The Metric Dimension of Unicyclic Graphs

Christopher Poisson and Ping Zhang[1]

Department of Mathematics and Statistics
Western Michigan University
Kalamazoo, MI 49008

Abstract

For an ordered set $W = \{w_1, w_2, \cdots, w_k\}$ of vertices and a vertex v in a graph G, the representation of v with respect to W is the k-vector $r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$, where $d(x, y)$ represents the distance between the vertices x and y. The set W is a resolving set for G if distinct vertices of G have distinct representations. A resolving set containing a minimum number of vertices is called a basis for G and the number of vertices in a basis is the (metric) dimension $\dim G$. A connected graph is unicyclic if it contains exactly one cycle. For a unicyclic graph G, tight bounds for $\dim G$ are derived. It is shown that all numbers between these bounds are attainable as the dimension of some unicyclic graph.

Keywords: resolving set, basis, dimension.

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1 Introduction

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. For an ordered set $W = \{w_1, w_2, \cdots, w_k\} \subseteq V(G)$ and a vertex v of G, we refer to the k-vector

$$r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$$

as the (*metric*) *representation of* v *with respect to* W. The set W is called a *resolving set* for G if distinct vertices of G have distinct representations. A resolving set containing a minimum number of vertices is called a *minimum resolving set* or a *basis* for G. The (*metric*) *dimension* $\dim G$ is the number

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of vertices in a basis for G. If $\dim G = k$, then G is called a *k-dimensional graph*.

To illustrate these concepts, we consider the graph G of Figure 1. The ordered set $W_1 = \{v_1, v_3\}$ is *not* a resolving set for G since $r(v_2|W_1) = (1,1) = r(v_4|W_1)$, that is, G contains distinct vertices having the same representation. On the other hand, $W_2 = \{v_1, v_2, v_3\}$ is a resolving set for G since the representations of the vertices of G with respect to W_2 are

$$r(v_1|W_2) = (0,1,1) \qquad r(v_2|W_2) = (1,0,1)$$
$$r(v_3|W_2) = (1,1,0) \qquad r(v_4|W_2) = (1,2,1).$$

However, W_2 is not a basis for G since $W_3 = \{v_1, v_2\}$ is also a resolving set. The representations for the vertices of G with respect to W_3 are

$$r(v_1|W_3) = (0,1) \qquad r(v_2|W_3) = (1,0)$$
$$r(v_3|W_3) = (1,1) \qquad r(v_4|W_3) = (1,2).$$

Since no single vertex constitutes a resolving set for G, it follows that W_3 is a basis for G and $\dim G = 2$. (The vertices in the basis W_3 are drawn as solid circles in Figure 1.)

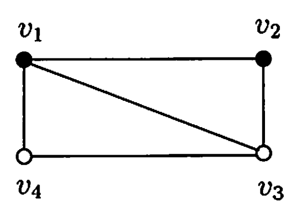

Figure 1: A graph G and a basis of G

The example just presented also illustrates an important point. When determining whether a given set W of vertices of a graph G is a resolving set for G, we need only investigate the vertices of $V(G) - W$ since $w \in W$ is the only vertex of G whose distance from w is 0.

The inspiration for these concepts stems from chemistry. It is often useful in chemistry to study the similarity (or difference) between two compounds. Two chemical compounds may differ structurally, yet behave similarly in certain situations. In [1] it is noted that if a set of functional groups or atoms in two compounds have the same distance to another functional group of atoms, then these compounds may be similar in certain instances.

A common but important problem in the study of chemical structures is to determine ways of representing a set of chemical compounds such that distinct compounds have distinct representations. The concepts of resolving set and minimum resolving set have previously appeared in the literature.

In [4] and later in [5], Slater introduced these ideas and used locating set for what we have called resolving set. He referred to the cardinality of a minimum resolving set in a graph G as its location number of G. Independently, Harary and Melter [2] investigated these concepts as well, but used metric dimension rather than location number, the terminology that we have adopted. A more generalized metric dimension in graphs has been studied in [3].

