The Metamorphosis of λ-fold Block Designs with Block Size Four into λ-fold Kite Systems

Selda Küçükçifçi and C. C. Lindner
Department of Discrete and Statistical Sciences
120 Math Annex
Auburn University
Auburn, Alabama 36849-5307
USA
email:kucuks1@mail.auburn.edu
email:lindncc@mail.auburn.edu

Abstract

A *kite* is a triangle with a tail consisting of a single edge. A *kite system* of order n is a pair (X, K), where K is a collection of edge disjoint kites which partitions the edge set of K_n (= the complete undirected graph on n vertices) with vertex set X. Let (X, B) be a block design with block size 4. If we remove a path of length 2 from each block in B, we obtain a partial kite-system. If the deleted edges can be assembled into kites the result is a kite system, called a *metamorphosis* of the block design (X, B). There is an obvious extension of this definition to λ-fold block designs with block size 4. In this paper we give a complete solution of the following problem: Determine all pairs (λ, n) such that there exists a λ-fold block design of order n with block size 4 having a metamorphosis into a λ-fold kite system.

JCMCC **40** (2002), pp. 241-252

1 Introduction

A λ-fold block design of order n with block size k is a pair (X, B), where B is a collection of edge disjoint copies of K_k (the complete undirected graph on k vertices) which partitions the edge set of λK_n (λ copies of K_n) with vertex set X.

If we remove a triangle from K_4 regardless of the triangle removed, the result is a star. Now let (X, B) be a λ-fold block design with block size 4 and define a collection of triangles B_1 and stars B_2 as follows: for each $b \in B$ partition b into a triangle and a star and place the triangle in B_1 and the star in B_2. Then (X, B_1) is a partial λ-fold triple system and a natural question to ask is whether or not the edges belonging to the stars in B_2 can be reassembled into a collection of triangles B_2^*. If this is possible then $(X, B_1 \cup B_2^*)$ is a λ-fold triple system and is called a *metamorphosis* of (X, B).

In everything that follows "λ-fold block design" will *always* mean "λ-fold block design with block size 4". If $\lambda = 1$ we will simply say "block design".

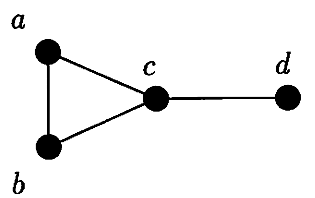
In [2] C. C. Lindner and A. Rosa gave a complete solution of the problem of constructing λ-fold block designs having metamorphoses into λ-fold triple systems.

In [4] C. C. Lindner and A. P. Street gave a complete solution of the metamorphosis problem of λ-fold block designs into λ-fold 4-cycle systems.

Recently, in [3] C. C. Lindner and A. Rosa looked at the metamorphosis problem when a single edge is removed from K_4 and gave a complete solution for the problem when $\lambda = 1$.

Now, if we remove a path of length two from K_4, the result is a *kite* (a triangle with a tail). Let (X, B) be a λ-fold block design and define a collection of paths of length two P, and kites K, as follows: for each $b \in B$ partition b into a path of length two and a kite and place the kites in K and the paths in P. Then (X, K) is a partial λ-fold kite system and we can ask whether or not the edges belonging to the paths in P can be reassembled into a collection of kites P^*. If this is possible then $(X, K \cup P^*)$ is a λ-fold kite system.

We will denote the kite



by (a, b, c)-d or (b, a, c)-d.

Example 1.1 *There is a metamorphosis of a block design of order* 16 *into a kite system.*

