# Pairwise Balanced Designs on $4s + 4$ blocks with longest block of cardinality $2s + 1$

R.G. Stanton

Department of Computer Science
University of Manitoba
Winnipeg, Canada R3T 2N2

ABSTRACT. The quantity $g^{(k)}(v)$ was introduced in [4] as the minimum number of blocks necessary in a pairwise balanced design on $v$ elements, subject to the condition that the longest block have cardinality $k$. When $k \geq (v-1)/2$, it is known that $g^{(k)}(v) = 1 + (v-k)(3k - v + 1)/2$, except for the case when $v \equiv 1 \pmod 4$ and $k = (v-1)/2$. This exceptional "case of first failure" was treated in [1] and [2]. In this paper, we discuss the structure of the "case of first failure" for the situation when $v = 4s + 4$.

## 1 Introduction

The formula $g^{(k)}(v) = 1 + (v-k)(3k - v + 1)/2$ holds for $v = 4s + 1$ and $k > 2s$. When $k = 2s$, the formula fails and the result for this case was given in [2]; the structure of the design in the case was discussed in [1].

In this paper, we consider the analogous situation for $v = 4s + 4$. The formula $g^{(k)}(v) = 1 + (v-k)(3k - v + 1)/2$ is valid for $k \geq 2s + 2$, and so it first fails in the case $k = 2s + 1$. It is natural to look at this case of first failure.

The $S$ bound in this case (cf. [3]) is given by

$$\begin{aligned} g^{(k)}(v) &\geq 1 + (v-k)(5k - v + 1)/6 \\ &= 1 + (2s+3)(3s+1)/3 \\ &= 1 + (6s^2 + 11s + 3)/3 \\ &= 1 + (2s^2 + 3s + 1) + \lceil 2s/3 \rceil. \end{aligned}$$

Now let $s = 3t + a$ $(a = 0, 1, 2)$. Then

$$\lceil 2s/3 \rceil = \lceil (6t + 2a)/3 \rceil = 2t + a.$$

JCMCC 40 (2002), pp. 161-165

So the $S$ bound is given by

$$S = 1 + (2s^2 + 3s + 1 + 2t + a) = 1 + S^*.$$

Now we use an approach similar to that used in [1]. We let $b_i$ be the number of blocks of length $i$ other than the base block of length $k$. Then

$$\begin{aligned} b_2 + b_3 + b_4 + \ldots &= S^* + \epsilon, \\ b_2 + 2b_3 + 3b_4 + \ldots &= (2s+1)(2s+3) + B_0, \\ b_2 + 3b_3 + 6b_4 + \ldots &= (2s+2)(4s+3) - s(2s+1). \end{aligned}$$

These equations simplify to

$$\begin{aligned} b_2 + b_3 + b_4 + \ldots &= 2s^2 + 3s + 1 + 2t + a + \epsilon, \\ b_2 + 2b_3 + 3b_4 + \ldots &= 4s^2 + 8s + 3 + B_0, \\ b_2 + 3b_3 + 6b_4 + \ldots &= 6s^2 + 13s + 6. \end{aligned}$$

Using multipliers 3, $-3$, 1, we obtain

$$b_2 + b_5 + 3b_6 + 6b_7 + \cdots = a + 3\epsilon - 3B_0.$$

We conclude that $b_2 \leq a + 3\epsilon - 3B_0$.

Now use multipliers 1, $-2$, 1; we obtain

$$b_4 + 3b_5 + 6b_6 + \cdots = 1 + 2t + a + \epsilon - 2B_0.$$

Hence $b_4 + b_6 + b_8 + \cdots \leq 1 + 2t + a + \epsilon - 2B_0$.

Adding our two inequalities gives the result that

$$\begin{aligned} b_2 + b_4 + b_6 + \ldots &\leq 1 + 2t + 2a + 4\epsilon - 5B_0 \\ &\leq 1 + 2t + 2a + 4\epsilon. \end{aligned}$$

But each point on the base block must occur with $2s + 3$ other points and this requires that at least one block of even length hang on each point of the base block. Thus, parity requires that

$$b_2 + b_4 + b_6 + \cdots \geq 2s + 1.$$

Thence $1 + 2t + 2a + 4\epsilon \geq 2s + 1 = 6t + 2a + 1$.

