On Equal Chromatic Partition of Interconnection Networks

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ABSTRACT. We introduce the concept of equal chromatic partition of networks. This concept is useful for deriving lower bounds and upper bounds for performance ratios of dynamic tree embedding schemes that arise in a wide range of treestructured parallel computations. We provide necessary and sufficient conditions for the existence of equal chromatic partitions of several classes of interconnection networks which include X-Nets, folded hypercubes, X-trees, n-dimensional tori and k-ary n-cubes. We use the pyramid network as an example to show that some networks do not have equal chromatic partitions, but may have near-equal chromatic partitions.

1 Introduction

Corloring all nodes of a graph with colors such that no two adjacent nodes have the same color is called the *vertex* (or *node*) *coloring* (or simply *coloring*). A graph that requires k different colors for its vertex coloring, and no less, is called a *k-chromatic graph*, and the number k is called the *chromatic number* of the graph. A closely related concept is graph partition. A graph is *p-partite* if its node set can be partitioned into k disjoint subsets such that no two nodes in any subset are adjacent. Such a partition is called a *p-partition*. Clearly, a graph G is p-partite if and only if $p \geq k$, where k is the chromatic number of G. A p-partition of a graph G is a *chromatic partition* of G if $p = k$. Graph coloring and partitioning have many applications in solving practical problems arising in, for instances, coding theory, state reduction of sequential circuits and VLSI layout.

In this paper, we consider a partition problem which has applications in interconnection networks of parallel computer systems. An interconnection network is characterized by a graph with its nodes and edges representing processors and communication channels connecting pairs of processors, respectively, in a parallel computer system. An interconnection network (or network for short) is bipartite if and only if it does not contain an odd cycle. This simple fact has been used to partition networks into two classes: bipartite and non-bipartite. Example bipartite networks include trees, hypercube, n-dimensional mesh, mesh of trees and the star graph. In this paper, we show the chromatic numbers of several useful networks are greater than 2. We are interested in the following network chromatic partition problem: find a chromatic partition of a given network such that the sizes of the subsets are as equal as possible. We call a k-partition of an n-node graph G an *equal k-partition of G* if the k subsets have the same size. Whenever possible, we want to find an equal chromatic partition of a given network. It is known that the hypercube and the star graph have equal chromatic partitions. We consider the partitions of several classes of interconnection networks which include X-Nets, folded hypercubes, X-trees, n-dimensional tori and k-ary n-cubes. We provide necessary and sufficient conditions for the existence of equal chromatic partitions of these networks. We use the pyramid network as an example to show that some networks do not have equal chromatic partitions, but may have near-equal chromatic partitions.

The network chromatic partition problem considered here has applications in optimizing performance ratios of dynamic tree embeddings into a given network. Dynamic tree embedding (DTE) schemes arise in tree-structured parallel computations such as divide-and-conquer algorithms, branch-and-bound computation, backtrack search algorithms, game-tree evaluation, functional and logical programs. In these applications the shape and the size of the tree representing a tree-structured computation are unpredictable at compile-time, and the tree grows during the course of computation. A DTE algorithm is an on-line algorithm that maps processes that are executable in parallel to processors. Several DTE algorithms have been developed and analyzed (e.g. [1, 3, 4, 6, 8, 9, 12, 10, 11]). In [10], lower bounds for performance ratios of DTEs on bipartite networks are given. In [11], the network partition concept is generalized to p-partite networks, and upper bounds for performance ratios of DTEs on such networks are derived. The class of DTE algorithms given in [11] are based on the equal p-partition, $p \geq k$, where k is the chromatic number of the graph. For different value p, these algorithms have different performances. Since an equal chromatic partition can be used to construct other equal partitions, our partition methods can be used as subalgorithms of the class of DTE algorithms given in [11]. Furthermore, our results can be combined with the analysis of [11] to assess the performances of these DTE algorithms on the networks discussed in this paper. We believe that the concept of equal chromatic network (graph) partition may have other applications.

2 Chromatic Partitions

In this section, we present our findings on chromatic partitions of several representative classes of interconnection networks. Our goal is to find such partitions with node subsets of sizes as equal as possible.

