On circulant G-matrices

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Abstract

Let X_1, X_2, X_3, X_4 be four type 1 $(1,-1)$ matrices on the same group of order n (odd) with the properties: (i) $(X_i - I)^T = -(X_i - I)$, $i = 1,2$, (ii) $X_i^T = X_i$, $i = 3,4$ and the diagonal elements are positive, (iii) $X_iX_j = X_jX_i$, and (iv) $X_1X_1^T + X_2X_2^T + X_3X_3^T + X_4X_4^T = 4nI_n$. Call such matrices G-matrices. If there exist circulant G-matrices of order n it can be easily shown that $4n - 2 = a^2 + b^2$, where a and b are odd integers. It is known that they exist for odd $n \leq 27$, except for $n = 11, 17$ for which orders they can not exist. In this paper we give for the first time all non-equivalent circulant G-matrices of odd order $n \leq 33$ as well as some new non-equivalent circulant G-matrices of order $n = 37, 41$. We note that no G-matrices were previously known for orders $31, 33, 37$ and 41. These are presented in tables in the form of the corresponding non-equivalent supplementary difference sets. In the sequel we use G-matrices to construct some F-matrices and orthogonal designs.

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1 Introduction and basic definitions

A $(1,-1)$ matrix of order n is called a Hadamard matrix if $HH^T = H^TH = nI_n$, where H^T is the transpose of H and I_n is the identity matrix of order n.

A $(1,-1)$ matrix A of order n is said to be of skew type if $A - I_n$ is skew-symmetric. Two $(1,-1)$ matrices A, B of order n are said to be amicable if $AB^T = BA^T$.

Let G be an additive abelian group of order n with elements $g_1, g_2, \ldots, g_n$ and X a subset of G. Define the type 1 $(1,-1)$ incidence matrix $M = (m_{ij})$ of order n of X to be

$$m_{ij} = \begin{cases} +1 & if\ g_j - g_i \in X \\ -1 & otherwise \end{cases}$$

and the type 2 $(1,-1)$ incidence matrix $N = (n_{ij})$ of order n of X to be

$$n_{ij} = \begin{cases} +1 & if\ g_j + g_i \in X \\ -1 & otherwise \end{cases}$$

In particular, if G is cyclic the matrices M and N are called circulant and back circulant respectively. In this case $m_{ij} = m_{1,j-i+1}$ *and* $n_{ij} = n_{1,i+j-1}$ respectively (indices should be reduced modulo n).

Definition 1 Let X_1, X_2, X_3, X_4 be four type 1 $(1,-1)$ matrices on the same group of order n (odd) with the properties

(i) $(X_i - I)^T = -(X_i - I),\ i = 1,2$

(ii) $X_i^T = X_i,\ i = 3,4$ and the diagonal elements are positive

(iii) $X_i X_j = X_j X_i$

(iv) $X_1 X_1^T + X_2 X_2^T + X_3 X_3^T + X_4 X_4^T = 4nI_n$

Call such matrices *G-matrices* of order n.

G-matrices were first introduced and applied to construct Hadamard matrices by J. Seberry Wallis [4]. In the present paper we are dealing with G-matrices which are circulant, so the condition (*iii*) is obviously satisfied. Hence, multiplying on the left by e^T (the $1 \times n$ vector of one's) and on the right by e both sides of (*iv*) we conclude that circulant G-matrices can only exist for orders n of which $4n = 1^2 + 1^2 + a^2 + b^2$, where a, b are the sums of the elements of the first row of symmetric matrices X_3 and X_4 respectively and they are odd integers. So, for example, they cannot exist for the following orders $\leq 100 : 11, 17, 29, 35, 39, 47, 53, 65, 67, 71, 81, 83, 89, 95$. Circulant G-matrices of order n, but only one solution for every case, were known for $n = 3, 5, 7, 9, 13, 15, 19, 21, 23, 25$ and 27, see [3, 4, 6, 7]. In this paper we give for the first time all non-equivalent circulant G-matrices of odd order $n \leq 33$ as well as some new non-equivalent circulant G-matrices of order $n = 37, 41$. We note that no G-matrices were previously known for orders $31, 33, 37$ and 41.

In section 2 we describe briefly the method of construction, in section 3 we use G-matrices to construct some F-matrices and orthogonal designs, and in section 4 we present in tables the new results.

2 Method of construction

In order to describe our constuction of G-matrices we need a few more definitions. Let n be a positive integer.

Definition 2 Four subsets S_0, S_1, S_2, S_3 of $\{1, 2, ..., n-1\}$ are called $4-(n; n_0, n_1, n_2, n_3; \lambda)$ *supplementary difference sets* (sds) modulo n if $|S_k| = n_k$ for $k = 0, 1, 2, 3$ and for each $m \in \{1, 2, ..., n-1\}$ we have $\lambda_0(m) + \ldots + \lambda_3(m) = \lambda$, where $\lambda_k(m)$ is the number of solutions (i, j) of the congruence $i - j \equiv m(mod\ n)$ with $i, j \in S_k$.

Suppose that S_k are $4-(n; n_0, n_1, n_2, n_3; \lambda)$ sds modulo n having the following additional properties:

$$n + \lambda = n_0 + n_1 + n_2 + n_3 \tag{1}$$

$$i \in S_k \Longleftrightarrow n - i \notin S_k, \ k = 0, 1 \tag{2}$$

$$i \in S_t \Longleftrightarrow n - i \in S_t, \ t = 2, 3 \tag{3}$$

where in (2) and (3) it is assumed that $i \in \{1, 2, \ldots, n-1\}$.

Let $a_k = (a_{k_0}, a_{k_1}, \ldots, a_{k_{n-1}})$, $k = 0, 1, 2, 3$, be the row vector defined by

$$a_{k_i} = \begin{cases} -1 & if\ \ i \in S_k \\ 1 & otherwise \end{cases}$$

Furthermore let A_k, $k = 0, 1, 2, 3$ be the circulant matrices with first row a_k. Then it is well-known (and can be easily verified) that A_0, A_1, A_2, A_3 are G-matrices of order n.

Let r be an integer relatively prime to n, and set

$$S_k' = \{ri\ (mod\ n) : i \in S_k\} \subset \{1, 2, \ldots, n-1\}$$

for $k = 0, 1, 2, 3$. These sets are also $4-(n; n_0, n_1, n_2, n_3; \lambda)$ sds modulo n satisfying the conditions (1), (2), (3). We shall say that such quadruples S_0, S_1, S_2, S_3 and S_0', S_1', S_2', S_3' are equivalent.

We now give a brief description of the method of computation used to find the necessary sds's. The numbers n_i are easy to determine (see [5]). We first generate a number of subsets of size n_i of $\{1, 2, ..., n\}$ having the required symmetry properties (2) or (3), and at the same time compute the corresponding set of differences. We store the multiplicities of these differences in a file, say f_i, saving only sets of differences with different multiplicities. After creating these files for each of the sizes $n_0, ..., n_3$, we try to match the items in the four files to produce an sds. This is done by examining items in two files only, say f_0 and f_1 and creating a new file in which we record the pairs which produce different total multiplicities of the differences. The procedure is repeated with the remaining two files f_2 and f_3. Finally the resulting two files are examined in order to find a perfect match.

The results that we have found applying this algorithm are presented in tables in section 4.

3 Constructions using G-matrices

First we note that we have

Lemma 1 *(see [3] or [7])* Suppose X_1, X_2, X_3, X_4 are four circulant G-matrices of odd order n, then there exists an $OD(4n; 1, 1, 2n-1, 2n-1)$.

Proof. Let $Y_1 = \frac{1}{2}(X_1 + X_2 - 2I)$, $Y_2 = \frac{1}{2}(X_1 - X_2)$, $Y_3 = \frac{1}{2}(X_3 + X_4)$, $Y_4 = \frac{1}{2}(X_3 - X_4)$. Then $Y_1^T = -Y_1$, $Y_2^T = -Y_2$, $Y_3^T = Y_3$, $Y_4^T = Y_4$, $Y_4Y_3^T = Y_3Y_4^T$, $Y_1Y_2^T = Y_2Y_1^T$, and $Y_1Y_1^T + Y_2Y_2^T + Y_3Y_3^T + Y_4Y_4^T = (2n-1)I_n$.
Let x_1, x_2, x_3, x_4 be commuting variables then $x_1I + x_3Y_1 + x_4Y_2, x_2I + x_4Y_1 - x_3Y_2, x_3Y_3 + x_4Y_4, x_4Y_3 - x_3Y_4$, are four circulant matrices which can be used in the Goethals-Seidel array to obtain the required $OD(4n; 1, 1, 2n-1, 2n-1)$. □

For convenient we set $S = \{3, 5, 7, 9, 13, 15, 19, 21, 23, 25, 27, 31, 33, 37, 41\}$. Then we have:

Corollary 1 Let $n \in S$. Then an $OD(4n; 1, 1, 2n-1, 2n-1)$ exists.

We recall the following definitions and results from [4]. The following theorem shows how the Williamson construction (the B_i) and the Goethals-Seidel construction (the A_i) may be combined to construct Hadamard matrices.

Theorem 1 *(see [4])* Suppose A_i and B_i, $i = 1, 2, 3, 4$ are type 1 $(1, -1)$ matrices of order a and b, respectively, which satisfy

(i) $A_iA_j = A_jA_i, \quad i, j = 1, 2, 3, 4$

(ii) $B_iB_j^T = B_jB_i^T, \quad i, j = 1, 2, 3, 4$

(iii) $\sum_{i=1}^{4}(A_i \times B_i)(A_i \times B_i)^T = 4abI_{ab}$

then H defined as

$$H = \begin{bmatrix} A_1 \times B_1 & A_2R \times B_2 & A_3R \times B_3 & A_4R \times B_4 \\ -A_2R \times B_2 & A_1 \times B_1 & A_4^TR \times B_4 & -A_3^TR \times B_3 \\ -A_3R \times B_3 & -A_4^TR \times B_4 & A_1 \times B_1 & A_2^TR \times B_2 \\ -A_4R \times B_4 & A_3^TR \times B_3 & -A_2^TR \times B_2 & A_1 \times B_1 \end{bmatrix}$$

is an Hadamard matrix of order $4ab$.

