Domination number of graphs with bounded diameter *

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Abstract

We prove that the domination number of every graph of diameter 2 on n vertices is at most $\left(\frac{1}{\sqrt{2}} + o(1)\right)\sqrt{n \log n}$ as $n \to \infty$ (with logarithm of base e). This result is applied to prove that if a graph of order n has diameter 2, then it contains a spanning caterpillar whose diameter does not exceed $\left(\frac{3}{\sqrt{2}} + o(1)\right)\sqrt{n \log n}$. These estimates are tight apart from a multiplicative constant, since there exist graphs of order n and diameter 2, with domination number not smaller than $\left(\frac{1}{2\sqrt{2}} + o(1)\right)\sqrt{n \log n}$. In contrast, in graphs of diameter 3, the domination number can be as large as $\lfloor \frac{n}{2} \rfloor$ (but not larger).

Our results concerning diameter 2 improve the previous upper bound of $O(n^{3/4})$, published by Faudree *et al.* in [*Discuss. Math. Graph Theory* 15 (1995), 111–118].

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1 Introduction

In a graph $G = (V, E)$ with vertex set V and edge set E, a set $X \subseteq V$ is a *dominating set* if every $v \in V \setminus X$ is adjacent to some $x \in X$. The smallest cardinality of a dominating set in G, denoted $\gamma(G)$, is termed the *domination number* of G.

This paper is motivated by the work [2] of Faudree *et al.* where the authors study the possible structures of spanning trees in graphs. In particular, their last assertion deals with spanning caterpillars in graphs of diameter 2. An upper bound of $O(n^{3/4})$ is presented there for both the domination number and the minimum diameter of a spanning caterpillar (or, equivalently, the length of a shortest dominating path plus 2) in all n-vertex graphs of diameter 2. The main goal of the present work is to improve these estimates to

$$\gamma(G) \le \left(\frac{1}{\sqrt{2}} + o(1)\right)\sqrt{n \log n}$$

and the upper bound $\left(\frac{3}{\sqrt{2}} + o(1)\right)\sqrt{n \log n}$ for the diameter of a spanning caterpillar (throughout the paper, 'log' denotes logarithm of base e). We also show that these new estimates are best possible apart from a multiplicative constant. More explicitly, we prove that there exists an infinite sequence of graphs of order n and diameter 2 whose domination number is at least $\left(\frac{1}{2\sqrt{2}} + o(1)\right)\sqrt{n \log n}$ as $n \to \infty$.

The situation changes dramatically when the diameter is assumed to be 3 (or larger). Then the domination number can be as large as linear in n, more precisely $\lfloor \frac{n}{2} \rfloor$, not only for connected but also for 2-connected graphs.

The domination number of graphs of diameter 2 is investigated in the next section, and the consequences on spanning caterpillars are presented in Section 3. The case of diameter ≥ 3 is considered in Section 4. Some open problems are discussed in the concluding section.

Notation We denote by $\gamma(n, d)$ the largest value of $\gamma(G)$ among graphs with n vertices and diameter at most d. Moreover, $\gamma_k(n, d)$ is defined similarly, with the additional assumption that the graphs considered are supposed to be k-connected.

2 Domination in graphs of diameter 2

In this section we prove the following result.

Theorem 1 $\left(\frac{1}{2\sqrt{2}}+o(1)\right)\sqrt{n\log n} \;\le\; \gamma(n,2) \;\le\; \left(\frac{1}{\sqrt{2}}+o(1)\right)\sqrt{n\log n}\,.$

Interestingly enough, both inequalities can be proved by probabilistic methods.

Proof of the upper bound Let $G=(V,E)$ be any graph on n vertices, with diameter 2. Suppose first that G contains a vertex v of degree at most $\frac{1}{\sqrt{2}}\sqrt{n\log n}$. Since the distance of each vertex from v is either 1 or 2, the set of neighbors of v dominates the entire G, thus $\gamma(G)\le\frac{1}{\sqrt{2}}\sqrt{n\log n}$ in this case.

Suppose next that the minimum degree in G exceeds $\frac{1}{\sqrt{2}}\sqrt{n\log n}$. The assertion for this case follows from the well-known more general result [3, Theorem 2.18] that the domination number of a graph on n vertices with minimum degree d is at most $\frac{1+\log d}{d}n$ (see [4] for a further generalization of this theorem, on vertex-partitioned graphs). For the sake of completeness, we present a short argument.

