# Multi-Fractional Domination

Peter J. Slater and Eric L. Trees
Mathematical Sciences
University of Alabama in Huntsville
Huntsville, Alabama
USA 35899

**ABSTRACT**: As an extension of the fractional domination and fractional domatic graphical parameters, multi-fractional domination parameters are introduced. We demonstrate the Linear Programming formulations, and to these formulations we apply the Partition Class Theorem, which is a generalization of the Automorphism Class Theorem. We investigate some properties of the multi-fractional domination numbers and their relationships to the fractional domination and fractional domatic numbers.

## 1 Introduction

For a graph $G = (V, E)$ of order $n$ and size $m$ with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G) = \{e_1, e_2, ..., e_m\}$, we define the open neighborhood of a vertex $v_i$ to be the set of all vertices adjacent to it, $N(v_i) = \{v_j \in V(G) : v_iv_j \in E(G)\}$, and its closed neighborhood is the set $N[v_i] = N(v_i) \cup \{v_i\}$. The closed neighborhood matrix $N = [n_{i,j}]$ is the binary $n$-by-$n$ matrix with $n_{i,j} = 1$ if $v_j \in N[v_i]$, and $n_{i,j} = 0$ otherwise. The degree of a vertex $v_i$ is the number of vertices to which $v_i$ is adjacent, $deg\, v_i = |N(v_i)|$. The minimum degree over all vertices in a graph $G$ is denoted by $\delta(G)$, and the maximum degree is denoted by $\Delta(G)$. If $D \subseteq V(G)$ and $|N[v] \cap D| \geq 1$ for every vertex $v \in V(G)$, then $D$ is said to be a dominating set of $G$. The domination number of a graph $G$, $\gamma(G)$, is the minimum cardinality of a dominating set in $G$. We call a function $f : V(G) \rightarrow \{0, 1\}$ a dominating function if for all $v \in V(G)$ we have $\sum_{w \in N[v]} f(w) \geq 1$. We note that a dominating function is merely the characteristic function of a dominating set. We will next consider a variation on the idea behind a dominating function.

As defined by Farber [4], a function $f : V(G) \rightarrow [0, 1]$ with $\sum_{w \in N[v]} f(w) \geq 1$ for all $v \in V(G)$ is called a fractional dominating function (fdf). The weight of $f$ is $w(f) = \sum_{v \in V(G)} f(v)$, and the minimum weight over all fractional dominating functions is called the fractional domination number of $G$, $\gamma_f(G)$.

JCMCC **40** (2002), pp. 171-181

Since every dominating function is also a fractional dominating function, we have that $\gamma_f(G) \le \gamma(G)$ for any graph $G$. If we define the function $g : V(G) \to [0,1]$ by $g(v) = \frac{1}{\delta(G)+1}$ for every vertex $v \in V(G)$, then $g$ is an fdf with $w(g) = \frac{n}{\delta(G)+1}$. Also, if $g : V(G) \to [0,1]$ is any fdf of the graph $G$, then we must have that $\sum_{w \in N[v]} g(w) \ge 1$. So, $n \le \sum_{v \in V(G)} 1 \le \sum_{v \in V(G)} \sum_{w \in N[v]} g(w) = \sum_{v \in V(G)} (deg\ v + 1) g(v) \le \sum_{v \in V(G)} (\Delta(G) + 1) g(v) = (\Delta(G) + 1) \sum_{v \in V(G)} g(v)$. Thus, we have established the following theorem and corollary of Grinstead and Slater [5].

**Theorem 1** [5]: If $G$ is a graph of order $n$ with minimum vertex degree $\delta(G)$ and maximum vertex degree $\Delta(G)$, then $n\frac{1}{\Delta(G)+1} \le \gamma_f(G) \le n\frac{1}{\delta(G)+1}$.