2.1 X-Nets

First, we consider the X-Net interconnection network used in the MasPar parallel computer. An $m \times n$ X-Net is an $m \times n$ two-dimensional mesh with additional diagonal links in each cycle of four nodes. Since it contains many 4-cliques, it cannot be k-partite such that $k < 4$, and furthermore, it is easy to see that the X-Net is not planar. However, we can use four colors c_i, $0 \le i \le 3$, to color its nodes in the way shown in Figure 1. It is simple to verify the following claim:

Theorem 1. *The chromatic number of an $m \times n$ X-Net, where $m \geq 2$ and $n \geq 2$, is 4, and such an XNet has an equal chromatic if and only if mn is a multiple of 4.*

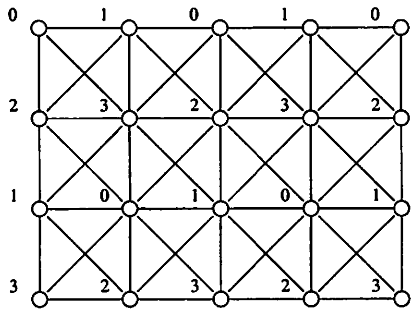


Figure 1. Four coloring of an 4×5 X-Net

2.2 Folded Hypercubes

An n-dimensional folded hypercube [2] is a graph of 2^n nodes, each being labeled by a unique n-bit binary number. In such a graph, two nodes are connected by an edge if and only if either their binary labels are distinct in exactly one bit position, or the label of one node is the complement of the label of the other. An n-dimensional folded hypercube, which has degree $(n+1)$, contains an n-dimensional hypercube, which has degree n, as a subgraph. The diameter of the n-dimensional folded hypercube is half of the diameter of the n-dimensional hypercube. Unlike the n-dimensional hypercube, the n-dimensional folded hypercube may not be bipartite.

We say that a node in the n-dimensional folded hypercube has an even (resp. odd) parity if its label contains an even (resp. odd) number of 1s. Divide nodes in the n-dimensional folded hypercube into two subsets V' and V'' such that V' contains all nodes of even parity, and V'' contains the remaining nodes. Clearly, $|V_1| = |V_2|$. If n is odd, then, by the definition of the n-dimensional folded hypercube, two nodes are not connected by an edge if they have the same parity. Therefore, the n-dimensional folded hypercube is bipartite if n is odd. If n is even, we partition the nodes into V' and V'' as in the case that n is odd. Clearly, two nodes in V' (resp. V'') are connected by an edge if and only if the label of one is the complement of the label of the other (note: the labels of both nodes have the same parity). Then, we can partition V' (resp. V'') into two subsets V_0 and V_1 (resp. V_2

and V_3) of equal size such that a bijection exists between them. That is, for every node u in V_0 (resp. V_2) there is exactly one node v in V_1 (resp. V_3) such that the label of v is the complement of the label of u, and vice versa. An example is shown in Figure 2. We summarize these findings by the following statement.

Theorem 2. *The chromatic number of the n-dimensional folded hypercube is 2, if n is odd, and 4, if n is even. Furthermore, the n-dimensional folded hypercube has an equal chromatic partition.*

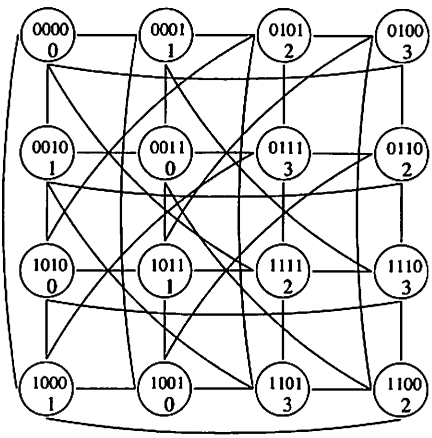


Figure 2. A 4-partition of an 4-dimensional folded hypercube. Each node has a 4-bit label. The single digit label i of a node indicates that the node belongs to subset V_i of a 4-partition