A formula for the dimension of trees that are not paths has been established in [1, 2, 4] In order to present this formula, we need additional definitions. A vertex of degree at least 3 in a tree T is called a *major vertex* of T. An end-vertex u of T is said to be a *terminal vertex of a major vertex* v of T if $d(u, v) < d(u, w)$ for every other major vertex w of T. The *terminal degree* $ter(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of T is an *exterior major vertex* of T if it has positive terminal degree. Let $\sigma(T)$ denote the sum of the terminal degrees of the major vertices of T, and let $ex(T)$ denote the number of exterior major vertices of T. For example, the tree T of Figure 2 has four major vertices, namely, v_1, v_2, v_3, v_4. The terminal vertices of v_1 are u_1 and u_2, the terminal vertices of v_3 are u_3, u_4, and u_5, and the terminal vertices of v_4 are u_6 and u_7. The major vertex v_2 has no terminal vertex and so v_2 is not an exterior major vertex of T. Therefore, $\sigma(T) = 7$ and $ex(T) = 3$. We can now state a formula for the dimension of a tree.

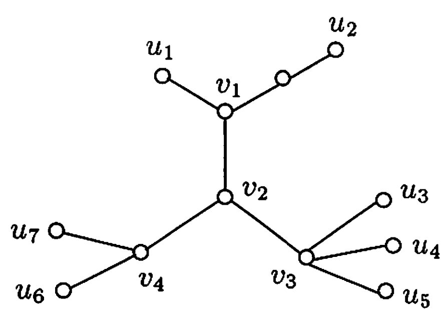

Figure 2: A tree with its major vertices

Theorem A *If T is a tree that is not a path, then*

$$\dim T = \sigma(T) - ex(T). \tag{1}$$

If $P_n : v_1, v_2, \cdots, v_n$ is a path of order $n \geq 2$, then $\sigma(P_n) = 0 = ex(P_n)$. So (1) is not true for paths. Since $d(v_1, v_i) = i - 1$ for all $1 \leq i \leq n$, it follows that $\{v_1\}$ is a basis of P_n and so $\dim P_n = 1$. Moreover, if G is a connected 1-dimensional graph of order $n \geq 2$, then there exists a vertex x of G such that $d(u,x) \neq d(v,x)$ for all $u, v \in V(G)$ and $u \neq v$. Since $1 \leq d(u,x) \leq n-1$ for all $u \in V(G)$ and $u \neq x$, it follows that $\{d(u,x) : u \in V(G)\} = \{1, 2, \cdots, n-1\}$. So there exists a vertex y in G such that $d(x,y) = n-1$ and $G = P_n$. Therefore, for a nontrivial connected graph G,

$$\dim G = 1 \quad \text{if and only if} \quad G = P_n. \tag{2}$$

A connected graph with exactly one cycle is called a *unicyclic graph*. Since every two consecutive vertices of the cycle C_n ($n \geq 3$) form a resolving set for C_n, it follows from (2) that

$$\dim C_n = 2 \ \text{ for all } n \geq 3. \tag{3}$$

Every unicyclic graph that is not a cycle is decomposable into a cycle and one or more trees, the dimensions of which are known. It is the goal of this paper to present tight bounds for the dimension of a unicyclic graph G that is not a cycle in terms of the dimensions of the cycle and the trees rooted on the cycle in G as well as the structure of G.

2 On the Dimension of Special Types of Graphs

Every unicyclic graph that is not a cycle can be obtained from a cycle and one or more trees by identifying some specified vertices on the cycle and on the trees. For example, the unicyclic graph G of Figure 3 is obtained from the cycle C_4 and the rooted tree T by identifying the vertices v_1 in C_4 and the root v_2 in T.

In order to establish the relationship between the dimension of a unicyclic graph and those of its cycle and rooted trees, we need additional definitions. Suppose that G_1 and G_2 are nontrivial connected graphs with $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. Then the *identification graph* $G = G[G_1, G_2, v_1, v_2]$ is obtained from G_1 and G_2 by identifying v_1 and v_2. Therefore, $v_1 = v_2$ in G. Such an example is shown in Figure 3. We first present a lemma that will be used on several occasions.

