A Note on Steiner-Distance-Hereditary Graphs

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ABSTRACT. In a graph, the Steiner distance of a set of vertices U is the minimum number of edges in a connected subgraph containing U. For $k \geq 2$ and $d \geq k-1$, let $S(k,d)$ denote the property that for all sets S of k vertices with Steiner distance d, the Steiner distance of S is preserved in any induced connected subgraph containing S. A k-Steiner-distance-hereditary (k-SDH) graph is one with the property $S(k,d)$ for all d. We show that property $S(k,k)$ is equivalent to being k-SDH, and that being k-SDH implies $(k+1)$-SDH. This establishes a conjecture of Day, Oellermann and Swart.

Distance is a fundamental concept in graphs. Indeed a whole book has been written on the subject [2]. One special family is distance-hereditary graphs. We are interested here in a generalisation of such graphs.

Distance-hereditary graphs are graphs such that for any pair of vertices x and y it holds that the distance between x and y is preserved in any induced connected subgraph containing x and y. Equivalently, for all pairs of vertices x and y, every induced x–y path has the same length. Distance-hereditary graphs have been well studied since their introduction by Howorka [5] in 1977; see for example [1, 3, 4, 6].

While studying distance-hereditary graphs, D'Atri and Moscarini [3] considered the relationship between normal distance and Steiner distance. If S is a set of vertices, then a **connection for** S is any induced connected subgraph containing S. A connection for S is **minimal** if the removal of any vertex destroys the property of being a connection for S. (That is, any vertex not in S is a cut-vertex.) The **Steiner tree** of a set S is a spanning tree of a connection of S of minimum order (number of vertices). The number of edges in the Steiner tree is the **Steiner distance** of S. So, if S is a pair of vertices then the Steiner distance is the normal distance.

A graph is said to be **Steiner-distance-hereditary** if for every set S of vertices the Steiner distance of S is preserved in any connection for S. Equivalently (since the Steiner tree of S in a minimal connection for S

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spans the connection), a graph is Steiner-distance-hereditary if for all sets S of vertices every minimal connection for S has the same order.

D'Atri & Moscarini [3], and later Day, Oellermann & Swart [4], proved:

Theorem 1 *A graph is distance-hereditary iff it is Steiner-distance-hereditary.*

For $k \geq 2$ and $d \geq k-1$, we define the property $S(k,d)$ as meaning that for all sets S of k vertices with Steiner distance d, the distance of S is preserved in any connection for S. We define the property $S(k)$ as meaning $S(k,d)$ for all d; Day et al. [4] introduced the property and called such graphs **k-Steiner-distance-hereditary**. Distance-hereditary graphs are the ones obeying $S(2)$.

Day et al. [4] conjectured as an extension to Theorem 1 that in general being k-Steiner-distance-hereditary implies being $(k+1)$-Steiner-distance-hereditary. This we show. We also show that there is a partial converse:

Theorem 2 *(1) For all $k \geq 2$ it holds that $S(k,k)$ is equivalent to $S(k)$.*
(2) For all $k \geq 2$ it holds that $S(k)$ implies $S(k+1)$.
(3) For all $k \geq 3$ it holds that $S(k)$ implies $S(k-1,d)$ for all $d \geq k$.

Bandelt and Mulder [1] showed that $S(2,2)$ is equivalent to $S(2)$. Theorem 2 is a consequence of the following three lemmas.

It is to be noted that property $S(k,k-1)$ is always satisfied; the interesting property is $S(k,k)$.

Lemma 1 *For $k \geq 2$ it holds that $S(k,k)$ implies $S(k+1,k+1)$.*

PROOF. Assume graph G obeys $S(k,k)$. Let $S \subseteq V(G)$ have cardinality $k+1$ and Steiner-distance $k+1$ in G. That is, the graph induced by S is not connected, but there exists a vertex x such that the graph induced by $S \cup \{x\}$ is connected.

Let H be any connection for S. We need to show that S has Steiner-distance $k+1$ in H. If H contains x, then we are done. So we may assume that $x \notin V(H)$. There are two cases.

1) *S is an independent set.* Let $s_1, s_2, s_3 \in S$. The set $S - \{s_1\}$ has cardinality and Steiner-distance k in G (x is a common neighbour), so by assumption there is a common neighbour u_1 of $S - \{s_1\}$ in H. Similarly, there is a common neighbour u_2 of $S - \{s_2\}$ in H.

Now, the graph induced by the set $S \cup \{u_1, u_2\}$ is connected. So, by assumption, there is a common neighbour u_3 of $S - \{s_3\}$ in the set $S \cup \{u_1, u_2\}$. Since S is independent, it follows that u_3 is one of u_1 or u_2. That is, u_3 is a common neighbour of all of S in H, as required.