Proof Let (X, B) be the block design given by $B = \{\{0,1,3,2\}, \{4,0,6,5\}, \{0,7,8,9\}, \{11,12,13,0\}, \{15,0,10,14\}, \{4,1,7,11\}, \{1,12,14,5\}, \{1,6,8,15\}, \{9,13,10,1\}, \{2,13,15,4\}, \{2,7,16,5\}, \{2,6,9,12\}, \{8,2,14,11\}, \{3,9,14,4\}, \{3,13,8,5\}, \{3,6,11,10\}, \{7,12,15,3\}, \{8,12,10,4\}, \{9,11,15,5\}, \{7,13,14,6\}\}$. Delete the paths $[a,b,c]$ (which is the path with edges $\{a,b\}$ and $\{b,c\}$) from each of the blocks $\{a,b,c,d\}$ belonging to B to obtain the kite (a,c,d)-b. (Call the resulting collection of kites K.) Now reassemble the set P of deleted paths into the collection of kites $P^* = \{(4,1,0)\text{-}6, (10,0,7)\text{-}1, (13,14,2)\text{-}7, (2,6,8)\text{-}13, (6,3,1)\text{-}12, (3,9,13)\text{-}15, (12,8,7)\text{-}13, (11,12,15)\text{-}0, (13,10,12)\text{-}14, (11,6,9)\text{-}14\}$. Then $(X, K \cup P^*)$ is a kite system and is a metamorphosis of (X, B). □

The object of this paper is the complete solution of the following problem. For which n and λ does there exist a λ-fold block design of order n having a metamorphosis into a λ-fold kite system of order n? It suffices to give a complete solution of this problem for $\lambda = 1,2,3,4,6$, and 12, since these solutions can be pasted together to obtain a complete solution for all other values of λ. Therefore we will organize our results into the sections $\lambda = 1,2,3,4,6$, and 12; followed by a summary.

2 Block designs with $\lambda = 1$

In this section block design means $\lambda = 1$. It is well known that the spectrum for block designs is precisely the set of all $n \equiv 1$ or 4 (mod 12) and the spectrum for kite systems ($\lambda = 1$) is $n \equiv 0$ or 1 (mod 8). Hence a necessary condition for a block design to have a metamorphosis into a kite system is $n \equiv 1$ or 16 (mod 24). We will show that this condition is sufficient.

Example 1.1 gives a solution for $n = 16$. The following example takes care of the cases $n = 25, 40, 49$ and 73.

Example 2.1 *There exist metamorphoses of block designs of order n into kite systems for* $n = 25, 40, 49$ *and* 73.

Proof **n=25.** Let $(Z_5 \times Z_5, B)$ be the block design with base blocks $\{(0,0),(0,1),(1,0),(2,2)\}$ and $\{(0,0),(0,2),(2,0),(4,4)\}$ (mod(5,5)). Delete the path $[(0,1),(0,0),(1,0)]$ from the first of these base blocks and $[(0,0),(4,4),(2,0)]$ from the second and reassemble the deleted edges into the kites $((0,0),(1,0),(1,1))$-$(4,2)$ (mod(5,5)).

n=40. Let $((Z_{13} \times \{0,1,2\}) \cup \{\infty\}, B)$ be the block design with base blocks $\{(0,0),(2,0),(0,1),(10,1)\}$, $\{(0,1),(2,1),(0,2),(10,2)\}$, $\{(0,2),(2,2),(1,0),(11,0)\}$, $\{(0,0),(8,0),(1,1),(2,1)\}$, $\{(0,1),(8,1),(1,2),(2,2)\}$, $\{(0,2),$

$(8,2),(2,0),(3,0)\}, \{(0,0),(6,0),(5,1),(9,1)\}, \{(0,1),(6,1),(5,2),(9,2)\}$, $\{(0,2),(6,2),(6,0),(10,0)\}, \{(0,0),(4,1),(8,2),\infty\}$ (mod 13).

Delete the paths $[(0,0),(0,1),(2,0)]$ from the first, $[(2,1),(0,1),(10,2)]$ from the second, $[(0,2),(2,2),(1,0)]$ from the third, $[(0,0),(2,1),(1,1)]$ from the fourth, $[(0,1),(1,2),(2,2)]$ from the fifth, $[(0,2),(3,0),(8,2)]$ from the sixth, $[(0,0),(9,1),(6,0)]$ from the seventh, $[(0,1),(5,2),(9,2)]$ from the eighth, $[(0,2),(6,0),(6,2)]$ from the ninth, and $[(0,0),(4,1),(8,2)]$ from the tenth base block.