**Corollary 2** [5]: If $G$ is regular of degree $r$, then $\gamma_f(G) = n\frac{1}{r+1}$.

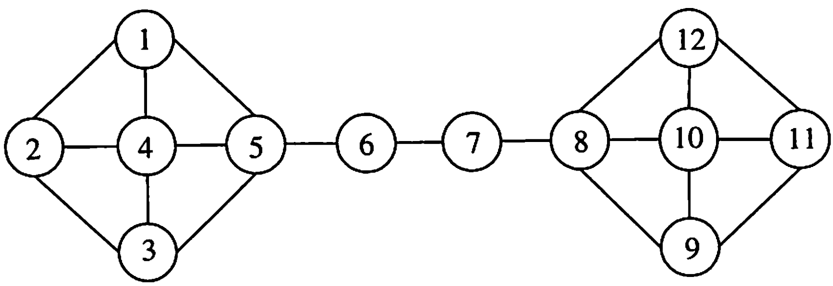


**Figure 1**: The graph $DB$

For a graph $G$, we say that a set $S$ of dominating functions on $G$ is disjoint if $\sum_{f \in S} f(v) \le 1$ for all $v \in V(G)$. This definition is equivalent to saying that dominating sets whose characteristic functions are those in $S$ are disjoint. In [2], Cockayne and Hedetniemi define the domatic number, $d(G)$, to be the maximum cardinality of a collection of disjoint dominating sets. They give several bounds on the domatic number, one of which is $d(G) \le \delta(G) + 1$ for any graph $G$. The graph $DB$ in Figure 1 has disjoint dominating sets $\{1,5,8,12\}$, $\{2,6,10\}$, and $\{4,7,11\}$, so $d(DB) = 3 = \delta(DB) + 1$. If a set $S$ of fractional dominating functions has $\sum_{f \in S} f(v) \le 1$ for all $v \in V(G)$ ("disjointness"), then the set $S$ is called a fractional dominating family (FDF). In [3], Rall defines the fractional domatic number $d_f(G)$ to be the maximum cardinality of an FDF. If we define

$g_i$: $V(G) \to [0,1]$ by $g_i(v) = \frac{1}{\delta(G)+1}$, then the set $\{g_1, g_2, g_3, ..., g_{\delta(G)+1}\}$ is an FDF of cardinality $\delta(G)+1$. If we let $S$ be any FDF of $G$, and $deg\, v = \delta(G)$, then $|S| = t = \sum_{i=1}^{t} 1 \leq \sum_{i=1}^{t} \sum_{w \in N[v]} f_i(w) = \sum_{w \in N[v]} \sum_{i=1}^{t} f_i(w) \leq \sum_{w \in N[v]} 1 = \delta(G)+1$. So, we have established Rall's result that $d_f(G) = \delta(G)+1$ for any graph $G$. In the example of Figure 1, we see that the minimum weight of one fdf is $\gamma_f(DB) = 3$ by placing values of $\frac{1}{2}$ on the vertices of the set {2, 4, 6, 7, 10, 11} and zeroes elsewhere. The minimum combined weight of two disjoint fdf's is $2\gamma_f(DB) = 6$, which can be achieved by using the same values from above for each of two fdf's. If we desire three disjoint fdf's, the minimum combined weight will be 10, which is achieved by placing values of $\frac{1}{3}$ on the vertices of the set {1, 2, 3, 5, 6, 7, 8, 9, 11, 12} and zeroes elsewhere for each of the three fdf's. We see that the combined weight of two disjoint fdf's is just twice the weight of one fdf, but for three disjoint fdf's we must exceed three times the weight of one. In the following section, we expand on these observations.

## 2 Multi-Fractional Domination

We define the k-fractional domination number of $G$ to be $\gamma_f^k(G) = min\sum_{i=1}^{k} w(f_i) = min\sum_{i=1}^{k} \sum_{v \in V} f_i(v)$, where $\{f_1, f_2, ..., f_k\}$ is an FDF. Using this definition, we now have for the graph $DB$ of Figure 1 that $\gamma_f^1(DB) = 3$, $\gamma_f^2(DB) = 6$, and $\gamma_f^3(DB) = 10$. A second example is the six cycle, for which $\gamma_f^1(C_6) = 2$, $\gamma_f^2(C_6) = 4$, and $\gamma_f^3(C_6) = 6$. Since we know that the maximum cardinality of a FDF is $d_f(G) = \delta(G)+1$, we can only define $\gamma_f^k(G)$ for $k \in \{1, 2, ..., \delta(G)+1\}$. Also, because all of the functions in an FDF of cardinality $k$ are fractional dominating functions, and the weight of each one is bounded below by $\gamma_f(G)$, we have the following observation. If $G$ is a graph, then $\gamma_f^k(G) \geq k\gamma_f(G)$ for $k \in \{1, 2, ..., \delta(G)+1\}$. In addition, we can always create an FDF of cardinality $1 \leq k \leq \delta(G)+1$ by allowing each of the $k$ functions to take on the value $\frac{1}{\delta(G)+1}$ on every vertex of $G$. Hence, we have the next two propositions.