2.3 Tori and k-ary n-Cubes

In this and subsequent subsections, we prove that the chromatic numbers of several classes of networks are equal to 3. For this purpose, we define three bijection functions f_i, $0 \leq i \leq 2$, on a set $C = \{c_0, c_1, c_2\}$ of three distinct colors:

$$f_a(c_b) = C_{(a+b) \bmod 3} \tag{1}$$

The $k_1 \times k_2 \ldots \times k_n$ n-dimensional torus is a graph $G = (V, E)$ that contains $\prod_{d=1}^{n} k_d$ nodes ($k_d > 2$ for $d = 1, 2, \ldots, n$). Each node a is labeled by a unique n-tuple $\langle a_1, a_2, \ldots, a_n \rangle$, $0 \leq a_d \leq k_d - 1$. Two nodes a and b are connected by an edge if and only if either $a_d = b_d + 1 \pmod{k_d}$ or $a_d = b_d - 1 \pmod{k_d}$ for some d, and $a_l = b_l$ for $l \neq d$. The $k_1 \times k_2 \ldots \times k_n$ n-dimensional torus can also be defined recursively as follows. A ring of size k_1 with nodes labeled $\langle 0 \rangle, \langle 1 \rangle, \ldots, \langle k_1 - 1 \rangle$ such that node $\langle i \rangle$ and node $\langle j \rangle$

are connected by an edge if and only if either $i = j+1 \mod k_1$ or $i = j-1 \pmod{k_1}$. The $k_1 \times k_2 \ldots \times k_n$, n-dimensional torus G is constructed using k_n copies $G_0, G_1, \ldots, G_{k_n-1}$ of the $k_1 \times k_2 \ldots \times k_{n-1}$ $(n-1)$-dimensional torus, and new edges connecting nodes from different copies. Each node in G is labeled by an n-tuple $\langle a_1, a_2, \ldots, a_n \rangle$. The connectivity of nodes in each copy G_i follows the definition of the the $k_1 \times k_2 \ldots \times k_{n-1}$ $(n-1)$-dimensional torus on the first $n-1$ elements (i.e. $(n-1)$-prefix) of the n-tuples. The connectivity of two nodes $a = \langle a_1, a_2, \ldots, a_n \rangle$ and $b = \langle b_1, b_2, \ldots, b_n \rangle$ from two different copies follows the definition of a ring of size k_n on the rightmost elements of the n-tuples. The added edges that connect nodes of different copies of the $(n-1)$-dimensional tours are called the *n-dimensional edges.*

Theorem 3. *The chromatic number of the $k_1 \times k_2 \ldots \times k_n$ n-dimensional torus is 2 if all k_d's, $1 \leq d \leq n$, are even, and 3, if at least one of k_d's is odd. The $k_1 \times k_2 \ldots \times k_n$ n-dimensional torus has an equal 2-partition if and only if all k_d's are even, and it has an equal 3-partition if and only if at least one of k_d's is a multiple of 3.*

Proof: Clearly, the $k_1 \times k_2 \ldots \times k_n$ n-dimensional torus has chromatic number 2 if and only if that all k_d, $1 \leq d \leq n$, are even since all its cycles are of even size. We show that if all k_d, $1 \leq d \leq n$, are even, the torus has an equal bipartition. Define

$$V_0 = \left\{ \langle a_1, a_2, \ldots, a_n \rangle \mid 1 \leq a_d \leq k_d, \sum_{d=1}^{n} a_d \text{ even} \right\},$$

and

$$V_1 = \left\{ \langle a_1, a_2, \ldots, a_n \rangle \mid 1 \leq a_d \leq k_d, \sum_{d=1}^{n} a_d \text{ odd} \right\}.$$

We claim that V_0 and V_1 give a partition. To see this, consider two nodes $\langle a_1, a_2, \ldots, a_n \rangle$ and $\langle b_1, b_2, \ldots, b_n \rangle$ in V_0 (resp. V_1). Assume that these two nodes are connected by an edge, i.e., the distance between them is one. Since $\langle a_1, a_2, \ldots, a_n \rangle \neq \langle b_1, b_2, \ldots, b_n \rangle$, there exists d such that $a_d \neq b_d$, either $a_d = b_d + 1 \mod k_d$ or $a_d = b_d - 1 \mod k_d$, and $a_{d'} = b_{d'}$ for all $d' \neq d$. This implies that $(a_1 + a_2 + \cdots + a_d)$ and $(b_1 + b_2 + \cdots + b_d)$ have different parity, and one of the two nodes must be in V_1 (resp. V_0). This is a contradiction to the assumption that both are from V_0 (resp. V_1). Now, let us look at the sizes of V_0 and V_1. It is well known that an n-dimensional torus has a Hamiltonian cycle. It is also clear that if a node is in V_0 (resp. V_1), then its neighbors are in V_1 (resp. V_0). Therefore, the nodes along this Hamiltonian cycle alternately belong to V_0 and V_1. Hence, $|V_0| = |V_1|$.