We will call the matrices $A_i \times B_i$, $i = 1,2,3,4$ of the theorem F-matrices and we will say H is a Goethals-Seidel like Hadamard matrix. The A_i will be called the GS-part and the B_i the W-part of the F-matrix. The following theorem shows how G-matrices may be used to construct F-matrices.

Theorem 2 *(see [4])* Let X_1, X_2, X_3, X_4 be G-matrices of order n. Suppose A, B, C are $(1,-1)$ matrices of order m which satisfy:

(i) AB^T, AC^T, BC^T are symmetric

(ii) $AA^T + BB^T + (4n-2)CC^T = 4nmI_m$.

Then

$A_1 = I \times A + (X_1 - I) \times C,\ A_2 = I \times B + (X_2 - I) \times C,\ A_3 = X_3 \times C,\ A_4 = X_4 \times C$

are F-matrices of order mn.

Corollary 2 Let $n \in S$. Suppose A, B, C are pairwise amicable $(1,-1)$ matrices of order m satisfying

$$AA^T + BB^T + (4n-2)CC^T = 4mnI_m$$

Then there are F-matrices of order mn and a Goethals-Seidel like Hadamard matrix of order $4mn$.

Theorem 3 *(see [3])* Let X_1, X_2, X_3, X_4 be G-matrices of order n. Suppose A, B, C, D are $(1,-1)$ matrices of order m which satisfy

(i) $AB^T, AC^T, AD^T, BC^T, BD^T, CD^T$ are symmetric

(ii) $AA^T + BB^T + (2n-1)CC^T + (2n-1)DD^T = 4nmI_m$

Then defining $Y_1 = (X_1+X_2-2I)/2$, $Y_2 = (X_1-X_2)/2$, $Y_3 = (X_3+X_4)/2$, and $Y_4 = (X_3 - X_4)/2$, we have that

$$B_1 = I \times A + Y_1 \times C + Y_2 \times D, \quad B_2 = I \times B + Y_1 \times D + Y_2 \times -C$$

$$B_3 = Y_3 \times C + Y_4 \times D, B_4 = Y_3 \times D + Y_4 \times -C$$

are F-matrices of order mn.

Corollary 3 Let $n \in S$. Suppose A, B, C, D are pairwise amicable $(1, -1)$ matrices of order m satisfying

$$AA^T + BB^T + (2n-1)CC^T + (2n-1)DD^T = 4mnI_m.$$

Then there are F-matrices of order mn and a Goethals-Seidel like Hadamard matrix of order $4mn$.

Theorem 4 *(see [4])* Suppose there exist G-matrices of order n. Further suppose ther exists a $(v, \frac{1}{2}(v-1), \frac{1}{4}(v-3))$ difference set.Then there exist F-matrices of order

$$(i)\ \tfrac{1}{2}v(v-1),\ n = \tfrac{1}{2}(v-1); \quad (ii)\ \tfrac{1}{4}v(v+1),\ n = \tfrac{1}{4}(v+1);$$

$$(iii)\ \tfrac{1}{4}v(v-3),\ n = \tfrac{1}{4}(v-3)$$

respectively.

Example 1 Applying theorem 4 for $n = 15, 19, 21, 23, 25, 27, 31, 33, 37, 41$ we have F-matrices of orders 465, 741, 885, 903, 945, 1081, 1275, 1425, 1485, 1501, 1743, 1827, 1953, 2093, 2185, 2211, 2475, 2575, 2775, 2889, 2997, 3403, 3813, 3937, 4323, 4455, 5439, 5587, 6683, 6847 of which orders no Williamson type matrices were yet known for 885, 903, 1425, 1743, 1953, 2093, 2889, 3403, 3813, 4323 and 6683 (see [2]).

Corollary 4 *(see [4])* Suppose there exist G-matrices of order n. Let p be a prime power and $p = 2n-1$ or $p = 2n+1$ or $p = 2n+3$. Then there exist F-matrices of order np

Example 2 Applying corollary 4 for $n = 15, 19, 21, 23, 25, 27, 31, 33, 37, 41$ we have F-matrices of orders 435, 465, 703, 779, 861, 903, 1081, 1127, 1225, 1325, 1431, 1891, 2211, 2701, 3321, 3403 of which orders no Williamson type matrices were yet known for 779, 903, 1127, 1325 and 3403.

Corollary 5 *(see [4])* Suppose there exist G-matrices of order n. Further suppose ther exists a (v, k, λ) difference set and $v \equiv 1 (mod\ 4)$ is a prime or a prime power. Then there exist F-matrices of order nv where

$$(i)\ v = 2n-1+2(k-\lambda); \quad (ii)\ v = 2n+1+2(k-\lambda); \quad (iii)\ v = 2n-1+4(k-\lambda)$$

respectively.

Example 3 Applying corollary 5 for

1. $(v,k,\lambda) = (37,9,2)$ and $n = 5$ we have F-matrices of orders 185,
2. $(v,k,\lambda) = (73,9,1)$ and $n = 21,29$ we have F-matrices of orders 1533 and 2117,
3. $(v,k,\lambda) = (101,25,6)$ and $n = 31$ we have F-matrices of orders 3131

of which orders no Williamson type matrices were yet known.

4 The non-equivalent supplementary difference sets

In table 1 we give for the first time all non-equivalent supplementary difference sets which satisfy the conditions (1), (2), (3) for all odd $n \leq 33$. In table 2 we give some new non-equivalent sds for $n = 37, 41$. Our computer search for G-matrices of orders 37 and 41 was incomplete. Hence there may exist additional solutions non-equivalent to those listed in Table 2.

Table 1.
All non-equivalent circulant G-matrices of odd order $n \leq 33$

n = 3; $4-(3;1,1,2,0;1)$

$S_0 = \{1\}, \quad S_1 = \{1\}, \quad S_2 = \{1,2\}, \quad S_3 = \emptyset$

n = 5; $4-(5;2,2,4,4;7)$

$S_0 = \{1,3\}, \quad S_1 = \{1,2\}, \quad S_2 = \{1,2,3,4\}, \quad S_3 = \{1,2,3,4\}$

n = 7; $4-(7;3,3,4,6;9)$

$S_0 = \{1,3,5\}, \quad S_1 = \{1,2,3\}, \quad S_2 = \{1,2,5,6\}, \quad S_3 = \{1,2,3,4,5,6\}$

n = 9; $4-(9;4,4,6,2;7)$

1. $S_0 = \{1,3,4,7\}, S_1 = \{1,2,4,6\}, S_2 = \{2,3,4,5,6,7\}, S_3 = \{2,7\}$,
2. $S_0 = \{1,3,4,7\}, S_1 = \{1,2,3,5\}, S_2 = \{1,2,3,6,7,8\}, S_3 = \{1,8\}$,

n = 13; $4-(13;3,6,6,6;8)$
No solution exists

n = 13; $4-(13;6,6,4,4;7)$

1. $S_0 = \{1,3,7,8,9,11\}, S_1 = \{3,4,5,7,11,12\}$,
   $S_2 = \{3,6,7,10\}, \quad S_3 = \{4,6,7,9\}$,
2. $S_0 = \{1,2,4,5,7,10\}, S_1 = \{1,2,3,6,8,9\}$,
   $S_2 = \{1,3,10,12\}, \quad S_3 = \{4,5,8,9\}$,