Lemma 2.1 *Let G_1 and G_2 be nontrivial connected graphs with $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ and let $G = G[G_1, G_2, v_1, v_2]$. If $W = W_1 \cup W_2$ is a resolving set of G, where $W_i \subseteq V(G_i)$ $(i = 1, 2)$, then $W_i \cup \{v_i\}$ is a resolving set of G_i for $i = 1, 2$.*

Proof. We show that $W_1 \cup \{v_1\}$ is a resolving set of G_1. Suppose, to the contrary, that this is not the case. Then there exist distinct vertices u and

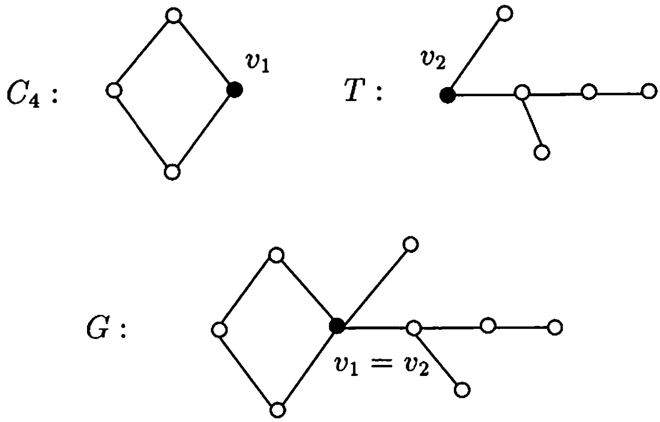

Figure 3: A unicyclic graph G obtained from a cycle and a rooted tree

v in $V(G_1)$ such that $r(u|W_1 \cup \{v_1\}) = r(v|W_1 \cup \{v_1\})$. This implies that $d_{G_1}(u, w_1) = d_{G_1}(v, w_1)$ for all $w_1 \in W_1$ and $d_{G_1}(u, v_1) = d_{G_1}(v, v_1)$. Then $d_G(u, w_2) = d_{G_1}(u, v_1) + d_{G_2}(v_2, w_2) = d_{G_1}(v, v_1) + d_{G_2}(v_2, w_2) = d_G(v, w_2)$ for all $w_2 \in W_2$ and so $r(u|W_2) = r(v|W_2)$. Therefore, $r(u|W_1 \cup W_2) = r(v|W_1 \cup W_2)$, contradicting the fact that $W = W_1 \cup W_2$ is a resolving set of G. Therefore, $W_1 \cup \{v_1\}$ is a resolving set for G_1. Similarly, $W_2 \cup \{v_2\}$ is a resolving set for G_1. ■

As an immediate consequence of Lemma 2.1, the following corollary gives a lower bound for the dimension of $G = G[G_1, G_2, v_1, v_2]$ in terms of $\dim G_1$ and $\dim G_2$.

Corollary 2.2 *Let G_1 and G_2 be nontrivial connected graphs with $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ and let $G = G[G_1, G_2, v_1, v_2]$. Then*

$$\dim(G) \geq \dim G_1 + \dim G_2 - 2.$$

Proof. Assume, to the contrary, that $\dim G \leq \dim G_1 + \dim G_2 - 3$. Let $W = W_1 \cup W_2$ be a basis of G, where $W_i \subseteq V(G_i)$, $i = 1, 2$. Then either $|W_1| \leq \dim G_1 - 2$ or $|W_2| \leq \dim G_2 - 2$, say the former. By Lemma 2.1, $W_1 \cup \{v_1\}$ is a resolving set of G_1 with cardinality at most $\dim G_1 - 1$, which is impossible. ■

The lower bound in Corollary 2.2 is sharp. For example, let $G_1 = C_4$, $G_2 = C_3$, and let G be obtained from G_1 and G_2 by identifying a vertex v_1 in G_1 and a vertex v_2 in G_2 (see Figure 4). Since $\dim C_4 = \dim C_3 = 2$ and $\dim G = 2$ (with a basis indicated in each of these graphs), it follows that $\dim G = 2 = \dim G_1 + \dim G_2 - 2$.