For $d>\frac{1}{\sqrt{2}}\sqrt{n\log n}$, we select a subset $X\subseteq V$ at random, by the rule

$$\mathsf{Prob}\,(v\in X)=\sqrt{\frac{\log n}{2n}}\,,$$

independently for all $v\in V$. The expected cardinality of this X is $\frac{1}{\sqrt{2}}\sqrt{n\log n}$.

Consider now the set Y of vertices not dominated by X. Let $v\in V$ be an arbitrary vertex, and denote the degree of v by d. Here $d>\frac{1}{\sqrt{2}}\sqrt{n\log n}$ holds by assumption. Thus, by the independent choice of the elements of X,

$$\mathsf{Prob}\,(v\in Y)=\mathsf{Prob}\,(v\notin X\wedge(u\notin X,\ \forall uv\in E))<\left(1-\sqrt{\frac{\log n}{2n}}\right)^{d}$$

$$<\left(\left(1-\sqrt{\frac{\log n}{2n}}\right)^{\sqrt{\frac{2n}{\log n}}}\right)^{\frac{1}{2}\log n}<e^{-\frac{1}{2}\log n}=\frac{1}{\sqrt{n}}\,.$$

Therefore, the expected cardinality of Y is smaller than $\sqrt{n}$. Since $X \cup Y$ is a dominating set, $\gamma(G) \le \left(\frac{1}{\sqrt{2}} + o(1)\right)\sqrt{n \log n}$ follows.

Proof of the lower bound Let $G = (V, E)$ be the *random graph* $G_{n,p}$, with n vertices and edge probability $p = p(n)$. Fix two distinct vertices $u, v \in V$. We are going to estimate the probability of being u and v at distance at least 3 apart.

For an arbitrary $z \in V \setminus \{u, v\}$,

$$\mathsf{Prob}\left((uz \notin E) \vee (vz \notin E)\right) = 1 - \mathsf{Prob}\left((uz \in E) \wedge (vz \in E)\right) = 1 - p^2 .$$

Moreover, this probability is not influenced by any other z'; i.e., it is totally independent of the presence or absence of paths $uz'v$, for all $z' \neq z$. Thus,

$$\mathsf{Prob}\left(\nexists\, z \in V : (uz \in E) \wedge (vz \in E)\right) = \left(1 - p^2\right)^{n-2} < (1 + o(1))\, e^{-np^2} .$$

Obviously, this probability becomes as small as $o(n^{-2})$ if we choose p to be

$$p = p(n) = \sqrt{\frac{(2+\epsilon)\log n}{n}}$$

for a suitable $\epsilon = \epsilon(n)$ tending to 0 sufficiently slowly. We take $G = G_{n,p}$ with this p; then, with probability $1 - o(1)$, each vertex pair has a common neighbor, hence the graph has diameter 2 almost surely.

We will prove that $\gamma(G) \ge \left(\frac{1}{2\sqrt{2}} - o(1)\right)\sqrt{n \log n}$ with probability $1 - o(1)$ as $n \to \infty$.

Let $m = m(n) \le \left(\frac{1}{2\sqrt{2}} - \epsilon'\right)\sqrt{n \log n}$, for an arbitrarily small but fixed $\epsilon' > 0$. Consider any m-element set $M \subset V$. The set of vertices adjacent to, but not contained in M will be denoted by $N(M)$. By definition, M dominates G if and only if $N(M) = V \setminus M$. The random choice of the edges implies that

$$\mathsf{Prob}\left(v \notin N(M)\right) = (1-p)^m = \left((1-p)^{\frac{1}{p}-1}\right)^{\frac{mp}{1-p}} > e^{-\frac{mp}{1-p}}$$

for each vertex $v \notin M$. Equivalently, for v fixed,

$$\mathsf{Prob}\left(v \in N(M)\right) < 1 - e^{-\frac{mp}{1-p}} .$$

Again by the independence of edges, there exist no dependencies among the events "$v \in N(M)$" for different vertices of $V \setminus M$ (because only the edges joining M with $N \setminus M$ are relevant in this context). Thus,

$$\mathsf{Prob}\,(N(M) = V \setminus M) < \left(1 - e^{-\frac{mp}{1-p}}\right)^{n-m} < e^{-(n-m)e^{-\frac{mp}{1-p}}},$$

the latter derived by the fact that $1 - x < e^{-x}$ holds for all $x > 0$. The choice of m and p implies

$$\frac{mp}{p-1} < (\tfrac{1}{2} - \epsilon'')\log n$$

for some fixed $\epsilon'' > 0$ and any sufficiently large n. Consequently, the bound obtained above does not exceed $e^{-(1+o(1))n^{1/2+\epsilon''}}$. On the other hand, there are

$$\binom{n}{m} < \left(\frac{en}{m}\right)^m = e^{m+m\log n - m\log m} < e^{m\log n}$$

different possible positions of M, therefore

$$\mathsf{Prob}\,(\exists\, M : |M| = m \wedge N(M) = V \setminus M) < e^{m\log n - (1+o(1))n^{1/2+\epsilon''}} = o(1).$$