**Proposition 3**: If $G$ is a graph, then $k\gamma_f(G) \leq \gamma_f^k(G) \leq k\frac{n}{\delta(G)+1}$ for $1 \leq k \leq \delta(G)+1$.

**Proposition 4**: If $G$ is regular of degree $r$, then $\gamma_f^k(G) = \frac{nk}{r+1}$ for $1 \leq k \leq \delta(G)+1$.

Consider the example of the complete multi-partite graph $K_{3,5,7,11,13}$, for which $n = 39$ and $\delta(K_{3,5,7,11,13}) + 1 = 27$. The values of $\gamma_f^k(K_{3,5,7,11,13})$ for $1 \leq k \leq 27$ are listed in Table 1 along with the values of the upper and lower bounds of Theorem 3. In addition, the plot in Figure 2 represents the same data, where the solid lines correspond to the bounds in Proposition 3. Note that $\gamma_f^k(K_{3,5,7,11,13}) = k\gamma_f(K_{3,5,7,11,13})$ precisely for $1 \leq k \leq 10$, and $\gamma_f^k(K_{3,5,7,11,13}) = k\frac{n}{\delta(K_{3,5,7,11,13})+1}$ only for $k = 27 = \delta(K_{3,5,7,11,13}) + 1$.

**Table 1:** $\gamma_f^k(K_{3,5,7,11,13})$ for $1 \leq k \leq 27$

| k | $k\gamma_f(G)$ | $\gamma_f^k(G)$ | $k\frac{n}{\delta(G)+1}$ | | k | $k\gamma_f(G)$ | $\gamma_f^k(G)$ | $k\frac{n}{\delta(G)+1}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 61/51 | 61/51 | 13/9 | | 15 | 305/17 | 55/3 | 65/3 |
| 2 | 122/51 | 122/51 | 26/9 | | 16 | 976/51 | 353/18 | 208/9 |
| 3 | 61/17 | 61/17 | 13/3 | | 17 | 61/3 | 188/9 | 221/9 |
| 4 | 244/51 | 244/51 | 52/9 | | 18 | 366/17 | 1046/47 | 26 |
| 5 | 305/51 | 305/51 | 65/9 | | 19 | 1159/51 | 1113/47 | 247/9 |
| 6 | 122/17 | 122/17 | 26/3 | | 20 | 1220/51 | 1180/47 | 260/9 |
| 7 | 427/51 | 427/51 | 91/9 | | 21 | 427/17 | 1247/47 | 91/3 |
| 8 | 488/51 | 488/51 | 104/9 | | 22 | 1342/51 | 1314/47 | 286/9 |
| 9 | 183/17 | 183/17 | 13 | | 23 | 1403/51 | 2113/71 | 299/9 |
| 10 | 610/51 | 610/51 | 130/9 | | 24 | 488/17 | 2244/71 | 104/3 |
| 11 | 671/51 | 119/9 | 143/9 | | 25 | 1525/51 | 2375/71 | 325/9 |
| 12 | 244/17 | 29/2 | 52/3 | | 26 | 1586/51 | 2506/71 | 338/9 |
| 13 | 793/51 | 142/9 | 169/9 | | 27 | 549/17 | 39 | 39 |
| 14 | 854/51 | 307/18 | 182/9 | | | | | |