If one of k_d's is odd, then the $k_1 \times k_2 \ldots \times k_n$ torus is not bipartite. We show that such a graph is 3-colorable. Our proof is by induction. Let c_1, c_1 and c_2 be three different colors. As the base, it is easy to see that

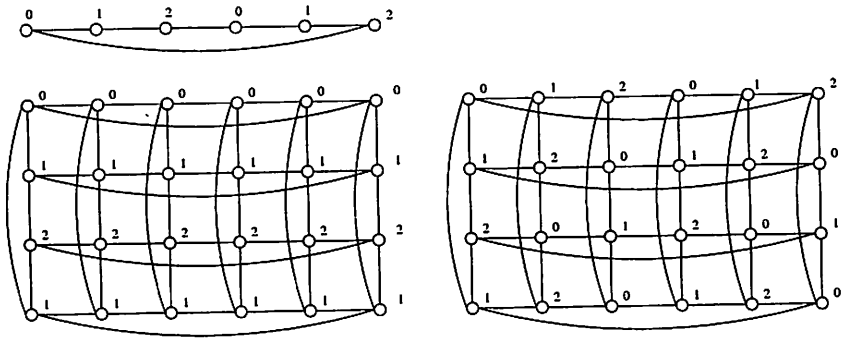
1-dimensional torus (a ring) is 3-colorable. Let a ring of size k be denoted by $(v_0, v_1, \ldots, v_{k-1})$, where v_i is connected to v_{i+1} for $0 \leq i < k-1$, and v_1 is connected to v_{k-1}. Clearly, the ring can be colored using three colors. Suppose that any $k_1 \times k_2 \ldots \times k_{n-1}$ $(n-1)$-dimensional torus is 3-colorable. We construct the $k_1 \times k_2 \ldots \times k_n$ n-dimensional torus by making k_n copies $G_0, G_1, \ldots, G_{n-1}$ of the $(n-1)$-dimensional torus, with all copies having the same 3-coloring (i.e. the corresponding nodes in all copies have the same color), and connecting nodes from different copies by rings of size k_n. Assume that g is a known 3-coloring of a ring $(v_0, v_1, \ldots, v_{n-1})$ of size k_n; i.e. $g(v_i)$ is the color of node v_i in the ring (refer to Figure 3(a)). Then, we use functions f_0, f_1 and f_2 (refer to (1)) to change the node colors in the copies as follows. If $g(v_i) = c_j$, then any node in G_i with color c_a is reassigned a new color $f_j(c_a)$. Clearly, the new colors of nodes in each copy G_i remain satisfying the 3-colorability, and all n-dimensional edges connect nodes of different colors. Therefore, we obtain a 3-coloring of the $(n+1)$-dimensional torus (see Figure 3(b) for an example). The induction is complete.