Reassemble the deleted edges into the kites $((0,0),(4,1),(2,1))$-$(6,0)$, $((0,2),(1,2),(0,0))$-$(5,2)$, $((0,0),(0,1),(10,2))$-$(8,2)$, $((1,2),(5,2),(0,1))$-$(2,0)$ and $((6,0),(0,2),(9,1))$-$(8,1)$ (mod 13).

n=49. Let (Z_{49}, B) be the cyclic block design with base blocks $\{0,1,3,8\}$, $\{0,4,18,29\}$, $\{0,6,21,33\}$ and $\{0,9,19,32\}$ (mod 49). Delete the path $[3,0,1]$ from the first, $[4,18,29]$ from the second, $[0,6,33]$ from the third and $[19,9,32]$ from the fourth base block.

Reassemble the deleted edges into the kites $(0,1,23)$-29 and $(3,0,14)$-24 (mod 49).

n=73. Let (Z_{73}, B) be the cyclic block design with base blocks $\{0,70,58,19\}$, $\{0,9,16,45\}$, $\{0,11,25,46\}$, $\{0,8,40,71\}$, $\{0,6,26,49\}$ and $\{0,55,68,72\}$ (mod 73).

Delete the path $[0,70,19]$ from the first, $[9,0,45]$ from the second, $[0,46,11]$ from the third, $[8,0,71]$ from the fourth, $[49,0,6]$ from the fifth, and $[0,72,68]$ from the sixth base block.

Reassemble the deleted edges into the kites $(0,6,7)$-5, $(0,51,2)$-37 and $(0,46,1)$-5 (mod 73). □

We will now handle the case $n \equiv 1 \pmod{24}$, $n \geq 97$, with the following construction.

The main ingredient we will need in our construction is a skew Room square with holes of size 4. In [1] it is proven that there exists a skew Room square of order $4k$ with holes of size 4 for every order $4k \geq 16$. This is the result that we need for the following construction.

The 24k+1 Construction Let $4k \geq 16$, $X = \{1,2,...,4k\}$ and let R be a skew Room square of order $4k$ with holes $H = \{h_1, h_2, ..., h_k\}$ of size 4. Let $S = \{\infty\} \cup (X \times Z_6)$ and define a collection of blocks B as follows:

(1) For each hole $h_i \in H$, define a copy of the block design of order 25 in Example 2.1 on $\{\infty\} \cup (h_i \times Z_6)$ and place these blocks in B.

(2) If x and y belong to different holes of H place the six blocks $\{(x,i),(y,i),(r,1+i),(c,4+i)\}$ in B, where $i \in Z_6$, the second coordinates are reduced modulo 6, and $\{x,y\}$ belongs to cell (r,c) of R.

It is straightforward to see that (S,B) is a block design. □

Lemma 2.2 *There exists a block design of order n having a metamorphosis into a kite system if* $n \equiv 1 \pmod{24}$.

Proof Example 2.1 takes care of the cases $n = 25, 49$, and 73, so we can assume $n \geq 97$. Hence $n = 6(4k) + 1 \geq 97$, $4k \geq 16$, and we can use the $24k + 1$ Construction.

(1) For each hole $h_i \in H$, delete the edges from the type (1) blocks as in Example 2.1, the case $n = 25$.

(2) Delete the paths $[(y,0),(x,0),(r,1)]$; $[(y,1),(x,1),(r,2)]$ and $[(y,5),(x,5),(r,0)]$ from all blocks of the form $\{(x,0),(y,0),(r,1),(c,4)\}$, $\{(x,1),(y,1),(r,2),(c,5)\}$ and $\{(x,5),(y,5),(r,0),(c,3)\}$. Delete the paths $[(x,2),(c,0),(y,2)]$; $[(x,3),(c,1),(y,3)]$ and $[(x,4),(r,5),(y,4)]$ from all blocks of the form $\{(x,2),(y,2),(c,0),(r,3)\}$, $\{(x,3),(y,3),(c,1),(r,4)\}$ and $\{(x,4),(y,4),(r,5),(c,2)\}$. The edges belonging to the deleted edges cover all edges not belonging to the same hole (i) between levels 0 and 2, (ii) between levels 1 and 3 and (iii) between levels 4 and 5, all pairs not belonging to the same hole (iv) on the level 0, (v) on the level 1 and (vi) on the level 5, half of the edges (vii) between levels 0 and 1, (viii) between levels 1 and 2 and (ix) between levels 0 and 5.

Reassemble the deleted edges in (1) as in Example 2.1, the case $n = 25$.

