $S = \{a, \frac{n}{2}, n-a\}$ is connected. Now each vertex ka in the even cycle is adjacent to the vertex $ka + \frac{n}{2}$ in the odd cycle. Also the vertex $ka + \frac{n}{2}$ is adjacent to $(ka + \frac{n}{2}) + a = (k+1)a + \frac{n}{2}$ and $(ka + \frac{n}{2}) - a = (k-1)a + \frac{n}{2}$. Therefore, the circulant on n vertices formed by $S = \{a, \frac{n}{2}, n-a\}$ is isomorphic to $P_2 \times C_{\frac{n}{2}}$. □

As an illustration of this isomorphism, consider the circulant graph with $n = 10$ vertices formed by $S = \{2, 5, 8\}$ (see Figure 4). Placing the odd numbered vertices in a cycle, the even numbered vertices in another cycle and properly connecting adjacent vertices reveals $P_2 \times C_5$ (see Figure 5).

Theorem 6 *Let n be even and $S = \{a, \frac{n}{2}, n-a\}$. If $gcd(n, a) = 2k$ where k is an integer greater than 1 and 4 does not divide $\frac{n}{k}$, then the circulant graph on n vertices formed by S has k components each isomorphic to $P_2 \times C_{\frac{n}{2k}}$.*

The proof to this theorem follows from Corollary 1 and Theorem 5.

4 Ranks of 3-Circulant Graphs.

Much is known about the eigenvalues of circulant matrices. In fact, a formula exists for them [7].

Theorem 7 *If A is an $n \times n$ circulant matrix with first row $[c_1, c_2, \ldots, c_n]$, then the eigenvalues of A are given by $\lambda_p = \sum_{i=1}^{n} c_i \omega^{(i-1)p}$, $p = 0, 1, \ldots, n-1$, where $\omega = e^{2\pi i/n}$.*

Applying this formula to our two basic types of 3-circulants completely determines their rank.

Theorem 8 *Let n be even and $S = \{1, \frac{n}{2}, n-1\}$. If G is the circulant on n vertices formed by S, then*

$$rank(G) = \begin{cases} n-2 & \text{if } n \equiv 0 mod 12 \\ n-4 & \text{if } n \equiv 6 mod 12 \\ n & \text{otherwise} \end{cases} .$$

Proof: Let n be even and G be the circulant on n vertices formed by $S = \{1, \frac{n}{2}, n-1\}$. To determine the rank of G it suffices to count the number of zero eigenvalues of $A(G)$. Let $\omega = e^{2\pi i/n}$. From the special structure of $A(G)$, we see that $\lambda_p = \sum_{i=1}^{n} c_i \omega^{(i-1)p}$ reduces to $\lambda_p = \omega^p + \omega^{p+m} + \omega^{p+2m} = \omega^p(1 + \omega^m + \omega^{2m})$, for $p = 0, 1, \ldots, (n-1)$, and therefore $m = 0, \frac{n-2}{2}, \frac{2(n-2)}{2}, \ldots, \frac{(n-1)(n-2)}{2}$.

For any of the eigenvalues to be zero, it is clear that $1+\omega^m+\omega^{2m} = 0$, and the quadratic formula implies that this is only possible when $\omega^m = e^{2\pi i/3}$ or $e^{4\pi i/3}$. Thus it becomes a combinatorial problem to discover when such an equality can hold.

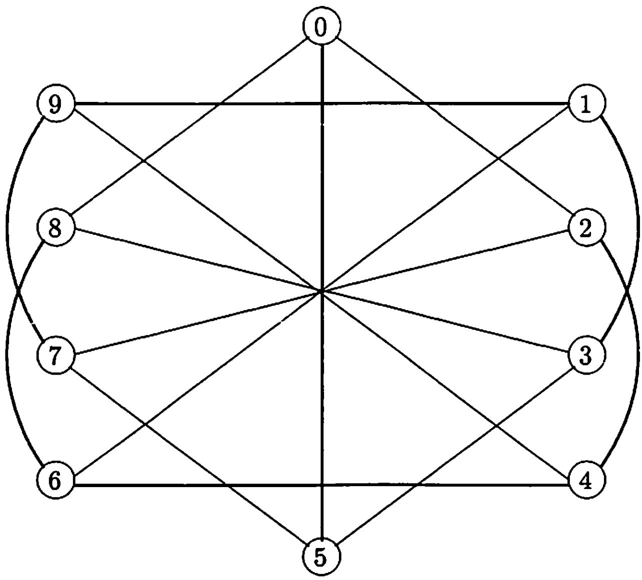

Figure 4: Circulant graph with 10 vertices formed by $S = \{2, 5, 8\}$.