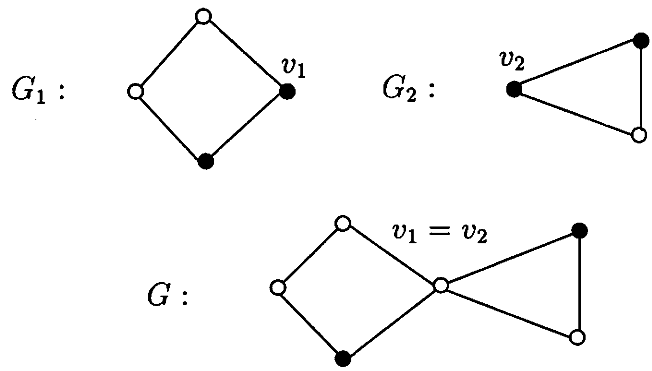

Figure 4: A graph G with $\dim G = \dim G_1 + \dim G_2 - 2$

For a set W of vertices of G, we define a relation R on $V(G)$ with respect to W by u R v if there exists $a \in \mathbf{Z}$ such that $r(v|W) = r(u|W) + (a, a, \cdots, a)$. Then R is an equivalence relation on $V(G)$. Let $[u]_W$ denote the equivalence class of u with respect to W. Then

$$v \in [u]_W \quad \text{if and only if} \quad r(v|W) = r(u|W) + (a, a, \cdots, a) \qquad (4)$$

for some $a \in \mathbf{Z}$. Next we present a lemma whose routine proof is omitted.

Lemma 2.3 *Let G be a nontrivial connected graph with $u, v \in V(G)$ and let W be a nonempty set of vertices of G.*

(*a*) *If $[v]_W \neq [u]_W$ in $V(G)$, then $r(x|W) \neq r(y|W)$ for all $x \in [v]_W$ and $y \in [u]_W$.*

(*b*) *If $[v]_W = \{v\}$ for all $v \in V(G)$, then W is a resolving set for G.*

The following lemma provides a way of constructing a resolving set of $G = G[G_1, G_2, v_1, v_2]$ from resolving sets of G_1 and G_2 that have certain prescribed properties. First, we need an additional definition. If a vertex v of a graph G belongs to some basis of G, then v is called a *basis vertex of* G.

Lemma 2.4 *Let G_1 and G_2 be nontrivial connected graphs with $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ and let $G = G[G_1, G_2, v_1, v_2]$. Suppose that G_1 contains a resolving set W_1 such that $[v_1]_{W_1} = \{v_1\}$ in G_1.*

(*a*) *If v_2 is not a basis vertex of G_2, then $W = W_1 \cup B_2$ is a resolving set for G for a basis B_2 of G_2.*

(*b*) *If v_2 is a basis vertex of G_2, then $W = W_1 \cup (B_2 - \{v_2\})$ is a resolving set for G, where B_2 is any basis for G_2 containing v_2.*

Proof. We first verify (a). Let $u, v \in V(G) - W$ and $u \neq v$. We show that $r(u|W) \neq r(v|W)$. We consider three cases.

Case 1. $u, v \in V(G_1)$. Then $r(u|W_1) \neq r(v|W_1)$ as W_1 is a resolving set for G_1, implying that $r(u|W) \neq r(v|W)$.

Case 2. $u, v \in V(G_2)$. Similarly, since B_2 is a basis for G_2, it follows that $r(u|B_2) \neq r(v|B_2)$ and so $r(u|W) \neq r(v|W)$.

Case 3. *Assume that one of u and v belongs to $V(G_1)$ and the other belongs to $V(G_2)$, say $u \in V(G_1)$ and $v \in V(G_2)$.* We may assume that that $u \neq v_1$ in G_1 and $v \neq v_2$ in G_2. Since $[v_1]_{W_1} = \{v_1\}$, it follows that $u \notin [v_1]_{W_1}$ and so $r(u|W_1) \neq r(v_1|W_1) + (a, a, \cdots, a)$ for all $a \in \mathbf{Z}$. On the other hand, $r(v|W_1) = r(v_1|W_1) + (b, b, \cdots, b)$, where $b = d(v_1, v)$. So $r(u|W_1) \neq r(v|W_1)$, implying that $r(u|W) \neq r(v|W)$. Therefore, W is a resolving set of G and so (a) is true.