Reassemble the edges in (i), (iv), and (vii) into the kites $((c,2),(y,0),(x,0))$-$(r,1)$, the edges in (ii), (v), and (viii) into the kites $((c,3),(y,1),(x,1))$-$(r,2))$ and the edges in (iii), (vi), and (ix) into the kites $((c,4),(y,5),(x,5))$-$(r,0))$. The resulting collection of kites is a metamorphosis of the block design (S,B) into a kite system. □

We will now handle the case $n \equiv 16 \pmod{24}$. In order to do this we will need the following example.

Lemma 2.3 *Let* (X,B) *be the block design of order* 16 *in Example 1.1. Then: (1) it is possible to choose a path of length* 2 *from each* $b \in B \setminus \{0,1,2,3\}$ *so that the edges belonging to these paths can be reassembled into* 8 *kites and the triangles* $(0,x_1,x_2)$, $(2,y_1,y_2)$*; where* x_1, x_2, y_1*, and* y_2 *are distinct. (2) It is also possible to choose a path of length* 2 *from each* $b \in B \setminus \{0,1,2,3\}$ *so that the edges belonging to these paths can be reassembled into* 9 *kites and* 2 *disjoint edges* $\{0,x\}$ *and* $\{2,y\}$.

Proof (1) Let (X,B) be the block design of order 16 in Example 1.1. Let P_1 consist of the paths [4, 5, 6]; [0, 7, 8]; [11, 12, 13]; [0, 10, 15]; [4, 7, 11]; [12, 5, 14]; [6, 8, 15]; [9, 13, 10]; [2, 13, 15]; [5, 7, 10]; [2, 6, 9]; [8, 2, 14]; [3, 9, 14]; [5, 3, 13]; [3, 6, 11]; [7, 12, 15]; [8, 12, 10]; [9, 11, 15] and [6, 14, 13].

Now reassemble these deleted edges into 8 kites and 2 triangles as follows:

(12, 8, 7)-11, (6, 2, 8)-15, (3, 6, 5)-12, (3, 13, 9)-14, (11, 12, 15)-13, (13, 12, 10)-15, (9, 11, 6)-14, (4, 7, 5)-14, (0, 7, 10), (2, 13, 14).

(2) Let (X, B) be the block design of order 16 in Example 1.1. Let P_2 consist of the paths $[4,5,6]; [8,7,9]; [11,12,13]; [10,15,0]; [7,4,11]; [12,5,14]; [15,6,8]; [9,13,10]; [13,2,4]; [2,5,7]; [2,6,9]; [8,2,14]; [3,9,14]; [5,3,13]; [3,6,11]; [7,12,15]; [8,12,10]; [9,11,15]; [7,13,14]$.

Now, reassemble these deleted edges into 9 kites and 2 disjoint edges as follows:

$(13,14,2)$-5, $(12,8,7)$-13, $(2,8,6)$-15, $(6,3,5)$-14, $(3,13,9)$-14, $(11,12,15)$-10, $(13,10,12)$-5, $(9,6,11)$-4, $(4,5,7)$-9, and the disjoint edges $\{0,15\}$ and $\{2,4\}$. □

Now, with Example 2.1 and Lemma 2.3 we can give a construction which handles all of the remaining cases.

The main ingredient we will need in this construction is a skew Room square with holes of size 2. Such a square exists for every order $2k \geq 10$ (see [1]) .

The 24k+16 Construction Let $4k+2 \geq 10$, $X = \{1,2,\ldots,4k+2\}$, and let R be a skew Room square of order $4k+2$ with holes $H = \{h_1, h_2, \ldots, h_{2k+1}\}$ of size 2. Let $S = \{a,b,c,d\} \cup (X \times Z_6)$ and define a collection of blocks B as follows:

(1) For the hole $h_1 \in H$, define a copy of the block design of order 16 in Example 1.1 on $\{a,b,c,d\} \cup (h_1 \times Z_6)$ and place these blocks in B.

(2) For each hole $h_i \in H \backslash \{h_1\}$, define a copy of the block design of order 16 in Lemma 2.3 on $\{a,b,c,d\} \cup (h_1 \times Z_6)$ containing $\{a,b,c,d\}$ as a block and place these blocks in B making sure to delete the block $\{a,b,c,d\}$.