We show that the $k_1 \times k_2 \ldots \times k_n$ n-dimensional torus has an equal 3-partition if and only if k_d for some d is a multiple of 3. If none of k_d's is a multiple of 3, then there does not exist an equal 3-partition for the torus. Without loss of generality, assume that k_n is a multiple of 3. Consider the inductive 3-coloring procedure described above. We construct the n-dimensional torus by making k_n copies $G_0, G_1, \ldots, G_{n-1}$ of the $(n-1)$-dimensional torus, with all copies having the same 3-coloring and each of the three colors c_0, c_1 and c_2 being used to color at least one node in each copy. Since k_n is a multiple of 3, we can color a ring $(v_0, v_1, \ldots, v_{n-1})$ of size k_n using the three colors alternatively as $(c_0, c_1, c_2, c_0, c_1, c_2, \ldots, c_0, c_1, c_2)$. Then, as before, we perform the following transformation: if $g(v_i) = c_j$, then any node in G_i with color c_a is reassigned a new color $f_j(c_a)$. We partition the k_n copies of the $(n-1)$-dimensional torus into $\frac{k_n}{3}$ groups: $(G_{3m}, G_{3m+1}, G_{3m+2})$, $0 \leq m \leq \frac{k_n}{3} - 1$. Let the number of nodes in each copy G_r that are colored with c_0, c_1 and c_2 be n_0, n_1, and n_2, respectively. We count the number of nodes in each group $(G_{3m}, G_{3m+1}, G_{3m+2})$ that are colored with each color after the transformation as follows: the number of nodes in G_{3m} colored with c_0, c_1 and c_2 are n_0, n_1 and n_2, respectively; the number of nodes in G_{3m+1} colored with c_0, c_1 and c_2 are n_1, n_2, and n_0, respectively; and the number of nodes in G_{3m+2} colored with c_0, c_1 and c_2 are n_2, n_0, and n_1, respectively. Clearly, the total number of nodes colored with each color in group $(G_{3m}, G_{3m+1}, G_{3m+2})$ is $(n_0 + n_1 + n_2)$, i.e. we have an equal 3-partition of $(G_{3m}, G_{3m+1}, G_{3m+2})$. Hence, for all groups, which form the n-dimensional torus, we obtain an equal 3-partition. □

Since the k-ary n-cube is a k-dimensional torus with $k_1 = k_2 = \cdots = k_n = k$, all arguments in the proof of above theorem are applicable to the

the k-ary n-cube. For the k-ary n-cube we have the following claim:

Corollary 1. *The chromatic number of the k-ary n-cube is 2 if k is even, and is 3, if k is odd. The k-ary n-cube has an equal chromatic bipartition if and only if k is even, and it has an equal chromatic 3-partition if and only if k is a multiple of 3.*



(a) (b)

Figure 3. 3-coloring of a 4×6 torus.
(a) The coloring of the six component rings of size 4, and the coloring of a ring of size 6. (b) Final coloring of the torus after transformation

2.4 X-Trees

A useful variation of the tree structure is the X-tree [7]. An X-tree is a supergraph of a complete binary tree with links added to connect consecutive nodes on the same level of the tree from left to right (see Figure 4 for an example). A $(2^m - 1)$-node X-tree has m levels, level 0 though level $m-1$, with the root in level 0. We show that an X-trees have equal chromatic partitions under certain conditions.

Theorem 4. *The chromatic number of m-level X-tree, $m > 1$, is 3. An m-level X-tree has an equal chromatic partition if and only if m is even. If m is odd, it has a chromatic partition (V_0, V_1, V_2) such that* $\max\{|V_0|, |V_1|, |V_2|\} - \min\{|V_0|, |V_1|, |V_2|\} = 1$.

Proof: Since the m-level X-tree, $m > 1$, contains cycles of odd number of nodes, it is not bipartite. We color the tree nodes using colors c_0, c_1 and c_2 in an inductive way. If $m = 2$, there are three nodes in the X-tree, and each is assigned a distinct color. If $m = 3$, we can color the tree nodes as in the left subtree of the X-tree of Figure 4. Suppose we have already colored the nodes for an X-tree T' of $m-1$ levels, where $m > 2$. The X-tree T of m levels is constructed by adding another $(m-1)$-level X-tree T'' to the right of T', and introducing an additional node u as the root of T. Node u is connected to the roots of T' and T'', and the nodes in the same level of T' and T'' are cannected following the definition of X-tree. We consider T'' as an mirror image of T', and the colors assigned to the nodes of T' are mapped to the nodes of T'' by this reflect mapping. Then, the roots of T' and T'' have the same color c_i. We select c_j from $C - \{c_i\}$ such that c_j is assigned to the least number of nodes of T' among the three colors, and use c_j to color u, which is the root of T (see Figure 4 (a)). Then, for each node of T'' colored c_k, we reassign it a new color $f_j(c_k)$ (see Figure 4 (b)). It is easy to verify that no two adjacent nodes in T have the same color (see Figure 4(b)), and therefore, we obtained a 3-partition (V_0, V_1, V_2) of T.