3. $S_0 = \{1,2,3,6,8,9\}$, $S_1 = \{1,2,4,5,6,10\}$,
$S_2 = \{1,5,8,12\}$, $S_3 = \{4,6,7,9\}$,

4. $S_0 = \{1,4,6,8,10,11\}$, $S_1 = \{1,2,4,5,6,10\}$,
$S_2 = \{4,6,7,9\}$, $S_3 = \{3,4,9,10\}$,

5. $S_0 = \{2,6,8,9,10,12\}$, $S_1 = \{1,2,3,5,6,9\}$,
$S_2 = \{3,5,8,10\}$, $S_3 = \{2,3,10,11\}$,

6. $S_0 = \{1,2,4,5,6,10\}$, $S_1 = \{1,2,3,4,6,8\}$,
$S_2 = \{2,5,8,11\}$, $S_3 = \{1,6,7,12\}$,

7. $S_0 = \{1,2,4,5,7,10\}$, $S_1 = \{1,2,3,4,6,8\}$,
$S_2 = \{1,5,8,12\}$, $S_3 = \{2,3,10,11\}$,

8. $S_0 = \{1,4,5,6,10,11\}$, $S_1 = \{1,2,3,4,6,8\}$,
$S_2 = \{3,5,8,10\}$, $S_3 = \{3,6,7,10\}$

<u>**n = 15; 4 − (15; 7, 7, 6, 4; 9)**</u>

1. $S_0 = \{1,3,6,7,10,11,13\}$, $S_1 = \{1,3,4,5,7,9,13\}$,
$S_2 = \{1,6,7,8,9,14\}$, $S_3 = \{5,6,9,10\}$,

2. $S_0 = \{1,4,6,8,10,12,13\}$, $S_1 = \{3,4,7,9,10,13,14\}$,
$S_2 = \{2,3,5,10,12,13\}$, $S_3 = \{6,7,8,9\}$,

3. $S_0 = \{1,3,7,9,10,11,13\}$, $S_1 = \{1,2,5,8,9,11,12\}$,
$S_2 = \{4,5,6,9,10,11\}$, $S_3 = \{4,6,9,11\}$,

4. $S_0 = \{1,2,4,7,9,10,12\}$, $S_1 = \{1,2,5,8,9,11,12\}$,
$S_2 = \{4,5,6,9,10,11\}$, $S_3 = \{1,3,12,14\}$,

5. $S_0 = \{1,2,5,7,9,11,12\}$, $S_1 = \{1,2,3,6,8,10,11\}$,
$S_2 = \{3,5,6,9,10,12\}$, $S_3 = \{1,2,13,14\}$,

6. $S_0 = \{1,2,4,6,7,10,12\}$, $S_1 = \{1,2,3,5,8,9,11\}$,
$S_2 = \{1,3,4,11,12,14\}$, $S_3 = \{5,6,9,10\}$,

7. $S_0 = \{1,3,4,6,8,10,13\}$, $S_1 = \{1,2,3,5,8,9,11\}$,
$S_2 = \{2,6,7,8,9,13\}$, $S_3 = \{5,6,9,10\}$,

8. $S_0 = \{1,2,4,5,6,8,12\}$, $S_1 = \{1,3,4,5,6,8,13\}$,
$S_2 = \{1,4,5,10,11,14\}$, $S_3 = \{1,7,8,14\}$,

9. $S_0 = \{1,2,6,8,10,11,12\}$, $S_1 = \{1,3,4,5,6,8,13\}$,
$S_2 = \{1,2,5,10,13,14\}$, $S_3 = \{1,7,8,14\}$,

10. $S_0 = \{1,2,4,5,6,8,12\}$, $S_1 = \{1,3,4,5,6,8,13\}$,
$S_2 = \{2,4,5,10,11,13\}$, $S_3 = \{2,7,8,13\}$,

11. $S_0 = \{1,2,6,8,10,11,12\}$, $S_1 = \{1,3,4,5,6,8,13\}$,
$S_2 = \{2,4,5,10,11,13\}$, $S_3 = \{4,7,8,11\}$,

12. $S_0 = \{1,5,7,9,11,12,13\}$, $S_1 = \{1,2,3,6,7,10,11\}$,
$S_2 = \{3,4,6,9,11,12\}$, $S_3 = \{5,7,8,10\}$,

13. $S_0 = \{1,3,4,5,6,8,13\}$, $S_1 = \{1,2,3,6,7,10,11\}$,
$S_2 = \{3,5,6,9,10,12\}$, $S_3 = \{2,4,11,13\}$,

14. $S_0 = \{1,2,3,5,6,8,11\}$, $S_1 = \{1,4,5,6,7,12,13\}$,
$S_2 = \{2,4,6,9,11,13\}$, $S_3 = \{5,6,9,10\}$,

15. $S_0 = \{3,5,6,7,11,13,14\}$, $S_1 = \{1,2,3,5,6,7,11\}$,
$S_2 = \{3,5,6,9,10,12\}$, $S_3 = \{1,4,11,14\}$,

16. $S_0 = \{1,2,4,5,7,9,12\}$, $S_1 = \{1,2,3,5,6,7,11\}$,
$S_2 = \{3,5,6,9,10,12\}$, $S_3 = \{1,7,8,14\}$,

17. $S_0 = \{1,2,3,6,8,10,11\}$, $S_1 = \{1,2,4,5,6,7,12\}$,
$S_2 = \{1,5,7,8,10,14\}$, $S_3 = \{4,7,8,11\}$,

18. $S_0 = \{1,2,3,6,8,10,11\}$, $S_1 = \{1,2,4,5,6,7,12\}$,
$S_2 = \{1,2,5,10,13,14\}$, $S_3 = \{2,4,11,13\}$,

19. $S_0 = \{1,3,5,6,7,11,13\}$, $S_1 = \{1,2,4,5,6,7,12\}$,
$S_2 = \{1,2,5,10,13,14\}$, $S_3 = \{1,7,8,14\}$,

20. $S_0 = \{1,3,5,6,7,11,13\}$, $S_1 = \{1,2,4,5,6,7,12\}$,
$S_2 = \{2,4,5,10,11,13\}$, $S_3 = \{4,7,8,11\}$,

21. $S_0 = \{1,5,6,8,11,12,13\}$, $S_1 = \{1,2,4,5,6,7,12\}$,
$S_2 = \{3,5,6,9,10,12\}$, $S_3 = \{2,4,11,13\}$,

22. $S_0 = \{1,2,3,5,8,9,11\}$, $S_1 = \{1,2,3,4,6,7,10\}$,
$S_2 = \{1,5,6,9,10,14\}$, $S_3 = \{4,6,9,11\}$,

23. $S_0 = \{1,3,7,9,10,11,13\}$, $S_1 = \{1,2,3,4,6,7,10\}$,
$S_2 = \{2,3,5,10,12,13\}$, $S_3 = \{3,7,8,12\}$,

24. $S_0 = \{1,4,6,8,10,12,13\}$, $S_1 = \{1,2,3,4,6,7,10\}$,
$S_2 = \{2,3,5,10,12,13\}$, $S_3 = \{2,3,12,13\}$,

25. $S_0 = \{3,4,7,9,10,13,14\}$, $S_1 = \{1,2,3,4,6,7,10\}$,
$S_2 = \{1,3,4,11,12,14\}$, $S_3 = \{3,5,10,12\}$,

26. $S_0 = \{1,3,6,7,10,11,13\}$, $S_1 = \{1,2,3,4,6,7,10\}$,
$S_2 = \{2,6,7,8,9,13\}$, $S_3 = \{3,5,10,12\}$,

27. $S_0 = \{1,2,5,7,9,11,12\}$, $S_1 = \{1,2,3,4,5,7,9\}$,
$S_2 = \{1,4,7,8,11,14\}$, $S_3 = \{4,5,10,11\}$,

28. $S_0 = \{1,3,5,8,9,11,13\}$, $S_1 = \{1,2,3,4,5,7,9\}$,
$S_2 = \{2,3,6,9,12,13\}$, $S_3 = \{4,5,10,11\}$,

29. $S_0 = \{1,2,5,7,9,11,12\}$, $S_1 = \{1,2,3,4,5,7,9\}$,
$S_2 = \{1,2,7,8,13,14\}$, $S_3 = \{2,5,10,13\}$,

30. $S_0 = \{1,2,4,5,7,9,12\}$, $S_1 = \{6,7,10,11,12,13,14\}$,
$S_2 = \{1,5,7,8,10,14\}$, $S_3 = \{2,7,8,13\}$,

31. $S_0 = \{1,2,4,5,7,9,12\}$, $S_1 = \{6,7,10,11,12,13,14\}$,
$S_2 = \{1,4,5,10,11,14\}$, $S_3 = \{2,4,11,13\}$,

32. $S_0 = \{1,3,5,8,9,11,13\}$, $S_1 = \{6,7,10,11,12,13,14\}$,
$S_2 = \{3,5,6,9,10,12\}$, $S_3 = \{2,7,8,13\}$

$\mathbf{n = 19}$; $4 - (19; 9, 9, 12, 6; 17)$

1. $S_0 = \{1,3,5,6,9,11,12,15,17\}$, $S_1 = \{1,2,4,6,7,8,9,14,16\}$,
   $S_2 = \{2,4,5,6,7,8,11,12,13,14,15,17\}$, $S_3 = \{5,6,9,10,13,14\}$,

2. $S_0 = \{1,5,7,10,11,13,15,16,17\}$, $S_1 = \{4,5,6,8,9,12,16,17,18\}$,
   $S_2 = \{1,4,6,7,8,9,10,11,12,13,15,18\}$, $S_3 = \{1,7,8,11,12,18\}$,

3. $S_0 = \{1,2,4,8,10,12,13,14,16\}$, $S_1 = \{2,8,9,12,13,14,15,16,18\}$,
   $S_2 = \{1,2,3,6,8,9,10,11,13,16,17,18\}$, $S_3 = \{1,2,7,12,17,18\}$,

4. $S_0 = \{1,2,5,7,8,9,13,15,16\}$, $S_1 = \{1,4,10,11,12,13,14,16,17\}$,
   $S_2 = \{2,3,5,6,7,8,11,12,13,14,16,17\}$, $S_3 = \{4,6,8,11,13,15\}$,

5. $S_0 = \{1,5,6,9,11,12,15,16,17\}$, $S_1 = \{1,2,3,4,5,8,10,12,13\}$,
   $S_2 = \{2,4,5,7,8,9,10,11,12,14,15,17\}$, $S_3 = \{3,5,9,10,14,16\}$,

6. $S_0 = \{1,4,5,7,9,11,13,16,17\}$, $S_1 = \{2,3,4,5,6,10,11,12,18\}$,
   $S_2 = \{1,2,3,4,6,7,12,13,15,16,17,18\}$, $S_3 = \{2,7,9,10,12,17\}$,

7. $S_0 = \{1,3,6,10,11,12,14,15,17\}$, $S_1 = \{2,3,10,11,12,13,14,15,18\}$,
   $S_2 = \{1,3,4,5,8,9,10,11,14,15,16,18\}$, $S_3 = \{3,7,9,10,12,16\}$,

8. $S_0 = \{1,2,4,6,10,11,12,14,16\}$, $S_1 = \{4,10,11,12,13,14,16,17,18\}$,
   $S_2 = \{1,2,4,7,8,9,10,11,12,15,17,18\}$, $S_3 = \{2,5,9,10,14,17\}$,