(3) If x and y belong to different holes of H, place the six blocks $\{(x,i),(y,i),(r,1+i),(c,4+i)\}$ in B, where $i \in Z_6$, the second coordinates are reduced modulo 6, and $\{x,y\}$ belongs to cell (r,c) of R.

It is straightforward to see that (S, B) is a block design. □

Lemma 2.4 *There exists a block design of order n having a metamorphosis into a kite system if $n \equiv 16 \pmod{24}$.*

Proof Examples 1.1 and 2.1 take care of the cases $n = 16$ and 40, so we can assume $n \geq 64$. Hence $n = 6(4k+2)+4 \geq 64$, $4k+2 \geq 10$, and we can use the $24k+16$ Construction.

(1) For the hole h_1, delete the edges from the type (1) blocks as in Example 1.1, the case $n = 16$.

(2) For each hole h_{2i}, $1 \leq i \leq k$, delete the edges from the type (2) blocks as in Lemma 2.3 (1).

(3) For each hole h_{2i+1}, $1 \leq i \leq k$, delete the edges from the type (2) blocks as in Lemma 2.3 (2).

(4) For type (3) blocks, delete the paths $[(y,0),(x,0),(r,1)]; [(y,1),(x,1),(r,2)]$ and $[(y,5),(x,5),(r,0)]$ from all blocks of the form $\{(x,0),(y,0),(r,1),(c,4)\}$, $\{(x,1),(y,1),(r,2),(c,5)\}$ and $\{(x,5),(y,5),(r,0),(c,3)\}$.

Delete the paths $[(x,2),(c,0),(y,2)]$; $[(x,3),(c,1),(y,3)]$ and $[(x,4),(r,5),(y,4)]$ from all blocks of the form $\{(x,2),(y,2),(c,0),(r,3)\}$, $\{(x,3),(y,3),(c,1),(r,4)\}$ and $\{(x,4),(y,4),(r,5),(c,2)\}$, as in the proof of Lemma 2.2 (2).

Reassemble the deleted edges in (1), (2), and (3) as in Example 1.1, Lemma 2.3 (1), and 2.3 (2), respectively.

Reassemble the deleted edges in (4) into the kites $((c,2),(y,0),(x,0))$-$(r,1)$, $((c,3),(y,1),(x,1))$-$(r,2)$, $((c,4),(y,5),(x,5))$-$(r,0)$ as in the proof of Lemma 2.2. The resulting collection of kites is a metamorphosis of the block design (S,B) into a kite system. □

Lemma 2.5 *There exists a block design of order n having a metamorphosis into a kite system if and only if $n \equiv 1$ or 16 (mod 24).*

Proof Lemmas 2.2 and 2.4. □

3 2-fold block designs

It is well known that the spectrum for 2-fold block designs is the set of all $n \equiv 1, 4, 7$ or 10 (mod 12) and that the spectrum for 2-fold kite systems is the set of all $n \equiv 0, 1, 4, 5, 8$ or 9 (mod 12). Hence $n \equiv 1$ or 4 (mod 12) is necessary for the existence of a 2-fold block design having a metamorphosis into a 2-fold kite system. The following lemma shows that this necessary condition is sufficient as well.

Lemma 3.1 *There exists a 2-fold block design of order n having a metamorphosis into a 2-fold kite system if and only if $n \equiv 1$ or 4 (mod 12).*

Proof Let (X,B) be a 2-fold block design with blocks $\{a,b,c,d\}$ and $\{a,b,c,d\}$. Delete the path $[a,b,c]$ from the first block and $[a,c,d]$ from the second block, and reassemble the deleted edges into the kite (a,b,c)-d. It follows that if we double each block of a block design ($\lambda = 1$), the result is a 2-fold block design having a metamorphosis into a 2-fold kite system. Since the spectrum for 2-fold block designs with block size 4 is $n \equiv 1$ or 4 (mod 12), we are done. □

4 3-fold block designs

Since the spectrum for 3-fold block designs is the set of all $n \equiv 0$ or 1 (mod 4) and the spectrum for 3-fold kite systems is the set of all $n \equiv 0$ or 1 (mod 8), a necessary condition for the existence of a 3-fold block design to have a metamorphosis into a 3-fold kite system is $n \equiv 0$ or 1 (mod 8). We will show that this condition is sufficient as well.