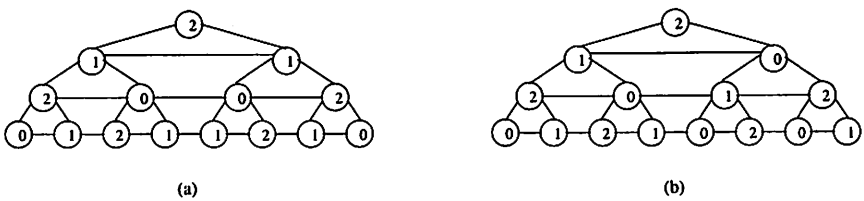


Figure 4. 3-coloring of a 4-level X-tree T.
(a) The coloring of two component 3-level X-trees T' and T'', and the root of T. (b) Final coloring of T after transformation

Now, we use this coloring scheme, as shown in Figure 4, to show that

$$\max\{|V_0|, |V_1|, |V_2|\} - \min\{|V_0|, |V_1|, |V_2|\} = \begin{cases} 0 & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd.} \end{cases} \quad (2)$$

Furthermore, if m is odd, the number of nodes with the color of the root is one more than the number of nodes colored by each of the two remaining colors. For $m = 2$ and $m = 3$, (2) is obviously true (for $m = 3$, refer to the left subtree of three levels shown in Figure 4). Suppose that (2) is true for $m - 1$ levels such that $m > 2$. If m is odd, then the left subtree has an equal 3-partition. After selecting any c_j from $C - \{c_i\}$, where c_i is the color of the root of T', to color the root of T, and applying transformation $f_j(c_k)$ to the colors of all nodes in the right subtree T'' of T, the number of nodes colored c_j is one more than the number of nodes colored by each of the remaining colors in T. If m is even, then by the induction hypothesis, the number of nodes with color c_i in subtree T', where c_i is the color of the root of T', is one more than the number of nodes colored by each of the two remaining colors. After selecting any c_j from $C - \{c_i\}$ to color the root of T, and applying transformation $f_j(c_k)$ to the colors of all nodes in the right subtree T'' of T, the number of nodes colored with the three different colors in T become the same. This completes the induction. The theorem directly follows from that fact that the number of nodes in an m-level X-tree is a multiple of 3 if and only if m is even. □

2.5 Pyramids

The $2^n \times 2^n$ pyramid network [7] consists of $n+1$ levels of 2-dimensional meshes; the mesh at level m, $0 \le m \le n$, is of size $2^m \times 2^m$. The node with indices (i, j) on the $2^m \times 2^m$ mesh is connected to the nodes with indices $(2i-1, 2j-1)$, $(2i-1, 2j)$, $(2i, 2j-1)$, and $(2i, 2j)$ on the $2^{m+1} \times 2^{m+1}$ mesh. The node at level 0 is called the root of the pyramid. The 4×4 pyramid network is shown in Figure 5. The pyramid network is a natural generalization of X-tree; they are closely related. It has been proven useful for many applications. For examples, finite difference methods for solving a system of partial differential equations, and parallel image processing. Since the pyramid contains odd cycles, it is not bipartite.

Theorem 5. *The chromatic number of the $2^n \times 2^n$ pyramid, $n \ge 1$, is* 3.

Proof: Using induction, we now show that the pyramid is 3-partite by coloring its nodes with three colors. A 2×2 pyramid has five nodes, four of them form a cycle, and the fifth node is connected to each of the four nodes in the cycle. We can use two colors to alternatively color the nodes along the 4-cycle, and use the third color to color the root. Suppose that we can use three colors to color the nodes of the $2^{n-1} \times 2^{n-1}$ pyramid P such that no two adjacent nodes in P have the same color. We label the boundary sides of P's mesh in each level with a, b, c and d in the order of traversing the boundary of the mesh. We construct a $2^n \times 2^n$ pyramid P' using four copies P_0, P_1, P_2 and P_3 of P as follows. P_0 is the same as P. P_1 is the mirror image of P_0. Then we add two more copies P_2 and P_3 of P, where P_2 is a mirror image of P_0 and P_3 is a mirror image of P_1. For each level of P_0, P_1, P_2 and P_3, except level 0, edges are added between P_0 and P_1, P_1 and P_3, P_2 and P_3, and P_0 and P_2 to connect corresponding nodes with the same boundary side label (a, b, c or d) to form a larger mesh. The roots of P_0, P_1, P_2 and P_3 are connected as a cycle, and furthermore, a new node u is connected to these four roots. Node u, which is the root of the constructed $2^n \times 2^n$ pyramid P', is assigned a color that distinct from the color of the roots of P_0, P_1, P_2 and P_3. An example of this construction is shown in Figure 5(a). Now, at every level of P', except the root level, there are adjacent nodes with the same color, and all four nodes in level 1 have the same color. Suppose the root of P', say node u, is assigned color c_i. Then, for all nodes in P_1 and P_2 (including their roots), we change their colors using functions of (1): if its original color is c_j, then its new color is $f_i(c_j)$. It is easy to verify that after this color change operation, no two adjacent nodes in the newly constructed $2^n \times 2^n$ pyramid P' have the same color, and consequently we obtain a 3-partition of the $2^n \times 2^n$ pyramid P' from a 3-partition of the $2^{n-1} \times 2^{n-1}$ pyramid P. □