We now verify (b). If $v_1 \in W_1$, then $W = W_1 \cup (B_2 - \{v_2\}) = W_1 \cup B_2$ in G. Then W is a resolving set of G by (a). So we assume that $v_1 \notin W_1$. Let $u, v \in V(G) - W$ and $u \neq v$. If (1) $u, v \in V(G_1)$ or (2) $u \in V(G_1)$ and $v \in V(G_2)$, then a similar argument to that employed in Cases 1 and 3 in the proof of (a) shows that $r(u|W_1) \neq r(v|W_1)$ and so $r(u|W) \neq r(v|W)$. Hence we assume that $u, v \in V(G_2)$. Let $B_2 = \{w_1, w_2, \cdots, w_k\}$, where $w_1 = v_2$ and $x \in W_1$. Let $W' = \{x, w_2, \cdots, w_k\} \subseteq W$. Observe that $d(v, x) = d(v, v_2) + d(v_2, x)$ in G. So $r(u|W') = r(u|B_2) + (d(v_2, x), 0, 0, \cdots, 0)$ and $r(v|W') = r(v|B_2) + (d(v_2, x), 0, 0, \cdots, 0)$. Since B_2 is a basis for G_2, it follows that $r(u|B_2) \neq r(v|B_2)$ and so $r(u|W') \neq r(v|W')$, implying that $r(u|W) \neq r(v|W)$. Therefore, W is a resolving set of G. ∎

For a nontrivial connected graph G, define a binary function $f_G : V(G) \to \mathbf{Z}$ by

$$f_G(v) = \begin{cases} \dim G & \text{if } v \text{ is not a basis vertex of } G \\ \dim G - 1 & \text{otherwise.} \end{cases} \tag{5}$$

The cardinality of the resolving set W in both (a) and (b) of Lemma 2.4 is $|W| = |W_1 \cup (B_2 - \{v_2\})| = |W_1| + f_{G_2}(v_2)$. This implies the following result.

Theorem 2.5 *Let G_1 and G_2 be nontrivial connected graphs with $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ and let $G = G[G_1, G_2, v_1, v_2]$. Suppose that G_1*

contains a resolving set W_1 *such that* $[v_1]_{W_1} = \{v_1\}$. *Then*

$$\dim G \le |W_1| + f_{G_2}(v_2) = \begin{cases} |W_1| + \dim G_2 & \text{if } v_2 \text{ is not a basis vertex of } G_2 \\ |W_1| + \dim G_2 - 1 & \text{otherwise.} \end{cases} \qquad (6)$$

In particular, if W_1 *is a basis for* G_1*, then*

$$\dim G \le \begin{cases} \dim G_1 + \dim G_2 & \text{if } v_2 \text{ is not a basis vertex of } G_2 \\ \dim G_1 + \dim G_2 - 1 & \text{otherwise.} \end{cases} \qquad (7)$$

In general, (7) is not true. For example, consider the graphs G_1, G_2, and $G = G[G_1, G_2, u_2, v_2]$ of Figure 5, where $G_1 = G_2 = P_3$ and $G = K_{1,4}$. By (2) and Theorem A, $\dim P_3 = 1$ and $\dim K_{1,4} = 3$ and so $\dim(G[G_1, G_2, u_2, v_2]) = 3 > 2 = 1 + \dim G_2$.

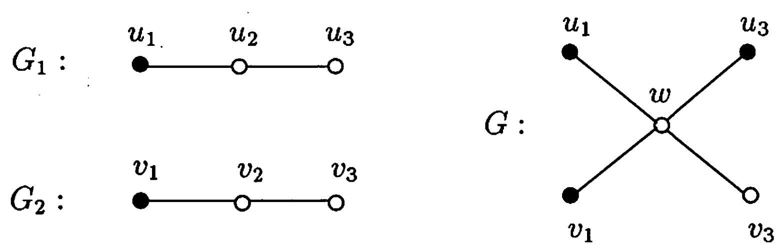

Figure 5: Graphs G_1, G_2, and G with $\dim G > \dim G_1 + \dim G_2$

3 Basis Vertices of a Tree

We have seen that the dimension of the graph $G = G[G_1, G_2, v_1, v_2]$ described in Section 2 not only depends on the dimensions of G_1 and G_2 but also on whether the vertex v_2 is a basis vertex of G_2. This implies that the dimension of a unicyclic graph G depends on (1) the dimensions of its subtrees rooted at the cycle of G, and (2) whether the roots of those subtrees T on the cycle C are basis vertices of T. Therefore, we need to determine which vertices in a tree T are the basis vertices of T. We begin with paths.