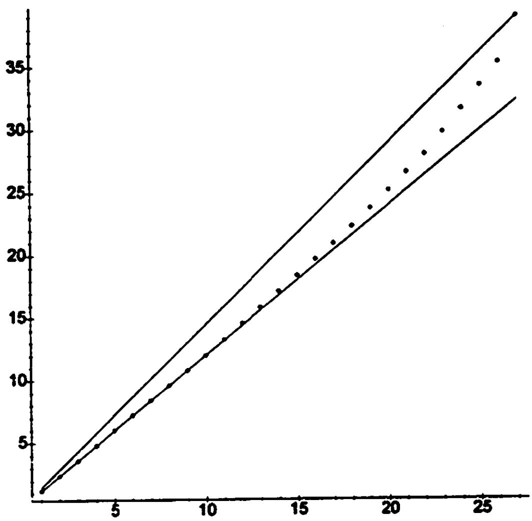


**Figure 2:** Plot of $\gamma_f^k(K_{3,5,7,11,13})$ for $1 \leq k \leq 27$

One item we want to investigate is the relationship between successive values of $\gamma_f^k(G)$ for a given graph. For instance, the values of $\gamma_f^k(K_{3,5,7,11,13})$ for $1 \leq k \leq 10$ lie on a line whose slope is $\frac{61}{51}$, which is the line defined by the lower bound of Proposition 3. Similarly, when $11 \leq k \leq 17$, $18 \leq k \leq 22$, and $23 \leq k \leq 26$ the values lie on lines of slope $\frac{23}{18}$, $\frac{67}{47}$, and $\frac{131}{71}$, respectively. It is also interesting to note that $\gamma_f^{27}(K_{3,5,7,11,13}) = n = 39$, yet for the graph $DB$ in Figure 1 we have $3\gamma_f^1(DB) = 9 < \gamma_f^3(DB) = 10 < n = 12$. We generalize this result in the next theorem.

**Theorem 5:** In a graph $G$ with order $n$, $\gamma_f^{\delta(G)+1}(G) = n$ if and only if every vertex has minimum degree or is adjacent to a vertex of minimum degree.

Proof: To prove the necessity, assume $\gamma_f^{\delta(G)+1}(G) = n$. Assume that there exists some vertex $v_0$ such that $N[v_0] \cap \{v | \; deg\, v = \delta(G)\} = \emptyset$. The functions $f_i : V(G) \rightarrow [0,1]$ for $1 \leq i \leq \delta(G)+1$ defined by $f_i(v) = \frac{1}{\delta(G)+1}$ for all $v \in V(G) - \{v_0\}$ and $f_i(v_0) = 0$ form an FDF. But, $\sum_{i=1}^{\delta(G)+1} w(f_i) = n - 1 < n$, a contradiction.

Next, we prove the sufficiency. In the case that $G$ is regular of degree $\delta(G)$, Proposition 4 states $\gamma_f^{\delta(G)+1}(G) = n$. So, assume that $G$ is not regular, but every vertex has minimum degree or is adjacent to a vertex of minimum degree. There exists an $FDF$ for which the value assigned to each given vertex is the same for each function in the $FDF$ (a consequence of the Partition Class Theorem and the LP formulation of $\gamma_f^k(G)$, which will be discussed in the next section). Let $f : V(G) \rightarrow [0,1]$ be such a solution. Since $(\delta(G)+1)f(v) \leq 1$ for every vertex $v$, then $f(v) \leq \frac{1}{\delta(G)+1}$. For contradiction, assume that the minimum of the set $\{f(v) : v \in V(G)\}$ is strictly less than $\frac{1}{\delta(G)+1}$. Let $v$ be a vertex which achieves the minimum. If $v$ is a minimum degree vertex, then $f(N[v]) < \frac{1}{\delta(G)+1} + \delta(G)\frac{1}{\delta(G)+1} = 1$, a contradiction that $f$ dominates. If $v$ is not a minimum degree vertex, then it must be adjacent to a minimum degree vertex $w$. Then, $f(N[w]) < \frac{1}{\delta(G)+1} + \delta(G)\frac{1}{\delta(G)+1} = 1$, a contradiction that $f$ dominates. So, $f(v) = \frac{1}{\delta(G)+1}$ for every vertex $v$ in $G$, and $\gamma_f^{\delta(G)+1}(G) = n$. □

Now that we have identified the class of graphs which do achieve the upper bound at $\delta(G)+1$, we would like to know if there are graphs which achieve the upper bound somewhere before $\delta(G)+1$, and what can happen to the values of $\gamma_f^k(G)$ before and after the upper bound is achieved. Before we endeavor to show this, we will introduce a different way of formulating domination, fractional domination, and multi-fractional domination.