We will begin with examples for $n = 8, 9, 17, 24, 32$, and 33 followed by a construction for the remaining cases.

Example 4.1 *There exist metamorphoses of a 3-fold block design of order n into a 3-fold kite system if $n = 8, 9, 17, 24, 32$, and 33.*

Proof **n=8.** Let $(\{\infty\} \cup Z_7, B)$ be the cyclic 3-fold block design with base blocks $\{0, 1, 3, \infty\}$ and $\{0, 1, 2, 4\}$ (mod 7). Delete the path $[1, 3, \infty]$ from the first base block and $[1, 0, 4]$ from the second base block. Reassemble the deleted edges into a partial cyclic 3-fold kite system with the base block $(1, 3, 0)$-∞ (mod 7).

n=9. Let (Z_9, B) be the cyclic 3-fold block design with base blocks $\{0, 1, 2, 4\}$ and $\{0, 1, 4, 6\}$ (mod 9). Delete the path $[0, 1, 2]$ from the first base block and $[4, 6, 0]$ from the second base block. Reassemble the deleted edges into a partial cyclic 3-fold kite system with the base block $(1, 2, 4)$-5 (mod 9).

n=17. Let (Z_{17}, B) be the cyclic 3-fold block design with base blocks $\{0, 1, 3, 7\}, \{0, 1, 5, 7\}, \{0, 1, 6, 9\}$ and $\{0, 2, 5, 9\}$ (mod 17). Delete the path $[0, 1, 3]$ from the first base block, $[0, 5, 7]$ from the second base block, $[1, 6, 9]$ from the third base block and $[0, 2, 9]$ from the fourth base block. Reassemble the deleted edges into a partial cyclic 3-fold kite system with the base blocks $(3, 0, 1)$-6 and $(5, 7, 0)$-2 (mod 17).

n=24. Let $(\{\infty\} \cup Z_{23}, B)$ be the cyclic 3-fold block design with base blocks $\{\infty, 0, 7, 10\}, \{1, 8, 12, 22\}, \{2, 5, 6, 11\}, \{3, 9, 14, 18\}, \{4, 16, 17, 19\}$, and $\{13, 15, 20, 21\}$ (mod 23). Delete the path $[0, 7, 10]$ from the first base block, $[8, 12, 22]$ from the second base block, $[5, 2, 11]$ from the third base block, $[3, 9, 14]$ from the fourth base block, $[16, 17, 19]$ from the fifth base block, and $[15, 20, 21]$ from the sixth base block. Reassemble the deleted edges into a partial cyclic 3-fold kite system with the base blocks $(0, 7, 10)$-5, $(0, 4, 5)$-6 and $(0, 6, 9)$-11 (mod 23).

n=32. Let $(\{\infty\} \cup Z_{31}, B)$ be the cyclic 3-fold block design with base blocks $\{\infty, 17, 22, 23\}, \{0, 4, 7, 20\}, \{2, 14, 25, 28\}, \{1, 8, 10, 16\}, \{5, 9, 18, 19\}$, $\{6, 11, 13, 21\}, \{30, 24, 12, 3\}$ and $\{15, 26, 27, 29\}$ (mod 31). Delete the path $[22, 23, 17]$ from the first base block, $[0, 4, 7]$ from the second base block, $[14, 25, 28]$ from the third base block, $[1, 8, 10]$ from the fourth base block, $[18, 9, 19]$ from the fifth base block, $[6, 11, 13]$ from the sixth base block, $[30, 24, 12]$ from the seventh base block and $[15, 26, 27]$ from the eighth base block. Reassemble the deleted edges into a partial cyclic 3-fold kite system with the base blocks $(0, 3, 10)$-16, $(0, 6, 9)$-20, $(0, 4, 5)$-7 and $(0, 11, 12)$-14 (mod 31).