Let $P_n : v_1, v_2, \cdots, v_n$ be a path of order $n \ge 2$. Then each of the two end-vertices v_1 and v_n of P_n is a basis for P_n. Thus every vertex of P_2 is a basis vertex of P_2. So we may assume that $n \ge 3$. Since $d(v_{i-1}, v_i) = d(v_i, v_{i+1}) = 1$ for all integers i with $2 \le i \le n-1$, it follows that v_i is not a basis vertex of P_n. This observation gives the following result.

Theorem 3.1 *A vertex v of a path P_n, $n \geq 2$, is a basis vertex if and only if v is an end-vertex of P_n.*

Next we consider trees that are not paths. The following result classifies the basis vertices of a tree. First we need an additional definition. For a cut-vertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v together with all edges joining v and $V(H)$ in G is called a *branch* of G at v.

Theorem 3.2 *Let T be a tree with p exterior major vertices $v_1, v_2, \cdots, v_p$. For each i $(1 \leq i \leq p)$, let $u_{i1}, u_{i2}, \cdots, u_{ik_i}$ be the terminal vertices of v_i, and let P_{ij} be the $v_i - u_{ij}$ path $(1 \leq j \leq k_i)$. Suppose that W is a set of vertices of T. Then W is a basis of T if and only if W contains exactly one vertex from each of the paths $P_{ij} - v_i$ $(1 \leq j \leq k_i$ and $1 \leq i \leq p)$ with exactly one exception for each i with $1 \leq i \leq p$ and W contains no other vertices of T.*

Proof. Let B be a basis for T and let v be an exterior major vertex of T. Suppose that $k = ter(v)$ and that $u_1, u_2, \cdots, u_k$ are the terminal vertices of v. Then the branch of T at v containing u_i $(1 \leq i \leq k)$ is a $v - u_i$ path P_i. We claim that B contains at least one vertex from each of the paths $P_i - v$ $(1 \leq i \leq k)$ with at most one exception. Assume, to the contrary, that two of these paths, say $P_1 - v$ and $P_2 - v$, contain no vertex of B. Let u_1' and u_2' be the vertices adjacent to v on P_1 and P_2, respectively. Since neither $P_1 - v$ nor $P_2 - v$ contains a vertex of B, it follows that $r(u_1'|B) = r(u_2'|B)$, contradicting the fact that B is a basis for T. Thus, as claimed, B contains at least one vertex from each of the paths $P_i - v$ $(1 \leq i \leq k)$ with at most one exception. This implies that

$$|B| \geq \sum (ter(v) - 1), \tag{8}$$

where the sum is taken over all exterior major vertices of T. If $v \in B$, then the inequality in (8) is, in fact, strict. However, $|B| = \dim(T)$ and $\sum(ter(v) - 1) = \sigma(T) - ex(T)$, which contradicts (1). This implies that B contains exactly one vertex from each path $P_i - v$ $(1 \leq i \leq k)$ with exactly one exception and W contains no other vertices of T.

We now verify the converse. Suppose that $ter(v_i) = k_i$ for all i with $1 \leq i \leq p$ and W is a set of vertices of T such that W contains exactly one vertex from each of the paths $P_{ij} - v_i$ $(1 \leq j \leq k_i)$ with exactly one exception for each i with $1 \leq i \leq p$. Then $|W| = \sigma(T) - ex(T)$. By Theorem A, it suffices to show that that W is a resolving set of T. In order to show this, let u be an arbitrary vertex of T and v any vertex of T different from u. We consider two cases.