## 3 IP/LP Formulations

Many graph theoretic subset parameters (as well as their fractional counterparts) have Integer Programming (IP) (respectively, Linear Programming (LP)) formulations. See [8] and [9] for several formulation examples. One example, the domination number can be IP-formulated as $\gamma(G) = min\sum_{i=1}^{n} x_i$ subject to $NX \geq \mathbf{1}_n$, where each $x_i \in \{0, 1, 2, 3, ...\}$, $X = [x_1, x_2, ..., x_n]^t$, $N$ is the closed neighborhood matrix, $\mathbf{1}_n$ is the column $n$-vector of all ones, and the decision variables determine a dominating function $f$ with $f(v_i) = x_i$. If we change $x_i \in \{0, 1, 2, 3, ...\}$ to $x_i \geq 0$, we have the formulation for $\gamma_f(G)$. In [4], Farber considers the problem of determining when $\gamma(G) = \gamma_f(G)$.

In (1) below, we give the LP formulation of $\gamma_f^k(G)$. Note that for any $k \in \{1, 2, 3, ..., \delta(G) + 1\}$, $kn$ is the number of decision variables. For instance, the graph $K_{3,5,7,11,13}$ will have from 39 decision variables when $k = 1$ to 1053 variables when $k = 27$.

(1) Let $G$ be a graph with order $n$ and $f_i(v_j) = x_{i,j}$ for $1 \leq i \leq k$ and $1 \leq j \leq n$. If $X_i = [x_{i,1}, x_{i,2}, ..., x_{i,n}]^t$ for $1 \leq i \leq k$, then

$$\gamma_f^k(G) = min\sum_{i=1}^{k}\sum_{j=1}^{n} x_{i,j}$$

subject to $NX_i \geq \mathbf{1}_n$ for $1 \leq i \leq k$,

$$\sum_{i=1}^{k} X_i \leq \mathbf{1}_n,$$

$x_{i,j} \geq 0$ for $1 \leq i \leq k$ and $1 \leq j \leq n$.

Clearly, the number of decision variables in such an LP problem can become extremely unwieldy. The amount of time to formulate the problem into LP software and the amount of memory space taken up by a formulation of this size would be undesirable.

The Automorphism Class Theorem (A.C.T.) in [11], determines conditions under which the solution to certain LP formulated graph problems will possess the same value on vertices in the same automorphism class. A generalization of this, the Partition Class Theorem (P.C.T.) in [12], gives conditions for general LP problems to possess equal values on certain groups of decision variables.

**Theorem 6**: **(P.C.T.)** [12]: Let $W_k = \{1, 2, ..., k\} = R_1 \cup R_2 \cup ... \cup R_h$, where $R_1 = \{r_0 + 1 = 1, 2, ..., r_1\}$, $R_2 = \{r_1 + 1, ..., r_2\}$, ... , $R_i = \{r_{i-1} + 1, ..., r_i\}$, ..., $R_h = \{r_{h-1} + 1, ..., r_h = k\}$, and let $W_j = \{1, 2, ..., j\} = S_1 \cup S_2 \cup ... \cup S_t$, where $S_1 = \{s_0 + 1 = 1, 2, ..., s_1\}$, $S_2 = \{s_1 + 1, ..., s_2\}$, ..., $S_i = \{s_{i-1} + 1, ..., s_i\}$, ... , $S_t = \{s_{t-1} + 1, ..., s_t = j\}$. Let $M \in \mathbb{R}^{j \times k}$

such that for $u, v \in S_i$, we have $f_{i,g} = \sum_{w \in R_g} M_{u,w} = \sum_{w \in R_g} M_{v,w}$ for $1 \leq i \leq t$ and $1 \leq g \leq h$, and for $u, v \in R_g$ we have $d_{i,g} = \sum_{w \in S_i} M_{w,u} = \sum_{w \in S_i} M_{w,v}$ for $1 \leq g \leq h$ and $1 \leq i \leq t$. Let $X = [a_1, a_2, ..., a_k]^t$ be a solution to one of the following linear programming problems

(i) $min \sum_{i=1}^{k} c_i x_i$ subject to $MX \geq B, x_i \geq 0$

or (ii) $max \sum_{i=1}^{k} c_i x_i$ subject to $MX \leq B, x_i \geq 0$.