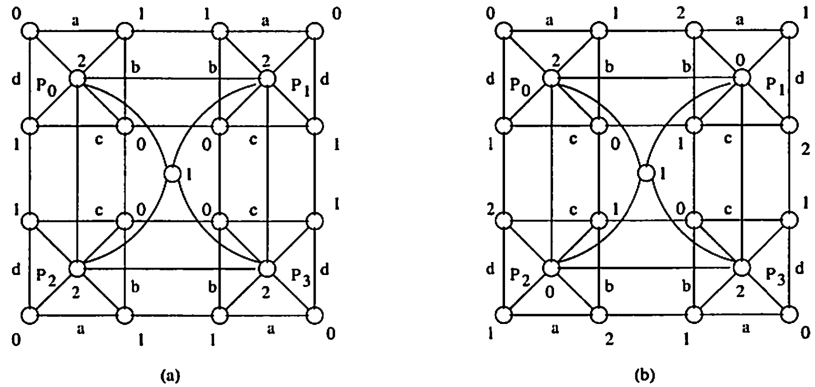


Figure 5. 3-coloring of a 4×4 pyramid.
(a) The coloring of four component 2×2 pyramids P_0, P_1, P_2 and P_3, and the root of P'. (b) Final coloring of P' after transformation

Let N_n denote the total number of nodes in the $2^n \times 2^n$ pyramid. Clearly, $N_n = \sum_{i=0}^{n-1} 4^i$, and N_n is a multiple of 3 if and only if $n+1$ is a multiple of 3. However, any pyramid does not have an equal chromatic partition.

Theorem 6. *Any pyramid does not have an equal chromatic partition.*

Proof: All 3-partitions (V_0, V_1, V_2) of the $2^n \times 2^n$ pyramid are equivalent in the sense that the relative sizes of the three node subsets are the same for all its 3-partitions. For brevity, we informally prove this fact. Consider a pyramid whose root is colored with color c_1. Then, in the next level, the four nodes must be colored with c_0 and c_2 in an alternative way along the cycle of four nodes. There are two such ways, and they are equivalent. Then, we go to the next level, we have a similar situation: the node colors of the previous level determines the colors of the nodes in the current level in a unique (under isomorphism) way. Such an argument can be applied from one level to the next level, level by level. Therefore, the relative sizes of the three node subsets are the same for all its 3-partitions.

The $2^n \times 2^n$ pyramid has an odd number of nodes. Suppose that the root of the pyramid is colored with c_1. Then, following the above arguments, the total numbers of nodes colored with c_1, c_1 and c_2 in the levels other than the root level are all even. Consequently, the parities of the numbers of nodes colored with three colors are not the same. Therefore, any pyramid network does not admit an equal 3-partition. □

Let $r_{\max/\min}$ denote the ratio $\max\{|V_0|, |V_1|, |V_2|\}/\min\{|V_0|, |V_1|, |V_2|\}$ of a 3-partition (V_0, V_1, V_2) of the pyramid network. As n becomes larger, the ratio $r_{\max/\min}$ of a 3-partition $(|V_0|, |V_1|, |V_2|)$ of the $2^n \times 2^n$ pyramid approaches 1 rapidly. For examples, for $n = 1$, $r_{\max/\min} = 2$, which is the largest possible; for $n = 5$, $r_{\max/\min} = 154/147$. Thus, for large n, the $2^n \times 2^n$ pyramid has a near-equal chromatic partition.