9. $S_0 = \{1,2,3,5,6,8,10,12,15\}$, $S_1 = \{1,2,3,4,5,6,9,11,12\}$,
   $S_2 = \{1,2,3,6,7,9,10,12,13,16,17,18\}$, $S_3 = \{2,4,9,10,15,17\}$

$\mathbf{n = 21}$; $4 - (21; 10, 10, 6, 10; 15)$

1. $S_0 = \{1,3,4,5,7,11,12,13,15,19\}$, $S_1 = \{4,6,7,8,11,12,16,18,19,20\}$,
   $S_2 = \{1,4,7,14,17,20\}$, $S_3 = \{1,2,3,4,8,13,17,18,19,20\}$,

2. $S_0 = \{2,4,5,6,7,9,10,13,18,20\}$, $S_1 = \{1,2,3,4,6,9,10,13,14,16\}$,
   $S_2 = \{5,7,10,11,14,16\}$, $S_3 = \{2,4,5,6,10,11,15,16,17,19\}$,

3. $S_0 = \{2,4,5,6,7,9,10,13,18,20\}$, $S_1 = \{1,2,3,5,6,8,9,10,14,17\}$,
   $S_2 = \{2,7,8,13,14,19\}$, $S_3 = \{2,4,5,6,8,13,15,16,17,19\}$,

4. $S_0 = \{1,2,4,7,9,10,13,15,16,18\}$, $S_1 = \{1,2,3,5,6,7,9,10,13,17\}$,
   $S_2 = \{2,9,10,11,12,19\}$, $S_3 = \{1,2,3,7,9,12,14,18,19,20\}$,

5. $S_0 = \{3,4,5,6,9,11,13,14,19,20\}$, $S_1 = \{1,2,4,5,6,7,9,10,13,18\}$,
   $S_2 = \{3,4,6,15,17,18\}$, $S_3 = \{1,3,7,8,10,11,13,14,18,20\}$,

6. $S_0 = \{1,4,5,6,8,9,11,14,18,19\}$, $S_1 = \{3,5,6,7,8,9,11,17,19,20\}$,
   $S_2 = \{1,5,10,11,16,20\}$, $S_3 = \{1,6,7,8,10,11,13,14,15,20\}$,

7. $S_0 = \{2,3,6,7,8,9,11,16,17,20\}$, $S_1 = \{1,2,3,4,5,7,8,10,12,15\}$,
   $S_2 = \{2,4,7,14,17,19\}$, $S_3 = \{2,4,5,6,10,11,15,16,17,19\}$,

8. $S_0 = \{1,3,5,8,9,11,14,15,17,19\}$, $S_1 = \{1,8,10,12,14,15,16,17,18,19\}$,
   $S_2 = \{4,5,8,13,16,17\}$, $S_3 = \{1,2,7,8,9,12,13,14,19,20\}$,

9. $S_0 = \{1,4,6,10,12,13,14,16,18,19\}$, $S_1 = \{3,4,5,10,12,13,14,15,19,20\}$,
   $S_2 = \{1,2,6,15,19,20\}$, $S_3 = \{1,3,4,6,7,14,15,17,18,20\}$,

10. $S_0 = \{1,2,4,5,6,8,11,12,14,18\}$, $S_1 = \{1,2,3,4,6,7,8,11,12,16\}$,
$S_2 = \{2,8,10,11,13,19\}$, $S_3 = \{1,4,6,7,8,13,14,15,17,20\}$,

11. $S_0 = \{1,5,7,10,12,13,15,17,18,19\}$, $S_1 = \{1,2,3,4,6,7,8,11,12,16\}$,
$S_2 = \{2,8,10,11,13,19\}$, $S_3 = \{3,4,5,7,10,11,14,16,17,18\}$,

12. $S_0 = \{1,2,3,4,7,9,11,13,15,16\}$, $S_1 = \{1,2,3,4,6,7,8,11,12,16\}$,
$S_2 = \{2,4,5,16,17,19\}$, $S_3 = \{1,4,6,7,10,11,14,15,17,20\}$,

13. $S_0 = \{1,2,3,4,6,8,11,12,14,16\}$, $S_1 = \{3,4,5,6,7,9,10,13,19,20\}$,
$S_2 = \{1,2,8,13,19,20\}$, $S_3 = \{2,3,5,7,10,11,14,16,18,19\}$,

14. $S_0 = \{1,5,6,8,10,12,14,17,18,19\}$, $S_1 = \{2,3,4,5,6,8,9,10,14,20\}$,
$S_2 = \{1,6,9,12,15,20\}$, $S_3 = \{2,5,6,7,8,13,14,15,16,19\}$,

15. $S_0 = \{1,3,4,5,6,9,10,13,14,19\}$, $S_1 = \{2,3,10,12,13,14,15,16,17,20\}$,
$S_2 = \{2,4,8,13,17,19\}$, $S_3 = \{1,4,6,7,8,13,14,15,17,20\}$,

16. $S_0 = \{1,2,4,6,9,10,13,14,16,18\}$, $S_1 = \{4,5,6,7,8,10,12,18,19,20\}$,
$S_2 = \{1,2,5,16,19,20\}$, $S_3 = \{2,3,5,7,8,13,14,16,18,19\}$,

17. $S_0 = \{1,3,4,6,9,10,13,14,16,19\}$, $S_1 = \{1,2,3,4,5,7,8,9,11,15\}$,
$S_2 = \{3,4,8,13,17,18\}$, $S_3 = \{3,4,5,7,9,12,14,16,17,18\}$,

18. $S_0 = \{1,2,6,9,11,13,14,16,17,18\}$, $S_1 = \{1,2,3,4,5,6,8,11,12,14\}$,
$S_2 = \{3,4,9,12,17,18\}$, $S_3 = \{1,3,7,8,10,11,13,14,18,20\}$,

19. $S_0 = \{1,2,6,9,11,13,14,16,17,18\}$, $S_1 = \{5,6,7,8,11,12,17,18,19,20\}$,
$S_2 = \{2,5,8,13,16,19\}$, $S_3 = \{1,2,3,5,7,14,16,18,19,20\}$,

20. $S_0 = \{1,4,7,9,11,13,15,16,18,19\}$, $S_1 = \{1,2,3,4,5,8,9,10,14,15\}$,
$S_2 = \{3,8,10,11,13,18\}$, $S_3 = \{3,5,6,7,10,11,14,15,16,18\}$,

21. $S_0 = \{1,4,8,10,12,14,15,16,18,19\}$, $S_1 = \{1,2,3,4,5,8,9,10,14,15\}$,
$S_2 = \{1,8,10,11,13,20\}$, $S_3 = \{1,4,7,8,9,12,13,14,17,20\}$,

22. $S_0 = \{1,5,7,9,10,13,15,17,18,19\}$, $S_1 = \{1,2,3,4,5,8,9,10,14,15\}$,
$S_2 = \{5,6,9,12,15,16\}$, $S_3 = \{2,4,7,9,10,11,12,14,17,19\}$,

23. $S_0 = \{1,2,3,5,6,8,9,10,14,17\}$, $S_1 = \{4,5,6,7,8,9,11,18,19,20\}$,
$S_2 = \{2,4,8,13,17,19\}$, $S_3 = \{1,4,6,7,10,11,14,15,17,20\}$

$\mathbf{n = 23}$; 4 − (23; 11, 11, 10, 16; 25)

1. $S_0 = \{2,4,5,6,8,9,10,11,16,20,22\}$,
$S_1 = \{3,5,6,7,8,9,12,13,19,21,22\}$,
$S_2 = \{1,3,4,7,11,12,16,19,20,22\}$,
$S_3 = \{1,2,3,4,7,9,10,11,12,13,14,16,19,20,21,22\}$,

2. $S_0 = \{1,2,5,7,12,13,14,15,17,19,20\}$,
$S_1 = \{1,3,7,8,9,10,11,17,18,19,21\}$,
$S_2 = \{2,4,7,8,11,12,15,16,19,21\}$,
$S_3 = \{2,4,5,7,8,9,10,11,12,13,14,15,16,18,19,21\}$,

3. $S_0 = \{1,2,5,6,7,9,12,13,15,19,20\}$,
$S_1 = \{2,3,4,5,6,7,10,12,14,15,22\}$,
$S_2 = \{2,4,5,7,11,12,16,18,19,21\}$,
$S_3 = \{2,4,5,6,7,8,10,11,12,13,15,16,17,18,19,21\}$,

4.
$$S_0 = \{1,2,5,7,11,13,14,15,17,19,20\},$$
$$S_1 = \{1,4,5,6,7,8,9,11,13,20,21\},$$
$$S_2 = \{1,4,6,7,8,15,16,17,19,22\},$$
$$S_3 = \{1,2,5,6,7,8,10,11,12,13,15,16,17,18,21,22\},$$

5.
$$S_0 = \{1,2,4,7,8,9,10,12,17,18,20\},$$
$$S_1 = \{1,3,4,5,7,8,9,10,11,17,21\},$$
$$S_2 = \{1,2,6,10,11,12,13,17,21,22\},$$
$$S_3 = \{1,2,4,6,7,8,10,11,12,13,15,16,17,19,21,22\},$$

6.
$$S_0 = \{1,2,3,5,6,8,10,12,14,16,19\},$$
$$S_1 = \{1,8,10,11,14,16,17,18,19,20,21\},$$
$$S_2 = \{1,2,4,9,10,13,14,19,21,22\},$$
$$S_3 = \{1,2,3,6,7,8,9,10,13,14,15,16,17,20,21,22\},$$

7.
$$S_0 = \{1,3,4,6,7,8,9,10,11,18,21\},$$
$$S_1 = \{1,5,6,7,8,9,12,13,19,20,21\},$$
$$S_2 = \{2,4,6,7,11,12,16,17,19,21\},$$
$$S_3 = \{1,2,4,5,6,8,10,11,12,13,15,17,18,19,21,22\},$$