Define $X^* = [a_1^*, a_2^*, ..., a_k^*]^t$ by $a_i^* = \sum_{w \in R_g} \frac{a_w}{|R_g|}$ for all $i \in R_g$ for each $1 \leq g \leq h$.

Then $X^*$ is a solution to (1) or (2), respectively, when $C = [c_1, c_2, ..., c_k]$ is constant on each $R_i$, and $B = [b_1, b_2, ..., b_j]^t$ is constant on each $S_i$.

Using the P.C.T. we can conclude that there exists an FDF, say $\{f_1, f_2, ..., f_k\}$, that achieves $\gamma_f^k(G)$ where all of the $f_i'$s are identical. We may now reduce the size of the LP formulation for $\gamma_f^k(G)$ to that which is given in (2).

(2) Let $G$ be a graph with order $n$ and $f_i(v_j) = x_j$ for $1 \leq i \leq k$ and $1 \leq j \leq n$. If $X = [x_1, x_2, ..., x_n]^t$, then

$$\gamma_f^k(G) = min\ k\sum_{j=1}^{n} x_j$$

subject to $NX \geq \mathbf{1}_n$,

$kx_j \leq 1$ for $1 \leq j \leq n$,

$x_j \geq 0$ for $1 \leq j \leq n$.

We now have an LP problem with only $n$ decision variables for any allowed value of $k$. This is clearly a significant decrease in the number of decision variables. We can easily change the value of $k$ in any LP solving package in order to generate all values of $\gamma_f^k(G)$ from nearly the same formulation. In (3), we show an example of the reduced LP formulation of $\gamma_f^k(K_{3,5,7,11,13})$. In this case, note that the number of decision variables is not $n = 39$. Because of the regularity of $K_{3,5,7,11,13}$, we are able to apply the P.C.T. a second time to the reduced formulation of (2) and obtain the formulation (3) with only five decision variables.

$$(3)\quad \gamma_f^k(K_{3,5,7,11,13}) = min\ k(3x_1 + 5x_2 + 7x_3 + 11x_4 + 13x_5)$$

$$\text{subject to}\quad \begin{bmatrix} 1 & 5 & 7 & 11 & 13 \\ 3 & 1 & 7 & 11 & 13 \\ 3 & 5 & 1 & 11 & 13 \\ 3 & 5 & 7 & 1 & 13 \\ 3 & 5 & 7 & 11 & 1 \end{bmatrix} X \geq \mathbf{1}_5$$

$$k\mathbf{I}_n X \leq \mathbf{1}_5$$

$$x_j \geq 0 \quad \text{for } 1 \leq j \leq 5$$

Within the LP formulation of $\gamma_f^k(G)$, the number $k$ is used only as a scalar multiple, so we could consider all real values $k \in [1, \delta(G) + 1]$. Although this extension has no apparent graph theoretic description for noninteger values of $k$, its analysis does give us some insight into the behavior of $\gamma_f^k(G)$ for the integral values. If we consider $k$ to be in the continuous interval $[1, 27]$, the plot of Figure 2 now becomes that in Figure 3.

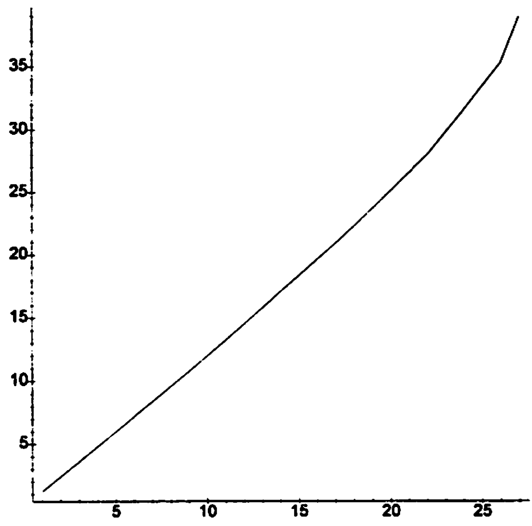


**Figure 3**: The plot of $\gamma_f^k(K_{3,5,7,11,13})$ where $k \in [1, 27]$.

Although it is difficult to see from Figure 3, the plot is the piecewise linear convex curve given in (4).