3 Concluding Remarks

We have presented necessary and sufficient conditions for the existence of equal chromatic partitions of several classes of interconnection networks. We also showed that the chromatic partition for the pyramid network is "unique", and it does not have an equal chromatic partition. We believe that our techniques can be used to find equal or near-equal chromatic partitions of other classes of networks, such as c-dimensional butterfly and wrapped butterfly networks.

For some applications, such as the algorithms for DTE, we may use a variation of equal chromatic partition without affecting the solutions. We may say that a network G has an equal chromatic partition $(V_0, V_1, \ldots, V_{k-1})$, where k is the chromatic number of G if $\max\{|V_0|, |V_1|, \ldots, |V_{k-1}|\} - \min\{|V_0|, |V_1|, \ldots, |V_{k-1}|\} \leq c$, where c is a constant. It is easy to see that for any n-node X-Net, there exists a 4-partition such that each of the four subsets has $\lceil \frac{n}{4} \rceil$ or $\lfloor \frac{n}{4} \rfloor$ nodes. As shown in Theorem 4, there exists a 3-partition for any n-node X-tree such that each of the three subsets has $\lceil \frac{n}{3} \rceil$ or $\lfloor \frac{n}{3} \rfloor$ nodes. Thus, under this new definition, all X-Nets and X-trees have equal chromatic partitions. We conjecture that, under this new definition, all $k_1 \times k_2 \ldots \times k_n$ n-dimensional tori and k-ary n-cubes have equal chromatic partitions, i.e. one can always find a chromatic partition (V_0, V_1, V_2) such that $\max\{|V_0|, |V_1|, |V_2|\} - \min\{|V_0|, |V_1|, |V_2|\} \leq c$, where c is a constant. However, if such a partition exists, its structure is not simple. For pyramid networks the value $\max\{|V_0|, |V_1|, |V_2|\} - \min\{|V_0|, |V_1|, |V_2|\}$ is not bounded as the size of the pyramid network increases.

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References

[1] W. Aiello and F.T. Leighton, Coding theory, hypercube embeddings, and fault tolerance, *Proc. 3rd ACM Symp. on Parallel Algorithms and Architectures* (1991).

[2] A. El-Amawy and S. Latifi, Properties and performance of folded hypercubes, *IEEE Trans. Parallel and Distributed Systems* **2**, **1** (1991), 31–42.

[3] S. Bhatt and J.-Y. Cai, Taking random walks to grow trees in hypercubes, *Journal of the ACM* **40**, No. 3 (1993), 741–764.

[4] S. Bhatt, D. Greenberg, T. Leighton, P. Liu, Tight bounds for on-line tree embeddings, *Proc. 2nd ACM-SIAM Symp. on Discrete Algorithms* (1991), 344–350.

[5] N. Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall (1974).

[6] R.M. Karp and Y. Zhang, Randomized parallel algorithms for backtrack search and branchand-bound computation, *Journal of the ACM* **40**, **3** (1993), 765–789.

[7] F.T. Leighton, *Introduction to Parallel Algorithms and Architectures: Arrays, Trees, Hypercube,* Morgan Kaufmann Publishers, Inc. (1992).

[8] F.T. Leighton, M.J. Newman, A.G. Ranade, and E.J. Schwabe, Dynamic tree embeddings in butterflies and hypercubes, *SIAM Journal on Computing* **21**, 4 (1992), 639–654.

[9] K. Li, Maintenance of tree structured computations on parallel and distributed computer systems, *Proceedings of the 11th ACM Symposium on Applied Computing* (1996), 337–343.

[10] K. Li, Y. Pan, H. Shen, G.H. Young and S.Q. Zheng, Lower bounds for dynamic tree embedding in bipartite networks, *IEEE Transactions on Parallel and Distributed Systems* **53**, **2** (1998), 119–143.

[11] H. Shen, K. Li, Y. Pan, G.H. Young and S.Q. Zheng, Performance ratios for dynamic tree embedding in k-partite networks by random walk, *Journal of Parallel and Distributed Computing* **50** (1998), 144–156.

[12] A.G. Ranade, Optimal speedup for backtrack search on a butterfly network, *Proc. of the 3rd ACM Symp. on Parallel Algorithms and Architectures* (1991), 40–48.