Thus, $\gamma(G) > m$ holds with probability $1 - o(1)$. The proof of the theorem is completed. □

3 Spanning caterpillars of small diameter

A *caterpillar* C is a tree whose non-leaf vertices induce a path, called the *spine* of C. One of the problems studied by Faudree *et al.* in [2] is to find estimates on the minimum diameter of spanning caterpillars in graphs of diameter 2. Improving on their estimtate of $O(n^{3/4})$, the following asymptotic solution can be derived from the results of the previous section.

Theorem 2 *Every graph of order n and diameter 2 contains a spanning caterpillar of diameter at most $\left(\frac{3}{\sqrt{2}} + o(1)\right)\sqrt{n\log n}$, and this upper bound is tight apart from a multiplicative constant.*

Proof Let us observe first that if an upper bound of $O(\sqrt{n\log n})$ is valid, then it is tight indeed. This follows by the fact that the spine of any spaning caterpillar C dominates the graph in question. Therefore, $\gamma(n,2)$ is a lower bound on the diameter of C in a worst-case graph G. On applying Theorem 1, the asymptotic estimate $\Omega(\sqrt{n \log n})$ follows.

Concerning the upper bound, let $G = (V, E)$ be any graph of order n and diameter 2. Choose a dominating set $X \subset V$ such that $|X| \le \gamma(n,2)$. By Theorem 1, the proof will be done if we show that G contains a spanning caterpillar of diameter at most $3\,|X| - 3$.

First we select a caterpillar $C \subseteq G$ with the largest number of vertices such that the following three conditions are satisfied:

(*i*) the spine S of C contains every $x \in X$ belonging to C,

(*ii*) the spine begins and ends in X,

(*iii*) consecutive vertices of X on S are at distance at most 3 on S.

We prove

Claim *A maximal caterpillar C with the properties (i)–(iii) contains all vertices of $V \setminus X$.*

Proof Suppose, for a contradiction, that some vertex $v \notin X$ does not belong to C. Since G has diameter 2, v is precisely at distance 2 apart from all vertices of S. In particular, there exists a path xuv from the last vertex x of S to v (and here $u \in V(C) \setminus V(S)$ holds). Moreover, v is dominated by some $x' \in X \setminus V(S)$, and $x' \notin V(C)$ because of (*i*). Thus, S can be extended with the path $xuvx'$ to obtain the spine of a larger caterpillar, contradicting the maximality of C and proving the claim.

Now, the condition (*iii*) implies that S has length at most $3\,|V(S) \cap X| - 3$. At this point the proof can be completed either by referring to Lemma 1 of [2], which states that if G has a caterpillar C of diameter d then it also contains a *spanning* caterpillar of diameter at most $d + |V \setminus V(C)|$; or, by directly choosing the longest path-extension S' of S such that also S' ends in X, and any two consecutive vertices of X on $(S' \setminus S) \cup \{x\}$ (where x is the last vertex of S) are at distance at most 2 on S'. The latter S' dominates the entire G, because otherwise any non-dominated $x' \in X$ would be at distance 2 from the end of S', thus S' would not be longest, contradicting the way it has been chosen. □

4 Graphs of diameter at least 3

In this section we present some simple constructions, which show that $d = 2$ is the unique value (apart from the trivial case $d = 1$ of complete graphs) for which $\gamma_k(n, d)$ is sublinear in n (where $k \geq 1$ is any fixed integer).

Example 1 $\gamma(n, d) = \lfloor \frac{n}{2} \rfloor$ *for every* $d \geq 3$ *and every* n.

Consider the graph obtained from the complete graph $K_{\lceil n/2 \rceil}$ by attaching pendant edges to $\lfloor \frac{n}{2} \rfloor$ of its vertices. For $n \geq 4$, this graph has diameter 3. Since a dominating set must contain a vertex from each pendant edge, we obtain that $\gamma(n, d)$ cannot be smaller than $\lfloor \frac{n}{2} \rfloor$.