Case 1. *Suppose that there is some exterior major vertex w of T and an terminal vertex x of w such that u lies on the $w - x$ path of T.* There are two subcases.

Subcase 1.1. *There exists $y \in W$ such that y lies on the $w - x$ path of T.* If v does not lie on the $w - x$ path, then $d(u, y) < d(v, y)$, implying that $r(u|W) \neq r(v|W)$. So assume that v lies on the $w - x$ path. If $d(u, y) \neq d(v, y)$, then $r(u|W) \neq r(v|W)$. If $d(u, y) = d(v, y)$, then y lies on the $u - v$ path of T. Since the degree of w is at least 3, there exists another terminal vertex x' of w and a vertex y' on the $w - x'$ path such that $y' \in W$. Then $d(u, y') \neq d(v, y')$. So $r(u|W) \neq r(v|W)$.

Subcase 1.2. *There is no vertex of W that lies on the $w - x$ path of T.* If there is a vertex $y \in W$ such that either v lies on the $u - y$ path or u lies on the $v - y$ path, then $d(v, y) < d(u, y)$ or $d(u, y) < d(v, y)$, respectively. In either case, $r(u|W) \neq r(v|W)$. Thus, we may assume that every path between v and a vertex of W does not contain u and every path between u and a vertex of W does not contain v. Necessarily, then, there exists an exterior major vertex w' and a terminal vertex x' of w' such that v lies on the $w' - x'$ path of T. Observe that $v \neq w'$, for otherwise there exists a terminal vertex x'' (it is possible that $x' = x''$) of w' such that the path $x'' - v$ contains a vertex y of W. Then v lies the $u - y$ path of T, which is a contradiction. Similarly, $x' \notin W$ and $u \neq w$.

If $w = w'$, then there exists $y \in W$ distinct from w such that y lies on the $w - x'$ path of T. Then $d(u, y) > d(v, y)$, implying that $r(u|W) \neq r(v|W)$. So $w \neq w'$. Since the degrees of both w and w' are at least 3, there exists a branch T_1 at w that does not contain x and a branch T_2 at w' that does not contain x'. Necessarily, both T_1 and T_2 must contain a vertex of W. Let z and z' be vertices of W belonging to T_1 and T_2, respectively. If $d(u, z') \neq d(v, z')$, then $r(u|W) \neq r(v|W)$. So we assume that $d(u, z') = d(v, z')$. In this case, $d(u, z) < d(v, z)$, implying that $r(u|W) \neq r(v|W)$.

Case 2. *For every exterior major vertex w of T and every terminal vertex x of w, u does not lie on the $w - x$ path of T.* Then there are at least two branches at u, say T_1 and T_2, each of which contains some exterior major vertex of terminal degree at least 2. Therefore, each of T_1 and T_2 contains a vertex of W. Let z and z' be the vertices of W in T_1 and T_2, respectively. First assume that $v \in V(T_1)$. Then the $v - z'$ path of T contains u, so $d(u, z') < d(v, z')$ and so $r(u|W) \neq r(v|W)$. We now assume that $v \notin V(T_1)$. Then the $v - z$ path of T contains u. Hence $d(u, z) < d(v, z)$, so $r(u|W) \neq r(v|W)$. Therefore, W is a resolving set of T. ■

The following result is an immediate consequence of Theorem 3.2

Corollary 3.3 *Let T be a tree that is not a path and let $v \in V(T)$. Then v is a basis vertex of T if and only if v is not an exterior major vertex of T.*

4 The Dimension of Unicyclic Graphs

In this section, we determine bounds for the dimension of unicyclic graphs. Let G be a unicyclic graph and let C be the unique cycle of G. Let u_1, u_2, $\cdots$, u_k be the distinct vertices of C with $\deg u_i \geq 3$, $1 \leq i \leq k$, and let T_i be the subtree of G rooted at u_i. A unicyclic graph G is said to be of *type 1* if $\Delta(G) = 3$ and for a vertex v of G, $\deg v = 3$ if and only if $v = u_i$ for some $1 \leq i \leq k$. Otherwise, G is of *type 2*. Therefore, a unicyclic graph G is type 1 if and only if every tree T_i of G is a path, one of whose end-vertex is $u_i \in V(C)$. We first present a lemma whose proof is routine and is therefore omitted.