8.
$$S_0 = \{1,4,7,9,10,11,15,17,18,20,21\},$$
$$S_1 = \{1,4,5,6,7,8,9,12,13,20,21\},$$
$$S_2 = \{2,3,8,10,11,12,13,15,20,21\},$$
$$S_3 = \{1,3,5,6,7,8,10,11,12,13,15,16,17,18,20,22\},$$

9.
$$S_0 = \{1,2,4,5,6,9,11,13,15,16,20\},$$
$$S_1 = \{1,2,6,7,8,9,10,12,18,19,20\},$$
$$S_2 = \{2,6,8,9,11,12,14,15,17,21\},$$
$$S_3 = \{1,2,3,4,5,6,9,11,12,14,17,18,19,20,21,22\},$$

10.
$$S_0 = \{1,2,4,8,11,13,14,16,17,18,20\},$$
$$S_1 = \{1,2,3,4,5,9,10,12,15,16,17\},$$
$$S_2 = \{3,7,8,9,11,12,14,15,16,20\},$$
$$S_3 = \{1,2,3,5,7,8,10,11,12,13,15,16,18,20,21,22\},$$

11.
$$S_0 = \{1,2,4,5,6,10,12,14,15,16,20\},$$
$$S_1 = \{4,5,11,13,14,15,16,17,20,21,22\},$$
$$S_2 = \{1,2,5,8,10,13,15,18,21,22\},$$
$$S_3 = \{2,4,5,6,7,9,10,11,12,13,14,16,17,18,19,21\},$$

12.
$$S_0 = \{1,4,5,6,9,11,13,15,16,20,21\},$$
$$S_1 = \{1,2,3,9,12,13,15,16,17,18,19\},$$
$$S_2 = \{1,4,5,7,9,14,16,18,19,22\},$$
$$S_3 = \{1,2,3,4,5,6,8,9,14,15,17,18,19,20,21,22\},$$

13.
$$S_0 = \{1,2,4,8,10,11,14,16,17,18,20\},$$
$$S_1 = \{1,2,3,4,5,7,8,9,12,13,17\},$$
$$S_2 = \{3,4,6,8,9,14,15,17,19,20\},$$
$$S_3 = \{1,2,3,5,6,7,8,10,13,15,16,17,18,20,21,22\},$$

14.
$$S_0 = \{1,5,6,9,11,13,15,16,19,20,21\},$$
$$S_1 = \{1,2,3,6,7,8,9,10,11,18,19\},$$
$$S_2 = \{2,4,7,9,10,13,14,16,19,21\},$$
$$S_3 = \{2,4,5,6,7,8,9,11,12,14,15,16,17,18,19,21\},$$

15. $S_0 = \{1,3,4,9,10,11,15,16,17,18,21\}$,
$S_1 = \{1,2,5,6,7,8,9,10,11,19,20\}$,
$S_2 = \{1,3,6,9,10,13,14,17,20,22\}$,
$S_3 = \{1,2,4,6,8,9,10,11,12,13,14,15,17,19,21,22\}$,

16. $S_0 = \{1,4,5,7,8,9,11,13,17,20,21\}$,
$S_1 = \{1,2,5,6,7,8,9,10,11,19,20\}$,
$S_2 = \{1,4,6,10,11,12,13,17,19,22\}$,
$S_3 = \{1,2,4,6,7,8,9,10,13,14,15,16,17,19,21,22\}$

n = 25; 4 − (25; 12, 12, 16, 16; 31)

1. $S_0 = \{1,2,3,5,8,9,10,13,14,18,19,21\}$,
$S_1 = \{1,2,3,4,6,7,8,10,11,13,16,20\}$,
$S_2 = \{2,4,5,6,7,8,10,12,13,15,17,18,19,20,21,23\}$,
$S_3 = \{1,2,4,5,6,8,9,10,15,16,17,19,20,21,23,24\}$,

2. $S_0 = \{1,2,3,5,6,8,11,12,15,16,18,21\}$,
$S_1 = \{1,2,3,4,6,7,8,9,11,13,15,20\}$,
$S_2 = \{2,3,4,6,7,8,10,11,14,15,17,18,19,21,22,23\}$,
$S_3 = \{1,2,4,6,7,8,9,10,15,16,17,18,19,21,23,24\}$,

3. $S_0 = \{1,4,5,9,11,13,15,17,18,19,22,23\}$,
$S_1 = \{4,7,8,13,14,15,16,19,20,22,23,24\}$,
$S_2 = \{1,4,6,7,9,10,11,12,13,14,15,16,18,19,21,24\}$,
$S_3 = \{1,2,3,4,8,10,11,12,13,14,15,17,21,22,23,24\}$,

4. $S_0 = \{1,2,3,4,7,10,12,14,16,17,19,20\}$,
$S_1 = \{1,2,8,11,12,15,16,18,19,20,21,22\}$,
$S_2 = \{1,3,5,6,8,9,10,11,14,15,16,17,19,20,22,24\}$,
$S_3 = \{3,4,5,7,8,9,10,12,13,15,16,17,18,20,21,22\}$,

5. $S_0 = \{1,2,4,6,10,13,14,16,17,18,20,22\}$,
$S_1 = \{1,3,4,5,6,7,10,11,12,16,17,23\}$,
$S_2 = \{1,2,4,6,9,10,11,12,13,14,15,16,19,21,23,24\}$,
$S_3 = \{1,2,6,7,8,9,10,12,13,15,16,17,18,19,23,24\}$,

6. $S_0 = \{1,5,6,8,9,11,13,15,18,21,22,23\}$,
$S_1 = \{1,2,3,4,5,8,9,10,12,14,18,19\}$,
$S_2 = \{1,4,5,6,7,8,10,11,14,15,17,18,19,20,21,24\}$,
$S_3 = \{1,2,3,4,5,7,9,10,15,16,18,20,21,22,23,24\}$,

7. $S_0 = \{1,4,5,7,9,12,14,15,17,19,22,23\}$,
$S_1 = \{1,2,6,12,14,15,16,17,18,20,21,22\}$,
$S_2 = \{2,3,4,5,6,8,10,11,14,15,17,19,20,21,22,23\}$,
$S_3 = \{3,4,5,6,8,9,10,12,13,15,16,17,19,20,21,22\}$,

8. $S_0 = \{1,2,4,7,8,9,10,12,14,19,20,22\}$,
$S_1 = \{4,5,7,8,9,10,11,13,19,22,23,24\}$,
$S_2 = \{1,2,3,5,6,8,10,11,14,15,17,19,20,22,23,24\}$,
$S_3 = \{2,4,5,6,9,10,11,12,13,14,15,16,19,20,21,23\}$,

9.
$S_0 = \{1,2,4,5,6,10,11,13,16,17,18,22\}$,
$S_1 = \{1,2,4,5,6,7,8,9,12,14,15,22\}$,
$S_2 = \{1,3,4,5,6,9,10,12,13,15,16,19,20,21,22,24\}$,
$S_3 = \{2,4,5,6,7,8,10,12,13,15,17,18,19,20,21,23\}$,

10.
$S_0 = \{1,2,4,7,11,13,15,16,17,19,20,22\}$,
$S_1 = \{2,3,4,5,6,7,9,13,14,15,17,24\}$,
$S_2 = \{1,2,3,5,6,7,10,12,13,15,18,19,20,22,23,24\}$,
$S_3 = \{1,4,5,6,7,10,11,12,13,14,15,18,19,20,21,24\}$,

11.
$S_0 = \{1,2,4,6,7,8,10,13,14,16,20,22\}$,
$S_1 = \{1,2,3,4,9,13,14,15,17,18,19,20\}$,
$S_2 = \{1,2,4,6,7,9,10,11,14,15,16,18,19,21,23,24\}$,
$S_3 = \{3,4,6,7,8,10,11,12,13,14,15,17,18,19,21,22\}$,

12.
$S_0 = \{1,3,4,6,8,9,13,14,15,18,20,23\}$,
$S_1 = \{5,6,7,8,9,11,12,15,21,22,23,24\}$,
$S_2 = \{1,2,3,4,5,7,9,11,14,16,18,20,21,22,23,24\}$,
$S_3 = \{1,2,3,4,5,8,9,12,13,16,17,20,21,22,23,24\}$,

13.
$S_0 = \{1,3,7,9,10,11,12,17,19,20,21,23\}$,
$S_1 = \{3,4,12,14,15,16,17,18,19,20,23,24\}$,
$S_2 = \{1,2,4,6,8,9,10,11,14,15,16,17,19,21,23,24\}$,
$S_3 = \{1,2,3,4,7,8,10,11,14,15,17,18,21,22,23,24\}$

n = 27; $4-(27;13,13,16,18;33)$

1.
$S_0 = \{1,3,4,5,6,8,9,10,12,13,16,20,25\}$,
$S_1 = \{3,4,7,8,9,10,13,15,16,21,22,25,26\}$,
$S_2 = \{1,3,5,6,8,11,12,13,14,15,16,19,21,22,24,26\}$,
$S_3 = \{1,2,3,4,5,9,11,12,13,14,15,16,18,22,23,24,25,26\}$,

2.
$S_0 = \{1,3,4,5,8,10,12,13,16,18,20,21,25\}$,
$S_1 = \{3,4,6,12,13,16,17,18,19,20,22,25,26\}$,
$S_2 = \{2,3,4,5,6,8,10,13,14,17,19,21,22,23,24,25\}$,
$S_3 = \{3,4,6,7,8,9,10,12,13,14,15,17,18,19,20,21,23,24\}$,