Hence $4\epsilon \geq 4t$, that is, $\epsilon \geq t$.

So we obtain a bound on $g - 1$ as

$$\begin{aligned} S + \epsilon &= 2s^2 + 3s + 1 + (2t + a) + t \\ &= 2s^2 + 4s + 1. \end{aligned}$$

We shall now show that this bound is met.

## 2 Attainment of the bound

If the bound $g-1=2s^2+4s+1$ is met, then

$$b_2+b_3+b_4+\ldots=2s^2+4s+1,$$
$$b_2+2b_3+3b_4+\ldots=4s^2+8s+3+B_0,$$
$$b_2+3b_3+6b_4+\ldots=6s^2+13s+6.$$

The previously obtained inequalities now become equalities.

So $b_2+b_4+b_6+\cdots=1+6t+2a=1+2s$, $b_2=3t+a-3B_0$, $b_4+b_6+\cdots=1+3t+a-2B_0$.

It follows that $B_0=0$, $b_2=3t+a$, $b_4+b_6+\cdots=1+3t+a$.

Furthermore, since

$$b_2+b_5+3b_6+6b_7+\cdots=3t+a-3B_0=3t+a,$$

it follows that

$$b_5=b_6=b_7=\cdots=0.$$

We thus find that, if the bound

$$g-1=2s^2+4s+1$$

is met, then $B_0=0$, that is, all other blocks meet the base block of length $2s+1$. Furthermore, $b_2=s$, $b_4=s+1$, and therefore

$$b_3=2s^2+4s+1-(2s+1)=2s^2+2s.$$

This information allows us to produce a construction for a design meeting the bound. We illustrate the construction for $s=4$, that is, for $g^{(9)}(20)=1+(49)$. However, the construction is perfectly general.

On each of the 9 base points of the long block, we must hang 11 others (call these 11 points $1,2,3,\ldots,8,9,0,\infty$).

By considering frequencies, we see that any point that occurs $r$ times in the 4 blocks of length 2 (singletons attached to the long block) must occur $r+1$ times in the 5 blocks of length 4 (triples hanging on the long block).

Suppose that $\infty$ occurs in the $s$ singletons and thus also in the $s+1$ triples. We write down the usual 1-factorization of $K_{10}$ (on $1,2,3,\ldots,8,9,0$) as

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 01 | 02 | 03 | 04 | 05 | 06 | 07 | 08 | $\underline{09}$ |
| 12 | 13 | 24 | 35 | 46 | 57 | 68 | 79 | 81 |
| 83 | 94 | 15 | 26 | 37 | 48 | 59 | 61 | 72 |
| 74 | 85 | 96 | 17 | 28 | 39 | 41 | 52 | 63 |
| $\underline{56}$ | 67 | $\underline{78}$ | 89 | 91 | $\underline{12}$ | 23 | $\underline{34}$ | 45. |

Now append $\infty$ to the pairs 12, 34, 56, 78, 90, and thus get 5 triples. Add a singleton $\infty$ to the remaining 4 columns. This gives a total of $45+4=49$ short blocks hanging on the long block.

In general, we have

$$\binom{2s+2}{2}+s=(s+1)(2s+1)+s=2s^2+4s+1$$

short blocks hanging on the long block. Thus we have shown that

$$g-1=2s^2+4s+1.$$

## 3 Alternative constructions

Not every possible assignment of singletons leads to a solution. For example, suppose there are three singletons $\infty_1$ and one singleton $\infty_2$ in the case $v=20$, $k=9$. Then 11 elements $\infty_1$, $\infty_2$, 1, 2, 3, 4, 5, 6, 7, 8, 9, hang on each point of the block of length 9. Element $\infty_1$ occurs in 4 triples and element $\infty_2$ occurs in 2 triples. So deletion of $\infty_1$ and $\infty_2$ leaves the usual factorization of $K_9$ on $\{1,2,\ldots,9\}$. This factorization has the form $(i,--,--,--,--)$ as $i$ ranges from 1 to 9.