n=33. Let (Z_{33}, B) be the cyclic 3-fold block design with base blocks $\{0, 6, 8, 28\}, \{0, 12, 16, 23\}, \{0, 5, 15, 19\}, \{0, 1, 2, 9\}, \{0, 6, 7, 15\}, \{0, 3, 5, 16\}$, $\{0, 3, 13, 17\}$ and $\{0, 3, 9, 21\}$ (mod 33). Delete the path $[6, 0, 8]$ from the first base block, $[12, 0, 16]$ from the second base block, $[5, 0, 15]$ from the third base block, $[1,0,9]$ from the fourth base block, $[0,7,6]$ from the fifth base block, $[5,0,3]$ from the sixth base block, $[3,0,13]$ from the seventh base block and $[3,0,9]$ from the eighth base block. Reassemble the deleted edges into a partial cyclic 3-fold kite system with the base blocks $(0,1,8)$-14, $(0,3,16)$-21, $(0,3,12)$-17 and $(0,1,10)$-13 (mod 33). □

Now, with Example 4.1 in hand we can give a construction which handles all of the remaining cases.

The 8k+r Construction Write $8k+r = 4(2k)+r$, where $2k \geq 10$, $r \in \{0,1\}$. Let $X = \{1,2,...,2k\}$, Z be a set of size r and R be a skew Room square of order $2k$ with holes $H = \{h_1, h_2, ..., h_k\}$ of size 2. Set $S = Z \cup (X \times Z_4)$ and define a collection of blocks B as follows:

(1) For each hole $h_i \in H$ define a copy of the 3-fold block design of order $8+r$ in Example 4.1 on $Z \cup (h_i \times Z_4)$ and place these blocks in B.

(2) If x and y belong to different holes of H place the 4 blocks $\{(x,i),(y,i),(r,i+1),(c,i+1)\}$ in B, where $i \in Z_4$, the second coordinates are reduced modulo 4, and $\{x,y\}$ belongs to cell (r,c) of R. and $\{x,y\}$ belongs to cell (r,c) of R.

(3) If x and y belong to different holes of H place the 2 blocks $\{(x,i),(y,i),(r,i+2),(c,i+2)\}$ in B, where $i \in \{0,1\}$ and $\{x,y\}$ belongs to cell (r,c) of R.

(4) Let $(X \times Z_4, T)$ be a transversal design of order $2k$ and strength 4 with holes of size 2 (which is equivalent to a pair of orthogonal latin squares of order $2k$ with holes of size 2). Since $2k \geq 10$, such a design exists [5]. Place the blocks of T in B.

Then (S,B) is a 3-fold block design. □

Lemma 4.2 *There exists a 3-fold block design of order n having a metamorphosis into a 3-fold kite system if and only if $n \equiv 0$ or 1 (mod 8).*

Proof The cases $n = 8, 9, 17, 24, 32$, and 33 are taken care of by Example 4.1. For the cases $n = 16$ and 25 taking three copies of block designs ($\lambda = 1$) of order 16 and 25 gives the solution. So we can assume that $n \geq 40$. Hence we can use the $8k+r$ Construction.

(1) For each hole $h_i \in H$, delete the paths from each block of a 3-fold block design of order 8 or 9 as in Example 4.1.

(2) Delete the path $[(x,i),(r,i+1),(y,i)]$ from each block in (2) of the $8k+r$ Construction.

(3) Delete the path $[(x,i),(y,i),(c,i+2)]$ from each block in (3) of the $8k+r$ Construction.

(4) Delete the path $[(x,0),(y,1),(z,3)]$ from each block in (4) of the $8k+r$ Construction, where the points $(x,0)$, $(y,1)$, and $(z,3)$ belong to the same block of T.

Now, reassemble the deleted edges in (1) as in Example 4.1, the case $n = 8$ if $r = 0$ and the case $n = 9$ if $r = 1$.

The edges belonging to the deleted paths in (2), (3) and (4) are all edges not belonging to the same hole (i) between levels 0 and 1, (ii) between levels 1 and 2, (iii) between levels 2 and 3, (iv) between levels 0 and 3 and (v) between levels 1 and 3, all pairs not belonging to the same hole (vi) on the level 0 and (vii) on the level 1, half of the edges (viii) between levels 0 and 2 and (ix) between levels 1 and 3 and finally (x) all edges not belonging to the same hole between levels 0 and 1. Reassemble the deleted edges in (i), (vi) and (viii) into the kites $((x,0),(r,1),(y,0))$-$(r,2)$, the edges in (ii), (vii) and (ix) into the kites $((x,1),(r,2),(y,1))$-$(r,3)$, the edges in (iii), (iv), (v) and (x) into the kites $((x,0),(y,1),(z,3))$-$(x,2)$. The resulting collection of kites is a metamorphism of the 3-fold block design (S, B) into a 3-fold kite system. □