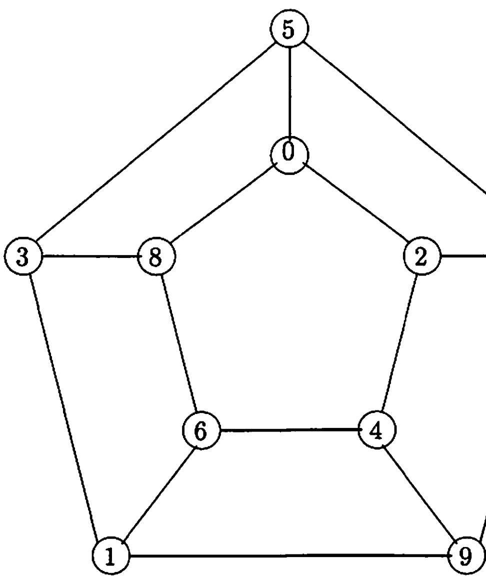

Figure 5: $P_2 \times C_5$.

First note that trying to solve $\omega^m = e^{2\pi i/3}$ or $e^{4\pi i/3}$ implies that $\frac{2\pi i p(n-2)}{2n} = \frac{2\pi i}{3} + 2\pi i k$ or $\frac{4\pi i}{3} + 2\pi i k$ for some integer k. This reduces to $\frac{p(n-2)}{2} = \frac{n}{3}$ or $\frac{2n}{3}$ modulo n. Recalling that n is even, the left hand side of this equation is an integer, and thus n must also be divisible by 3 or else there are no solutions. Therefore the rank of G is n unless n is divisible by 6.

Knowing that the only possibility for rank deficiency is when n is divisible by 6, we examine the two cases: $n \equiv 0 \bmod 12$, and $n \equiv 6 \bmod 12$.

If $n \equiv 0 \bmod 12$, then $n = 12L$, and we attempt solve

$$\frac{p(12L-2)}{2} = \frac{12L}{3} \text{ or } \frac{24L}{3} \bmod 12L.$$

Noting that p must be even, since both sides of this equation are even numbers, and doing some algebraic simplification yields

$$p = 8L \text{ or } 4L \bmod 12L.$$

Therefore, there are exactly two zero eigenvalues, and the rank of $G = n-2$.

If $n \equiv 6 \bmod 12$, then $n = 12L-6$, and we attempt solve

$$\frac{p(12L-6-2)}{2} = \frac{(12L-6)}{3} \text{ or } \frac{2(12L-6)}{3} \bmod 12L-6.$$

Here p could be even or odd, so we consider each in turn. If p is even, then the algebra reveals

$$p = 8L-4 \bmod 12L-6, \text{ or}$$

$$p = 4L-2 \bmod 12L-6.$$

If p is odd, then similar manipulation gives

$$p = 2L-1 \bmod 12L-6, \text{ or}$$

$$p = 10L-5 \bmod 12L-6.$$

Therefore, there are exactly four zero eigenvalues, and the rank of $G = n-4$. □

If a connected 3-circulant graph G is not isomorphic to a 3-circulant with $S = \{1, \frac{n}{2}, n-1\}$, then we have shown that it must be isomorphic to $P_2 \times C_{\frac{n}{2}}$. The following two results completely characterize the rank of such G.

The following result of Bevis, Domke and Miller [3] address the rank of $P_m \times C_n$.

Theorem 9 *Let* $g = \gcd(m+1, n)$, *then*
$$rank(P_m \times C_n) = \begin{cases} mn+1-g & \text{if } n \text{ odd} \\ mn+2-g & \text{if } g \text{ even}, \frac{n}{g} \text{ odd} \\ mn+2-2g & \text{if } \frac{n}{g} \text{ even} \end{cases}.$$

We can apply this theorem to our specific 3-circulant case as follows.

Corollary 2 *Let* $n \equiv 2(mod4)$, *then*
$$rank(P_2 \times C_{\frac{n}{2}}) = \begin{cases} n & \text{if } n \not\equiv 0 mod3 \\ n-2 & \text{if } n \equiv 0 mod3 \end{cases}.$$

Proof: Since $n \equiv 2(\text{mod}4)$, then $\frac{n}{2}$ is odd. Also, $g = \gcd(3, \frac{n}{2}) \in \{1, 3\}$. Thus, $\text{rank}(P_2 \times C_{\frac{n}{2}}) = 2(\frac{n}{2}) + 1 - g$, and the result holds. □

5 Conclusion

We have completely described the class of 3-circulant graphs. If G is a 3-circulant graph, then it is either connected or is made up of isomorphic copies of connected 3-circulant components. Every connected 3-circulant graph can then be classified as one of two basic types, isomorphic to either the 3-circulant formed by $S = \{1, \frac{n}{2}, n-1\}$ or $P_2 \times C_{\frac{n}{2}}$. We have shown that the rank of $A(G)$ is determined in every case. Future investigations will attempt to generalize these properties to circulants with vertices of higher degree.

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