3.
$S_0 = \{1,2,4,9,11,14,15,17,19,20,21,22,24\}$,
$S_1 = \{2,3,4,5,7,12,13,16,17,18,19,21,26\}$,
$S_2 = \{1,3,5,6,9,10,11,12,15,16,17,18,21,22,24,26\}$,
$S_3 = \{1,2,3,4,5,6,7,10,13,14,17,20,21,22,23,24,25,26\}$,

4.
$S_0 = \{2,3,4,6,7,8,9,11,14,15,17,22,26\}$,
$S_1 = \{2,3,6,8,9,10,11,12,13,20,22,23,26\}$,
$S_2 = \{1,2,4,6,7,11,12,13,14,15,16,20,21,23,25,26\}$,
$S_3 = \{2,3,4,6,8,9,10,12,13,14,15,17,18,19,21,23,24,25\}$,

5.
$S_0 = \{1,5,9,11,14,15,17,19,20,21,23,24,25\}$,
$S_1 = \{3,5,8,9,10,11,12,13,20,21,23,25,26\}$,
$S_2 = \{1,3,4,5,6,10,11,13,14,16,17,21,22,23,24,26\}$,
$S_3 = \{1,2,3,4,5,6,9,10,12,15,17,18,21,22,23,24,25,26\}$,

6. $$\begin{aligned} S_0 &= \{1,2,3,7,10,11,13,15,18,19,21,22,23\},\\ S_1 &= \{1,2,3,4,5,7,8,9,11,12,14,17,21\},\\ S_2 &= \{1,2,5,6,7,8,10,12,15,17,19,20,21,22,25,26\},\\ S_3 &= \{2,3,4,6,7,8,9,10,12,15,17,18,19,20,21,23,24,25\}, \end{aligned}$$

7. $$\begin{aligned} S_0 &= \{1,2,3,5,8,9,10,12,13,16,20,21,23\},\\ S_1 &= \{1,8,10,14,15,16,18,20,21,22,23,24,25\},\\ S_2 &= \{1,2,3,7,8,10,11,13,14,16,17,19,20,24,25,26\},\\ S_3 &= \{1,2,3,4,7,9,11,12,13,14,15,16,18,20,23,24,25,26\}, \end{aligned}$$

8. $$\begin{aligned} S_0 &= \{1,2,4,7,8,12,13,16,17,18,21,22,24\},\\ S_1 &= \{2,3,5,13,15,16,17,18,19,20,21,23,26\},\\ S_2 &= \{2,3,5,7,9,10,11,13,14,16,17,18,20,22,24,25\},\\ S_3 &= \{2,3,4,5,7,10,11,12,13,14,15,16,17,20,22,23,24,25\}, \end{aligned}$$

9. $$\begin{aligned} S_0 &= \{1,2,4,5,7,10,14,15,16,18,19,21,24\},\\ S_1 &= \{1,2,6,7,8,9,10,12,13,16,22,23,24\},\\ S_2 &= \{2,3,5,7,9,10,11,13,14,16,17,18,20,22,24,25\},\\ S_3 &= \{1,2,3,4,5,6,7,11,13,14,16,20,21,22,23,24,25,26\}, \end{aligned}$$

10. $$\begin{aligned} S_0 &= \{1,3,8,10,12,14,16,18,20,21,22,23,25\},\\ S_1 &= \{1,5,10,11,12,13,18,19,20,21,23,24,25\},\\ S_2 &= \{3,4,6,7,9,10,12,13,14,15,17,18,20,21,23,24\},\\ S_3 &= \{1,2,3,4,7,8,11,12,13,14,15,16,19,20,23,24,25,26\}, \end{aligned}$$

11. $$\begin{aligned} S_0 &= \{1,2,3,4,6,9,12,13,16,17,19,20,22\},\\ S_1 &= \{1,3,4,5,6,7,8,9,12,13,16,17,25\},\\ S_2 &= \{1,2,4,6,8,10,11,12,15,16,17,19,21,23,25,26\},\\ S_3 &= \{1,2,3,4,8,9,10,11,13,14,16,17,18,19,23,24,25,26\}, \end{aligned}$$

12. $$\begin{aligned} S_0 &= \{1,3,4,5,7,9,10,11,14,15,19,21,25\},\\ S_1 &= \{5,6,12,14,16,17,18,19,20,23,24,25,26\},\\ S_2 &= \{1,2,3,4,7,11,12,13,14,15,16,20,23,24,25,26\},\\ S_3 &= \{1,2,4,6,7,9,10,11,13,14,16,17,18,20,21,23,25,26\}, \end{aligned}$$

13. $$\begin{aligned} S_0 &= \{1,3,4,6,8,9,10,14,15,16,20,22,25\},\\ S_1 &= \{1,2,3,4,5,6,10,12,13,16,18,19,20\},\\ S_2 &= \{1,2,4,5,6,8,9,13,14,18,19,21,22,23,25,26\},\\ S_3 &= \{1,2,3,4,5,6,8,10,13,14,17,19,21,22,23,24,25,26\}, \end{aligned}$$

14. $$\begin{aligned} S_0 &= \{3,5,6,7,8,11,14,15,17,18,23,25,26\},\\ S_1 &= \{1,2,3,4,5,7,8,9,10,13,15,16,21\},\\ S_2 &= \{2,3,4,6,7,8,10,12,15,17,19,20,21,23,24,25\},\\ S_3 &= \{1,3,4,6,7,8,9,10,13,14,17,18,19,20,21,23,24,26\}, \end{aligned}$$

15. $$\begin{aligned} S_0 &= \{1,4,5,6,8,10,14,15,16,18,20,24,25\},\\ S_1 &= \{1,2,3,6,13,15,16,17,18,19,20,22,23\},\\ S_2 &= \{1,4,5,7,8,10,12,13,14,15,17,19,20,22,23,26\},\\ S_3 &= \{1,2,6,7,8,9,10,12,13,14,15,17,18,19,20,21,25,26\}, \end{aligned}$$

16. $$\begin{aligned} S_0 &= \{1,2,4,5,7,10,12,14,16,18,19,21,24\},\\ S_1 &= \{1,2,3,4,5,6,8,9,11,12,13,17,20\},\\ S_2 &= \{2,4,5,6,7,8,11,12,15,16,19,20,21,22,23,25\},\\ S_3 &= \{1,3,4,5,6,9,10,11,12,15,16,17,18,21,22,23,24,26\}, \end{aligned}$$

17.
$S_0 = \{1,3,8,9,12,13,16,17,20,21,22,23,25\}$,
$S_1 = \{1,2,3,11,13,15,17,18,19,20,21,22,23\}$,
$S_2 = \{1,2,4,5,8,10,11,12,15,16,17,19,22,23,25,26\}$,
$S_3 = \{1,2,3,4,7,9,11,12,13,14,15,16,18,20,23,24,25,26\}$,

18.
$S_0 = \{2,4,5,12,14,16,17,18,19,20,21,24,26\}$,
$S_1 = \{6,7,12,13,16,17,18,19,22,23,24,25,26\}$,
$S_2 = \{1,4,5,6,7,9,10,13,14,17,18,20,21,22,23,26\}$,
$S_3 = \{1,2,4,5,6,8,10,12,13,14,15,17,19,21,22,23,25,26\}$,

19.
$S_0 = \{1,2,3,5,7,9,12,13,16,17,19,21,23\}$,
$S_1 = \{1,9,10,11,14,15,19,20,21,22,23,24,25\}$,
$S_2 = \{1,2,4,5,6,8,11,13,14,16,19,21,22,23,25,26\}$,
$S_3 = \{1,2,6,7,8,9,10,12,13,14,15,17,18,19,20,21,25,26\}$,

20.
$S_0 = \{1,3,4,7,11,12,14,17,18,19,21,22,25\}$,
$S_1 = \{5,6,7,8,9,11,12,13,17,23,24,25,26\}$,
$S_2 = \{1,4,6,7,8,10,11,12,15,16,17,19,20,21,23,26\}$,
$S_3 = \{1,3,6,7,8,9,11,12,13,14,15,16,18,19,20,21,24,26\}$

$\mathbf{n = 31}$; $4-(31;15,15,16,10;25)$

1.
$S_0 = \{3,4,6,13,16,17,19,20,21,22,23,24,26,29,30\}$,
$S_1 = \{2,3,5,6,7,8,9,11,16,17,18,19,21,27,30\}$,
$S_2 = \{2,4,5,6,10,11,13,15,16,18,20,21,25,26,27,29\}$,
$S_3 = \{3,5,9,10,14,17,21,22,26,28\}$,

2.
$S_0 = \{3,4,5,6,9,11,12,13,14,15,21,23,24,29,30\}$,
$S_1 = \{1,2,3,6,7,9,10,11,12,13,14,16,23,26,27\}$,
$S_2 = \{2,5,6,7,9,11,12,14,17,19,20,22,24,25,26,29\}$,
$S_3 = \{1,3,7,12,15,16,19,24,28,30\}$,

3.
$S_0 = \{1,2,4,5,6,8,10,13,14,16,19,20,22,24,28\}$,
$S_1 = \{1,2,3,6,7,9,10,11,12,13,14,16,23,26,27\}$,
$S_2 = \{1,4,6,7,8,11,14,15,16,17,20,23,24,25,27,30\}$,
$S_3 = \{6,7,8,12,14,17,19,23,24,25\}$,

4.
$S_0 = \{1,4,6,10,11,14,16,18,19,22,23,24,26,28,29\}$,
$S_1 = \{1,2,3,4,9,14,16,18,19,20,21,23,24,25,26\}$,
$S_2 = \{1,4,7,8,12,13,14,15,16,17,18,19,23,24,27,30\}$,
$S_3 = \{1,3,10,11,14,17,20,21,28,30\}$,