The fact that equality holds, follows now from the general inequality $\gamma(G) \leq \lfloor \frac{n}{2} \rfloor$ valid for *all* connected graphs G. A short proof of this upper bound can be obtained by induction on n, choosing a longest path $v_1 v_2 \cdots v_s$ in a spanning tree T of G and removing v_2 with all leaves of T joined to it (including v_1).

Example 2 $\gamma_2(n, d) \geq \lfloor \frac{n-1}{3} \rfloor + 1$ *for every* $d \geq 3$ *and every* $n \geq 7$.

Denote $q = \lfloor \frac{n-1}{3} \rfloor$. Consider q mutually disjoint paths $v_1^i v_2^i v_3^i$ $(1 \leq i \leq q)$ of length 2, together with $r = n - 3q \geq 1$ further vertices $u_1, \ldots, u_r$. Let $\{v_1^1, \ldots, v_1^q\}$ induce a complete graph, and join all v_3^i to all u_j $(1 \leq i \leq q$, $1 \leq j \leq r)$. This graph is 2-connected, has diameter 3, and its domination number is $q + 1$.

For later reference, we denote this graph by $G(q, r)$.

Example 3 $\gamma_k(n, d) \geq \left\lfloor \dfrac{n - \lceil \frac{k}{2} \rceil}{\lfloor \frac{3k}{2} \rfloor} \right\rfloor + 1$ *for every* $d \geq 3$, *every* $k \geq 2$, *and every* $n \geq 5k$. *As a consequence, graphs of arbitrarily high (but fixed) connectivity can have their domination number linear in* n *as* $n \to \infty$.

The basic structure for these graphs is provided by the previous construction, but now we substitute mutually disjoint independent sets S_x for the original vertices x. (By "substitution" we mean that the vertices of S_x are completely joined to the vertices of S_y if xy is an edge in the original graph; and there will be no edges between S_x and S_y if x and y are nonadjacent.)

For a general k, we choose $q = \left\lfloor \frac{n - \lceil \frac{k}{2} \rceil}{\lceil \frac{3k}{2} \rceil} \right\rfloor$ and start with the graph $G(q,1)$. In the substitution, we let $|S_x| = \lfloor \frac{k}{2} \rfloor$ if $x = v_1^i$ or $x = v_2^i$, $|S_x| = \lceil \frac{k}{2} \rceil$ if $x = v_3^i$, and finally $|S_{u_1}| = n - q \lfloor \frac{3k}{2} \rfloor$.

By the choice of q, we have $|S_{u_1}| \geq \lceil \frac{k}{2} \rceil$; and $q \geq 3$ for $n \geq 5k$. Thus, as one can easily check, the graph G obtained is k-connected. (As a matter of fact, for k even, the condition $q \geq 2$ — and hence $n \geq 7k/2$ — is already sufficient.) Moreover, $\gamma(G) = q + 1$, similarly to the previous example.

5 Concluding remarks

(1) The edge probability $p(n)$ in the proof of the lower bound in Theorem 1 cannot be chosen to be essentially smaller. The reason is that the diameter of $G_{n,p}$ with $p = \sqrt{\frac{(2-c)\log n}{n}}$ is almost surely greater than 2, for every constant $c > 0$ (see e.g. [1], Theorem 10 in Section X.2, p. 233). Therefore, in order to prove a larger lower bound on $\gamma(n,2)$ — if there exists any — would probably require a more constructive approach. We summarize the related open questions as follows.

Problem 1 *Prove that the limit*

$$\lim_{n\to\infty} \frac{\gamma(n,2)}{\sqrt{n \log n}}$$

exists, and determine its value.

(2) Concerning graphs of larger diameter and higher connectivity, we raise

Problem 2 *Prove that the limit*

$$\lim_{n\to\infty} \frac{\gamma_k(n,d)}{n}$$

exists for every fixed $k \geq 2$ *and* $d \geq 3$*, and determine its value. Is it independent of* d *for* $d \geq d_0(k)$*, or depends on both* k *and* d *?*

Note that high connectivity requires large minimum degree as well; thus, the above expression cannot exceed $\frac{1+\log k}{k}$. On the other hand, we have seen in the previous section that it is not smaller than $\frac{2}{3k}$ for k even, and $\frac{2}{3k-1}$ for k odd.

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