5 4-fold block designs

The spectrum for 4-fold block designs is the set of all $n \equiv 1 \pmod 3$, while the spectrum for 4-fold kite systems consists of all $n \geq 4$. Hence $n \equiv 1 \pmod 3$ is a necessary condition for the existence of a 4-fold block design having a metamorphosis into a 4-fold kite system. The following lemma shows that this necessary condition is also sufficient.

Lemma 5.1 *There exists a 4-fold block design of order n having a metamorphosis into a 4-fold kite system if and only if $n \equiv 1 \pmod 3$.*

Proof As was pointed out in Section 3, the spectrum for 2-fold block designs is the set of all $n \equiv 1 \pmod 3$. So let (X, B) be any such design and let $(X, 2B)$ be the 4-fold block design obtained from B by doubling each block. The remainder of the proof is exactly the same as the proof of Lemma 3.1. □

6 6-fold block designs

The spectrum for 6-fold block designs is the set of all $n \geq 4$ and the spectrum for 6-fold kite systems is the set of all $n \equiv 0$ or $1 \pmod 4$. Hence a necessary condition for the existence of a 6-fold block design having a metamorphosis into a 6-fold kite system is $n \equiv 0$ or $1 \pmod 4$. The following lemma shows that this necessary condition is also sufficient.

Lemma 6.1 *There exists a 6-fold block design of order n having a metamorphosis into a 6-fold kite system if and only if $n \equiv 0$ or $1 \pmod 4$.*

Proof As was pointed out in Section 4, the spectrum for 3-fold block designs is the set of all $n \equiv 0$ or $1 \pmod 4$. So let (X, B) be any such design and let $(X, 2B)$ be the 6-fold block design obtained from B by doubling each block. The remainder of the proof is exactly the same as the proof of Lemma 3.1. □

7 12-fold block designs

The necessary condition for a 12-fold block design of order n to have a metamorphosis into a 12-fold kite system is $n \geq 4$, no restrictions. The following lemma shows that this necessary condition is also sufficient.

Lemma 7.1 *There exists a 12-fold block design of every order $n \geq 4$ having a metamorphosis into a 12-fold kite system.*

Proof As was pointed out in Section 6, the spectrum for 6-fold block designs is the set of all $n \geq 4$. So let (X, B) be any such design and let $(X, 2B)$ be the 12-fold block design obtained from B by doubling each block. The remainder of the proof is exactly the same as the proof of Lemma 3.1.

□

8 Summary

Since we can paste together solutions of $\lambda = 1, 2, 3, 4, 6$, and 12 to obtain solutions for all other values of λ, the following table gives a summary of the results in this paper.

$\lambda \pmod{12}$	spectrum for λ-fold block designs having a metamorphosis into λ-fold kite systems
0	all $n \geq 4$
1,5,7,11	1 or 16 (mod 24)
2,10	1 or 4 (mod 12)
3,9	0 or 1 (mod 8)
4,8	1 (mod 3)
6	0 or 1 (mod 4)

Theorem 8.1 *The above table provides necessary and sufficient conditions for the existence of a λ-fold block design having a metamorphosis into a λ-fold kite system.* □

References

[1] Chen Kejun and Zhu Lie, *On the existence of skew Room frames of type t^u*, Ars Combinatoria, 43 (1996), 65-79.

[2] C. C. Lindner and A. Rosa, *The metamorphosis of λ-fold block designs with block size four into λ-fold triple systems*, JSPI, (to appear).

[3] C. C. Lindner and A. Rosa, *The metamorphosis of block designs with block size four into $(K_4 \setminus e)$-designs*, (submitted for publication).

[4] C. C. Lindner and Anne Street, *The metamorphosis of λ-fold block designs with block size four into λ-fold 4-cycle systems*, Bulletin of the ICA, 28 (2000), 7-18.

[5] D. R. Stinson and Zhu Lie, *On the existence of MOLS with equal-sized holes*, Aequat. Math., 33 (1987), 96-105.