5.
$S_0 = \{1,2,5,6,7,9,12,13,14,15,20,21,23,27,28\}$,
$S_1 = \{1,2,4,5,6,7,8,9,11,16,17,18,19,21,28\}$,
$S_2 = \{1,2,4,5,6,8,12,14,17,19,23,25,26,27,29,30\}$,
$S_3 = \{2,3,6,12,14,17,19,25,28,29\}$,

6.
$S_0 = \{1,2,6,8,9,11,12,13,16,17,21,24,26,27,28\}$,
$S_1 = \{1,2,3,4,5,6,7,9,10,11,14,16,18,19,23\}$,
$S_2 = \{1,3,5,6,7,12,13,15,16,18,19,24,25,26,28,30\}$,
$S_3 = \{3,4,6,11,14,17,20,25,27,28\}$,

7.
$S_0 = \{1,3,7,8,10,14,16,18,19,20,22,25,26,27,29\}$,
$S_1 = \{2,6,7,8,9,10,11,12,13,16,17,26,27,28,30\}$,
$S_2 = \{1,3,6,9,11,12,13,14,17,18,19,20,22,25,28,30\}$,
$S_3 = \{4,5,8,12,13,18,19,23,26,27\}$,

8.
$S_0 = \{1,2,4,5,6,7,8,10,12,14,15,18,20,22,28\}$,
$S_1 = \{5,6,7,8,9,10,13,14,15,19,20,27,28,29,30\}$,
$S_2 = \{1,2,5,6,7,10,12,14,17,19,21,24,25,26,29,30\}$,
$S_3 = \{3,6,8,9,12,19,22,23,25,28\}$

n = 33; $4-(33;16,16,20,12;31)$

1.
$S_0 = \{1,2,5,8,10,12,13,14,15,16,22,24,26,27,29,30\}$,
$S_1 = \{1,3,5,7,8,11,14,15,16,20,21,23,24,27,29,31\}$,
$S_2 = \{1,2,3,7,8,9,10,11,13,16,17,20,22,23,24,25,26,30,31,32\}$,
$S_3 = \{1,2,3,4,7,14,19,26,29,30,31,32\}$,

2.
$S_0 = \{2,3,4,5,6,9,10,11,12,15,16,19,20,25,26,32\}$,
$S_1 = \{1,2,3,4,7,9,12,14,16,18,20,22,23,25,27,28\}$,
$S_2 = \{4,5,6,7,8,10,11,13,14,15,18,19,20,22,23,25,26,27,28,29\}$,
$S_3 = \{2,6,10,12,13,15,18,20,21,23,27,31\}$,

3.
$S_0 = \{2,4,5,6,7,8,10,12,13,17,18,19,22,24,30,32\}$,
$S_1 = \{1,2,4,6,7,10,13,14,17,18,21,22,24,25,28,30\}$,
$S_2 = \{3,6,7,8,9,10,11,13,15,16,17,18,20,22,23,24,25,26,27,30\}$,
$S_3 = \{2,6,7,8,14,16,17,19,25,26,27,31\}$,

4.
$S_0 = \{1,2,3,4,9,12,14,16,18,20,22,23,25,26,27,28\}$,
$S_1 = \{1,2,5,6,8,11,13,17,18,19,21,23,24,26,29,30\}$,
$S_2 = \{3,4,6,8,9,10,11,12,13,16,17,20,21,22,23,24,25,27,29,30\}$,
$S_3 = \{1,9,10,13,14,16,17,19,20,23,24,32\}$,

5.
$S_0 = \{2,6,8,10,11,12,17,18,19,20,24,26,28,29,30,32\}$,
$S_1 = \{1,2,4,6,7,10,11,12,14,17,18,20,24,25,28,30\}$,
$S_2 = \{5,6,7,8,9,10,12,13,15,16,17,18,20,21,23,24,25,26,27,28\}$,
$S_3 = \{3,4,8,11,13,16,17,20,22,25,29,30\}$,

6.
$S_0 = \{1,6,8,13,14,17,18,21,22,23,24,26,28,29,30,31\}$,
$S_1 = \{1,2,5,8,10,11,15,17,19,20,21,24,26,27,29,30\}$,
$S_2 = \{3,5,7,8,9,10,11,12,14,15,18,19,21,22,23,24,25,26,28,30\}$,
$S_3 = \{3,4,7,9,15,16,17,18,24,26,29,30\}$,

7.
$S_0 = \{5,7,8,9,10,11,14,15,16,20,21,27,29,30,31,32\}$,
$S_1 = \{1,2,4,6,7,9,10,11,14,17,18,20,21,25,28,30\}$,
$S_2 = \{2,3,5,7,8,9,10,11,13,16,17,20,22,23,24,25,26,28,30,31\}$,
$S_3 = \{1,3,4,5,9,12,21,24,28,29,30,32\}$,

8.
$S_0 = \{1,2,3,5,6,7,8,12,13,14,15,16,22,23,24,29\}$,
$S_1 = \{1,2,4,5,7,10,13,17,18,19,21,22,24,25,27,30\}$,
$S_2 = \{2,4,6,8,9,11,12,13,15,16,17,18,20,21,22,24,25,27,29,31\}$,
$S_3 = \{1,2,3,9,12,16,17,21,24,30,31,32\}$,

9.
$$\begin{aligned}
S_0 &= \{1,5,7,9,14,15,17,20,21,22,23,25,27,29,30,31\},\\
S_1 &= \{1,3,5,6,8,12,13,14,17,18,22,23,24,26,29,31\},\\
S_2 &= \{1,2,3,5,6,7,8,13,14,16,17,19,20,25,26,27,28,30,31,32\},\\
S_3 &= \{7,8,9,11,12,15,18,21,22,24,25,26\},
\end{aligned}$$

10.
$$\begin{aligned}
S_0 &= \{1,7,8,10,12,16,18,19,20,22,24,27,28,29,30,31\},\\
S_1 &= \{1,4,5,6,8,11,13,17,18,19,21,23,24,26,30,31\},\\
S_2 &= \{1,6,7,8,9,10,12,13,15,16,17,18,20,21,23,24,25,26,27,32\},\\
S_3 &= \{5,7,8,11,12,16,17,21,22,25,26,28\},
\end{aligned}$$

11.
$$\begin{aligned}
S_0 &= \{1,3,7,8,9,10,13,17,18,19,21,22,27,28,29,31\},\\
S_1 &= \{1,4,6,8,9,11,12,13,17,18,19,23,26,28,30,31\},\\
S_2 &= \{1,3,5,6,11,12,13,14,15,16,17,18,19,20,21,22,27,28,30,32\},\\
S_3 &= \{5,8,9,12,13,15,18,20,21,24,25,28\},
\end{aligned}$$

12.
$$\begin{aligned}
S_0 &= \{2,6,7,10,14,15,16,20,21,22,24,25,28,29,30,32\},\\
S_1 &= \{1,3,4,5,7,9,10,14,15,17,20,21,22,25,27,31\},\\
S_2 &= \{1,2,5,7,10,11,12,13,14,15,18,19,20,21,22,23,26,28,31,32\},\\
S_3 &= \{3,5,12,14,15,16,17,18,19,21,28,30\},
\end{aligned}$$

13.
$$\begin{aligned}
S_0 &= \{1,2,3,4,5,6,8,9,11,12,13,17,18,19,23,26\},\\
S_1 &= \{1,3,4,8,11,12,13,15,17,19,23,24,26,27,28,31\},\\
S_2 &= \{1,2,3,4,7,8,10,13,15,16,17,18,20,23,25,26,29,30,31,32\},\\
S_3 &= \{1,2,3,9,11,15,18,22,24,30,31,32\},
\end{aligned}$$

14.
$$\begin{aligned}
S_0 &= \{1,2,3,5,8,9,10,12,13,15,16,19,22,26,27,29\},\\
S_1 &= \{1,4,6,8,9,10,12,13,17,18,19,22,26,28,30,31\},\\
S_2 &= \{1,2,3,7,9,12,13,14,15,16,17,18,19,20,21,24,26,30,31,32\},\\
S_3 &= \{1,9,10,11,14,16,17,19,22,23,24,32\},
\end{aligned}$$

15.
$$\begin{aligned}
S_0 &= \{1,2,7,8,10,13,15,17,19,21,22,24,27,28,29,30\},\\
S_1 &= \{1,3,5,6,9,10,11,13,14,15,17,21,25,26,29,31\},\\
S_2 &= \{4,5,6,7,9,10,11,13,14,16,17,19,20,22,23,24,26,27,28,29\},\\
S_3 &= \{3,4,5,6,9,13,20,24,27,28,29,30\},
\end{aligned}$$

16.
$$\begin{aligned}
S_0 &= \{1,3,5,7,15,16,19,20,21,22,23,24,25,27,29,31\},\\
S_1 &= \{1,3,6,8,9,10,14,17,18,20,21,22,26,28,29,31\},\\
S_2 &= \{1,2,3,4,5,9,10,11,13,14,19,20,22,23,24,28,29,30,31,32\},\\
S_3 &= \{3,4,7,8,10,13,20,23,25,26,29,30\},
\end{aligned}$$

17.
$$\begin{aligned}
S_0 &= \{1,2,3,7,11,16,18,19,20,21,23,24,25,27,28,29\},\\
S_1 &= \{1,3,4,6,7,11,13,17,18,19,21,23,24,25,28,31\},\\
S_2 &= \{2,3,5,6,7,12,13,14,15,16,17,18,19,20,21,26,27,28,30,31\},\\
S_3 &= \{3,6,9,11,13,16,17,20,22,24,27,30\},
\end{aligned}$$