When we adjoin $\infty_1$ and $\infty_2$, we get 9 blocks in different factors as

$$\infty_2 1, \infty_2 2, \infty_2 3, \infty_1 4, \infty_1\infty_2 5, \infty_2 6, \infty_2 7, \infty_2 8, \infty_1 9.$$

One of the factors containing $\infty_1 4$ and $\infty_1 9$ must contain a singleton $\infty_2$ and the other must contain a tripleton $\infty_2--$. But this tripleton has to be $\infty_2 49$ and this is a contradiction.

This argument is perfectly general and shows that it is never possible to have $s-1$ singletons $\infty_1$ and one singleton $\infty_2$.

On the other hand, suppose that the 4 singletons are $\infty_1$, $\infty_1$, $\infty_2$, $\infty_2$. For convenience, write $\infty_1$ and $\infty_2$ as $a$ and $b$. Then we can immediately start the factors as

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $a1$ | $a2$ | $a3$ | $a4$ | $ab5$ | $b6$ | $b7$ | $b8$ | $b9$ |
| $b$ | $b$ | $b14$ | $b23$ | | $a$ | $a$ | $a69$ | $a78$ |

It is straightforward to complete the factors containing $a3$, $a4$, $b8$, $b9$, and then proceed to the others to obtain a solution as

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $a1$ | $a2$ | $a3$ | $a4$ | $ab5$ | $b6$ | $b7$ | $b8$ | $b9$ |
| $b$ | $b$ | $b14$ | $b23$ | 16 | $a$ | $a$ | $a69$ | $a78$ |
| 26 | 17 | 52 | 51 | 27 | 18 | 19 | 57 | 56 |
| 37 | 39 | 67 | 68 | 38 | 29 | 28 | 12 | 13 |
| 48 | 46 | 89 | 79 | 49 | 47 | 45 | 34 | 24 |
| 59 | 58 | | | | 35 | 36 | | |

We conclude by noting that $g^{(3)}(8) = 12$ is an exception to the general result that $g^{(2s+1)}(4s+4) = 1 + (2s^2 + 4s) + 1$. The general result only holds for $s > 1$.

Also, the results of this section show that the solution for $g^{(5)}(12)$ is given uniquely by adjoining 2 singletons $\infty$ to two factors of $K_6$ and using $\infty$ to form triples on the other 3 factors.

The solution for $g^{(7)}(16)$ is rather interesting. One has the solutions obtained from a factorization of $K_8$ by adjoining $\infty$ to form 3 singletons and 4 triples. The considerations of this section show that using $K_7$ with singletons $\infty_1$, $\infty_1$, $\infty_2$, does not lead to a solution.

However, we point out that solutions do exist when there are 3 distinct singletons (call them $a$, $b$, $c$). The result may have a triple $abc$, as follows.

| | | | | | | |
|---|---|---|---|---|---|---|
| $a$ | $b$ | $c$ | $abc$ | $a12$ | $b34$ | $c56$ |
| $b6$ | $a4$ | $a5$ | 16 | $b5$ | $a6$ | $a3$ |
| $c4$ | $c1$ | $b2$ | 23 | $c3$ | $c2$ | $b1$ |
| 13 | 26 | 14 | 45 | 46 | 15 | 24 |
| 25 | 35 | 36 | | | | |

Or it may have no triple $abc$, as shown below.

| | | | | | | |
|---|---|---|---|---|---|---|
| $a$ | $b$ | $c$ | $ab1$ | $ac2$ | $b34$ | $c56$ |
| $bc$ | $a5$ | $a4$ | $c3$ | $b5$ | $a6$ | $a3$ |
| 16 | $c4$ | $b6$ | 26 | 13 | $c1$ | $b2$ |
| 24 | 12 | 15 | 45 | 46 | 25 | 14 |
| 35 | 36 | 23 | | | | |

## References


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[4] R.G. Stanton, J.L. Allston, and D.D. Cowan, Pair-Coverings with Restricted Largest Block Length, *Ars Combinatoria* **11** (1981), 85–98.