2) *S is not an independent set.* Then there exists a vertex y in a nontrivial component of the graph induced by S such that the graph induced by $(S - \{y\}) \cup \{x\}$ is connected. Thus the set $S - \{y\}$ has cardinality and Steiner-distance k in G, and so has Steiner-distance k in H by the assumption. But y has a neighbour in S and so can be added to the Steiner tree for $S - \{y\}$ for the cost of one edge. Thus S has Steiner-distance $k+1$ in H, as required. QED

Lemma 2 *For all $d \geq k \geq 3$ it holds that $S(k,d)$ implies $S(k-1,d)$.*

PROOF. Assume graph G obeys $S(k,d)$. Let $S \subseteq V(G)$ have cardinality $k-1$ and Steiner-distance d in G. Say this is achieved by Steiner tree T^*. Let H be any connection for S.

Let z be any vertex of T^* not in S but adjacent to a vertex of S (exists since $d \geq k$). Then the graph induced by $V(H) \cup \{z\}$ is connected. So it contains a tree T of Steiner-distance d for $S \cup \{z\}$. Let m be a vertex of T not in $S \cup \{z\}$ (exists since $d \geq k$); necessarily $m \in V(H)$. Then the set $S \cup \{m\}$ has Steiner-distance d in G, and therefore $S \cup \{m\}$ and hence S has Steiner-distance d in H, as required. QED

Lemma 3 *For $d \geq k \geq 2$ it holds that $S(k,d)$ implies $S(k,d+1)$.*

PROOF. Assume graph G obeys $S(k,d)$. Let $S \subseteq V(G)$ have cardinality k and Steiner-distance $d+1$ in G. Say this is achieved by Steiner tree T^*. Let y be an end-vertex of T^*; of course, $y \in S$. Let z be the neighbour of y in T^*. Let $S' = (S - y) \cup \{z\}$. Then S' has Steiner distance d in G. (The tree $T^* - y$ shows the distance at most d; any smaller then adding y would contradict the distance of S.)

Let H be any connection for S. We need to show that S has Steiner-distance $d+1$ in H. Assume $z \in S$. Then since we have property $S(k-1,d)$ (by the above lemma), the set S' has Steiner-distance d in H. So S has Steiner-distance $d+1$ in H (by adding edge yz to the Steiner tree for S').

So assume $z \notin S$. By assumption, the set S' has Steiner-distance d in the graph induced by $V(H) \cup \{z\}$; say by Steiner tree T'. Let y' be an end-vertex of T' such that neither it nor its neighbour in T' is z. (Such a vertex exists since $d \geq k$ implies that T' is not a star centered at z.) Let z' be the neighbour of y' in T'.

Let $S'' = (S - y') \cup \{z'\}$. Then S' has distance d in G. (Adding the edge yx to the tree $T' - y'$ shows that distance at most d.) Furthermore, H contains S''. So S'' has distance d in H, and thus S has distance $d+1$ in H (by adding edge $y'z'$ to the Steiner tree for S''), as required. QED

There are some limits to these implications. For instance, graphs which satisfy property $S(k)$ but not property $S(k-1)$ also show that property $S(k,k)$ does not imply property $S(k-1,k-1)$. One such example is the cycle C_{k+2} on $k+2$ vertices for $k \geq 3$. Another example of a graph which satisfies property $S(k)$ but not property $S(k-1)$ for $k \geq 4$ is obtained by taking the complete bipartite graph $K(k,k)$ and removing a perfect matching. (Details omitted.)

Also, the line graph $L(K_n)$ of the complete graph (vacuously) satisfies property $S(2,3)$ but does not satisfy property $S(3,4)$ for n large.

Another example graph G is obtained by taking a set X of six elements and making $V(G)$ the set of the 15 subsets of X of cardinality 3, with two vertices of G adjacent iff the subsets overlap in two elements. It can be shown that G satisfies property $S(2,3)$ but not property $S(2,2)$.

Finally, we note that the results in this paper imply a polynomial-time recognition algorithm for k-Steiner-distance-hereditary graphs. To see this, it suffices to show that we can verify whether $S(k,k)$ holds for any fixed k. We look at all sets S of cardinality k and consider only those of Steiner-distance k. These are the S that do not induce a connected subgraph, but there exists a vertex x such that $S \cup \{x\}$ induces a connected subgraph. To check whether the distance of S is preserved in every connection for it, remove all those vertices which are adjacent to every component of S; if what remains is connected then the distance of S is not preserved in every connection for it, otherwise S is okay.

References

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