18.
$$\begin{aligned}
S_0 &= \{2,3,5,7,9,17,18,19,20,21,22,23,25,27,29,32\},\\
S_1 &= \{1,3,4,9,11,13,14,17,18,21,23,25,26,27,28,31\},\\
S_2 &= \{1,2,7,8,9,10,12,13,14,16,17,19,20,21,23,24,25,26,31,32\},\\
S_3 &= \{3,6,10,11,15,16,17,18,22,23,27,30\},
\end{aligned}$$

19.
$$\begin{aligned}
S_0 &= \{4,7,8,9,10,11,12,14,15,16,20,27,28,30,31,32\},\\
S_1 &= \{1,3,7,9,10,14,17,18,20,21,22,25,27,28,29,31\},\\
S_2 &= \{2,3,4,8,10,11,12,13,14,16,17,19,20,21,22,23,25,29,30,31\},\\
S_3 &= \{2,3,5,7,12,15,18,21,26,28,30,31\},
\end{aligned}$$

20.
$S_0 = \{1,2,3,4,7,10,12,14,15,16,20,22,24,25,27,28\}$,
$S_1 = \{1,4,8,10,11,12,15,17,19,20,24,26,27,28,30,31\}$,
$S_2 = \{3,5,6,8,9,10,11,13,14,15,18,19,20,22,23,24,25,27,28,30\}$,
$S_3 = \{3,7,12,13,14,15,18,19,20,21,26,30\}$,

21.
$S_0 = \{2,3,6,8,10,17,18,19,20,21,22,24,26,28,29,32\}$,
$S_1 = \{1,3,4,5,6,8,10,13,14,17,18,21,22,24,26,31\}$,
$S_2 = \{1,2,3,4,5,8,9,10,11,14,19,22,23,24,25,28,29,30,31,32\}$,
$S_3 = \{4,5,6,10,12,15,18,21,23,27,28,29\}$,

22.
$S_0 = \{1,2,3,9,10,12,14,15,17,20,22,25,26,27,28,29\}$,
$S_1 = \{1,3,5,6,7,9,10,13,17,18,19,21,22,25,29,31\}$,
$S_2 = \{2,4,6,7,8,9,11,12,15,16,17,18,21,22,24,25,26,27,29,31\}$,
$S_3 = \{2,9,12,13,14,15,18,19,20,21,24,31\}$

n = 33; $4-(33;16,16,18,22;39)$

1.
$S_0 = \{1,2,4,5,6,8,10,12,13,17,18,19,22,24,26,30\}$,
$S_1 = \{1,3,4,5,8,10,13,16,18,19,21,22,24,26,27,31\}$,
$S_2 = \{2,5,6,7,8,9,12,15,16,17,18,21,24,25,26,27,28,31\}$,
$S_3 = \{2,3,4,8,10,11,12,13,14,15,16,17,18,19,20,21,22,23,25,29,30,31\}$,

2.
$S_0 = \{4,5,7,8,9,10,11,12,14,15,16,20,27,30,31,32\}$,
$S_1 = \{1,2,4,6,9,13,15,16,19,21,22,23,25,26,28,30\}$,
$S_2 = \{2,3,4,6,7,10,12,15,16,17,18,21,23,26,27,29,30,31\}$,
$S_3 = \{1,2,3,4,5,9,11,12,13,14,16,17,19,20,21,22,24,28,29,30,31,32\}$,

3.
$S_0 = \{1,2,3,6,7,9,15,17,19,20,21,22,23,25,28,29\}$,
$S_1 = \{1,2,4,5,7,9,10,12,13,14,16,18,22,25,27,30\}$,
$S_2 = \{2,5,7,8,9,13,14,15,16,17,18,19,20,24,25,26,28,31\}$,
$S_3 = \{1,2,3,6,7,8,9,10,11,13,16,17,20,22,23,24,25,26,27,30,31,32\}$,

4.
$S_0 = \{1,2,3,4,7,8,9,10,12,13,15,17,19,22,27,28\}$,
$S_1 = \{1,2,4,5,6,9,12,13,15,17,19,22,23,25,26,30\}$,
$S_2 = \{1,3,4,6,7,8,9,10,15,18,23,24,25,26,27,29,30,32\}$,
$S_3 = \{2,4,5,6,7,8,10,11,12,15,16,17,18,21,22,23,25,26,27,28,29,31\}$,

5.
$S_0 = \{2,3,7,9,12,13,14,15,16,22,23,25,27,28,29,32\}$,
$S_1 = \{1,3,5,6,7,11,12,14,15,17,20,23,24,25,29,31\}$,
$S_2 = \{2,6,8,9,10,13,14,15,16,17,18,19,20,23,24,25,27,31\}$,
$S_3 = \{1,2,4,6,7,8,9,10,11,13,14,19,20,22,23,24,25,26,27,29,31,32\}$,

6.
$S_0 = \{3,5,6,7,8,9,10,13,16,18,19,21,22,29,31,32\}$,
$S_1 = \{1,3,8,9,12,13,15,17,19,22,23,26,27,28,29,31\}$,
$S_2 = \{1,2,3,6,7,8,9,10,14,19,23,24,25,26,27,30,31,32\}$,
$S_3 = \{1,3,4,5,7,8,9,10,11,14,16,17,19,22,23,24,25,26,28,29,30,32\}$,

7.
$S_0 = \{1,4,5,6,13,16,18,19,21,22,23,24,25,26,30,31\}$,
$S_1 = \{1,3,5,6,7,9,10,11,14,15,17,20,21,25,29,31\}$,
$S_2 = \{2,3,5,6,8,12,13,14,15,18,19,20,21,25,27,28,30,31\}$,
$S_3 = \{1,2,3,5,7,8,9,10,11,12,14,19,21,22,23,24,25,26,28,30,31,32\}$,

8.
$S_0 = \{1,3,4,8,9,11,13,14,15,17,21,23,26,27,28,31\}$,
$S_1 = \{1,4,8,9,10,12,15,17,19,20,22,26,27,28,30,31\}$,
$S_2 = \{1,2,3,6,11,12,13,14,15,18,19,20,21,22,27,30,31,32\}$,
$S_3 = \{3,4,6,8,9,10,12,13,14,15,16,17,18,19,20,21,23,24,25,27,29,30\}$,

9.
$S_0 = \{1,3,5,6,11,12,14,16,18,20,23,24,25,26,29,31\}$,
$S_1 = \{1,3,6,7,8,10,14,17,18,20,21,22,24,28,29,31\}$,
$S_2 = \{4,6,7,8,9,10,13,14,15,18,19,20,23,24,25,26,27,29\}$,
$S_3 = \{1,2,3,4,5,6,7,10,11,13,14,19,20,22,23,26,27,28,29,30,31,32\}$

Table 2.

Some non-equivalent circulant G-matrices of orders 37, 41

n = 37; $4-(37;18,18,16,24;39)]$

1.
$S_0 = \{1,2,3,4,5,13,18,20,21,22,23,25,26,27,28,29,30,31\}$,
$S_1 = \{1,2,4,6,8,9,12,14,15,18,20,21,24,26,27,30,32,34\}$,
$S_2 = \{1,5,6,7,8,11,15,16,21,22,26,29,30,31,32,36\}$,
$S_3 = \{2,3,5,7,8,10,11,12,14,15,16,18,19,21,22,23,25,26,27,29,30,32,34,35\}$

2.
$S_0 = \{1,4,5,6,7,8,13,17,19,21,22,23,25,26,27,28,34,35\}$,
$S_1 = \{1,3,5,7,8,10,11,14,15,16,19,20,24,25,28,31,33,35\}$,
$S_2 = \{2,4,5,6,8,13,16,18,19,21,24,29,31,32,33,35\}$,
$S_3 = \{1,2,3,4,5,6,8,9,10,11,16,17,20,21,26,27,28,29,31,32,33,34,35,36\}$

3.
$S_0 = \{1,4,7,10,11,12,15,17,19,21,23,24,28,29,31,32,34,35\}$,
$S_1 = \{1,3,4,7,9,11,12,16,17,19,22,23,24,27,29,31,32,35\}$,
$S_2 = \{3,7,8,14,15,16,17,18,19,20,21,22,23,29,30,34\}$,
$S_3 = \{1,2,3,4,6,8,12,13,14,15,16,17,20,21,22,23,24,25,29,31,33,34,35,36\}$

4.
$S_0 = \{1,2,4,5,6,7,11,14,16,17,18,22,24,25,27,28,29,34\}$,
$S_1 = \{1,2,3,5,10,13,15,19,20,21,23,25,26,28,29,30,31,33\}$,
$S_2 = \{2,3,5,6,8,9,10,14,23,27,28,29,31,32,34,35\}$,
$S_3 = \{1,2,5,6,7,8,9,11,13,14,15,17,20,22,23,24,26,28,29,30,31,32,35,36\}$

n = 41; $4-(41;20,20,16,16;31)]$

1.
$S_0 = \{1,3,5,7,8,13,16,17,18,19,21,26,27,29,30,31,32,35,37,39\}$,
$S_1 = \{1,6,7,10,12,13,15,16,17,19,20,23,27,30,32,33,36,37,38,39\}$,
$S_2 = \{3,9,10,11,14,15,16,18,23,25,26,27,30,31,32,38\}$,
$S_3 = \{1,2,5,6,7,9,15,17,24,26,32,34,35,36,39,40\}$

2.
$S_0 = \{2,4,5,6,10,11,13,14,15,16,17,18,19,21,23,31,34,35,36,40\}$,
$S_1 = \{2,3,4,8,9,12,15,17,19,22,23,25,27,29,30,32,33,36,37,38\}$,
$S_2 = \{2,3,7,11,13,16,17,18,25,26,27,30,32,36,40,41\}$,
$S_3 = \{2,4,5,6,7,14,17,20,23,26,29,36,37,38,39,41\}$

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