$$\text{(4)} \qquad \gamma_f^k(K_{3,5,7,11,13}) = \begin{cases} \frac{61}{51}k & 1 \le k \le \frac{51}{5} \\ \frac{23}{18}k - \frac{5}{6} & \frac{51}{5} \le k \le \frac{87}{5} \\ \frac{67}{47}k - \frac{160}{47} & \frac{87}{5} \le k \le \frac{221}{10} \\ \frac{131}{71}k - \frac{900}{71} & \frac{221}{10} \le k \le \frac{161}{6} \\ 13k - 312 & \frac{161}{3} \le k \le 27 \end{cases}.$$

In fact, the function $\gamma_f^k(G)$ on $[1, \delta(G)+1]$ will always be a piecewise linear convex function.

**Theorem 7**: If $M$ is an $m$-by-$n$ nonnegative real matrix with all row sums at least one, $C = [c_1, c_2, ..., c_n]$, $\mathbf{1}_n = [1, 1, ..., 1]^t$, $X = [x_1, x_2, ..., x_n]^t$, then

$$\begin{aligned} &Z(k) = minimum(kCX) \\ &\text{subject to } \; MX \ge \mathbf{1}_n \\ &\qquad\qquad\quad X \le (1/k)\mathbf{1}_n \\ &\qquad\qquad\quad x_i \ge 0 \text{ for } 1 \le i \le n \end{aligned}$$

is a piecewise linear and convex function for $1 \le k \le m = min\{r_i : r_i$ is the sum of the entries in the $i$th row of $M\}$.

, Proof: By parametric linear programming [1], we have

$$\begin{aligned} &Y(\lambda) = minimum(CX) \\ &\text{subject to } \; MX \ge \mathbf{1}_n \\ &\qquad\qquad\quad X \le (1-\lambda)\mathbf{1}_n \\ &\qquad\qquad\quad x_i \ge 0 \text{ for } 1 \le i \le n \end{aligned}$$

is a piecewise linear and convex function for $0 \le \lambda \le 1 - \frac{1}{m}$. If $\lambda = \frac{k-1}{k}$, then $Z(k) = kY(\frac{k-1}{k})$. Since $Y(\lambda)$ is piecewise linear and convex then $Y$ can be written as follows :

$$Y(\lambda) = \begin{cases} m_1\lambda + b_1 & [0, \lambda_1] \\ m_2\lambda + b_2 & [\lambda_1, \lambda_2] \\ \vdots & \\ m_i\lambda + b_i & [\lambda_{i-1}, \lambda_i] \\ \vdots & \\ m_p\lambda + b_p & [\lambda_{p-1}, \lambda_p = 1 - \frac{1}{m}] \end{cases}$$

where $b_i = m_{i-1}\lambda_{i-1} - m_i\lambda_{i-1} + b_{i-1}$ for $2 \le i \le p$, and $m_1 < m_2 < ... < m_p$, but

$$Z(k) = kY(\tfrac{k-1}{k}) = \begin{cases} (m_1 + b_1)k - m_1 & [1, k_1] \\ (m_2 + b_2)k - m_2 & [k_1, k_2] \\ \vdots & \\ (m_i + b_i)k - m_i & [k_{i-1}, k_i] \\ \vdots & \\ (m_p + b_p)k - m_p & [k_{p-1}, k_p = m] \end{cases}$$

where $k_i = \frac{1}{1-\lambda_i}$. Hence $Z$ is piecewise linear on $[1, m]$. But, we need to show that $Z$ is continuous on $[1, m]$, that is, adjacent linear pieces of $Z$ have a common intersection. Consider the following:

$$\begin{aligned}
q &= (m_i + b_i)k_{i-1} - m_i \\
&= (m_i + (m_{i-1}\lambda_{i-1} - m_i\lambda_{i-1} + b_{i-1}))k_{i-1} - m_i \\
&= (m_i + m_{i-1}(\tfrac{k_{i-1}-1}{k_{i-1}}) - m_i(\tfrac{k_{i-1}-1}{k_{i-1}}) + b_{i-1})k_{i-1} - m_i \\
&= m_ik_{i-1} + m_{i-1}(\mathrm{k}_{i-1} - 1) - m_i(k_{i-1} - 1) + b_{i-1}k_{i-1} - m_i \\
&= m_ik_{i-1} + m_{i-1}k_{i-1} - m_{i-1} - m_ik_{i-1} + m_i + b_{i-1}k_{i-1} - m_i \\
&= m_{i-1}k_{i-1} - m_{i-1} + b_{i-1}k_{i-1} \\
&= (m_{i-1} + b_{i-1})k_{i-1} - m_{i-1}
\end{aligned}$$

for $2 \leq i \leq p$. Therefore, $Z$ is a continuous piecewise linear function on $[1, m]$. We only need show that Z is also convex, that is, $m_i + b_i > m_{i-1} + b_{i-1}$. Consider the following.

$$\begin{array}{ll}
\lambda_{i-1} & < \ 1 \\
(m_{i-1} - m_i)\lambda_{i-1} & > \ m_{i-1} - m_i \\
(m_{i-1} - m_i)\lambda_{i-1} + m_i & > \ m_{i-1} \\
m_{i-1}\lambda_{i-1} - m_i\lambda_{i-1} + m_i + b_{i-1} & > \ m_{i-1} + b_{i-1} \\
m_i + (m_{i-1}\lambda_{i-1} - m_i\lambda_{i-1} + b_{i-1}) & > \ m_{i-1} + b_{i-1} \\
m_i + b_i & > \ m_{i-1} + b_{i-1}
\end{array}$$

Hence, the slopes of the piecewise linear curve are strictly increasing, which implies that $Z$ is convex.□

**Corollary 8**: A graph $G$ must have $\gamma_f^k(G) = k\frac{n}{\delta(G)+1}$ (upper bound of Theorem 3) for all $k \in \{1, 2, ..., \delta(G) + 1\}$, or only achieve the upper bound at $k = \delta(G) + 1$.

## 4 Future Work

Although Theorem 7 gives a complete view of the behavior of the multi-fractional domination numbers, we are developing a generalization of the fractional domatic and domatic numbers. Also, there is work being done by Hedetniemi, Hedetniemi, Markus, and Slater [10] to define interesting multiple domination parameters, such as minimizing the cardinality of the union of two disjoint dominating sets, that is related to the work of Grinstead and Slater [6,7] and the results of this paper.

**References**

[1] M.S. Bazaraa, J.J. Jarvis and H.D. Sherali, Duality and sensitivity analysis, in Linear Programming and Network Flows Second Edition, John Wiley and Sons, Inc., 1990, 294-301.

[2] E.J. Cockayne and S.T. Hedetniemi, Towards a theory of domination in graphs, *Networks* **7** (1977), 247-261.

[3] D.F. Rall, A fractional version of domatic number, *Congressus Numerantium* **74** (1990), 100-106.

[4] M. Farber, Domination, independent domination, and duality in strongly chordal graphs, *Discrete Applied Math.* **7** (1984), 115-130.

[5] D.L. Grinstead and P.J. Slater, Fractional domination and fractional packing in graphs, *Congressus Numerantium* **71** (1990), 153-172.

[6] D.L. Grinstead and P.J. Slater, On minimum dominating sets with minimum intersection, *Discrete Math.* **86** (1990), 239-254.

[7] D.L. Grinstead and P.J. Slater, On minimum intersection of minimum dominating sets in series-parallel graphs, Graph Theory, Combinatorics and Applications, eds. Y. Alari, G. Chartrand, O.R. Oellermann and A. Schwenk, John Wiley and Sons, Inc., 1991.

[8] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, Inc., 1998.

[9] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, eds., Domination in Graphs Advanced Topics, Marcel Dekker, Inc., 1998.

[10] S.M. Hedetniemi, S.T. Hedetniemi, L. Markus and P.J. Slater, in preparation.

[11] P.J. Slater, The automorphism class theorem for LP-based graphical parameters, *Bulletin of the ICA*, Vol. 20 (1997), 19-26.

[12] P.J. Slater and E.L. Trees, A matrix partition linear programming theorem, *Bulletin of the ICA*, Vol. 28 (2000), 75-84.