Lemma 4.1 *Let C_n be a cycle of order $n \geq 3$ and diameter $d = \lfloor n/2 \rfloor$. If n is odd, let $W_1 = \{u, v\}$ with $d(u, v) = d$, while if n is even, let $W_2 = \{u, v, w\}$, where $d(u, v) = d$ and w is any other vertex of C_n. Then $[v]_{W_i} = \{v\}$ for all $v \in V(C_n)$, where $i = 1, 2$.*

The following theorem concerns the dimension of a unicyclic graph of type 1.

Theorem 4.2 *Let G be a unicyclic graph of type 1 with unique cycle C of order n. Then $\dim G = 2$ if n is odd and $\dim G \leq 3$ if n is even.*

Proof. Since G is not a path, $\dim G \geq 2$ by (2). It remains to verify the upper bound. Let u_1, u_2, $\cdots$, u_k the vertices of C with $\deg u_i \geq 3$, and let P_i be the path rooted on C with end-vertex u_i, $1 \leq i \leq k$. We consider two cases.

Case 1. *n is odd.* Let $W = \{x, y\}$ with $d(x, y) = diam\ C$. Then $[v]_W = \{v\}$ for all vertices $v \in V(C) - \{u_1, u_2, \cdots, u_k\}$ and $[u_i]_W = \{u_i\} \cup V(P_i)$ for all i. Therefore, the $k+1$ sets $V(C) - \{u_1, u_2, \cdots, u_k\}$ and $[u_i]_W$, $1 \leq i \leq k$, form a partition of $V(G)$. We claim that W is a resolving set of G. Let $u, w \in V(G) - W$ and $u \neq w$. If there exist two distinct equivalent classes $[s]_W$ and $[t]_W$, $s, t \in V(C)$, such that $u \in [s]_W$ and $w \in [t]_W$, then $r(u|W) \neq r(w|W)$ by Lemma 2.3 (a). So we assume that u and v belong to the same equivalence class in $V(G)$. Since $[v]_W = \{v\}$ for all vertices $v \in V(C) - \{u_1, u_2, \cdots, u_k\}$, it follows that $u, w \in [u_i]_W$ for some i ($1 \leq i \leq k$). This implies that $u, w \in V(P_i)$. Since $r(u|W) = r(w|W) + (a, a, \cdots, a)$, where $a = d(u, w) > 0$, it follows that $r(u|W) \neq r(w|W)$, which is a contradiction. Therefore, W is a resolving set for G and so $\dim G = 2$.

Case 2. *n is even.* Choose $W = \{x, y, w\}$ with $d(x,y) = diam\ C$ and where w is any other vertex of C. Then an argument similar to that employed in Case 1 shows that W is a resolving set of G. Therefore, $\dim G \le |W| = 3$. ■

Both 2 and 3 are attainable as the dimensions of unicyclic graphs of type 1 that contain an even cycle. For example, the unicyclic graph G_1 of Figure 6 has dimension 2 and the unicyclic graph G_2 of Figure 6 has dimension 3.

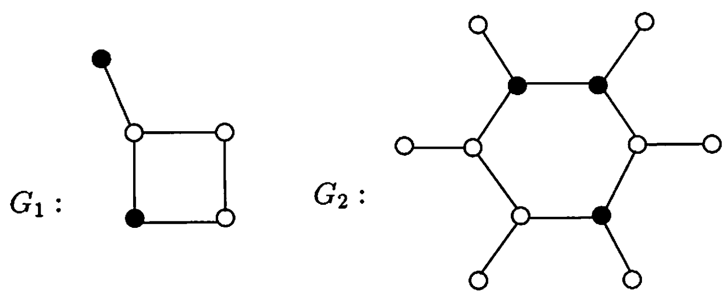

Figure 6: Unicyclic graphs of type 1 with dimensions 2 and 3

Next we consider the dimensions of unicyclic graphs of type 2.

Theorem 4.3 *Let G be unicyclic graph of type 2 with cycle C such that $u_1, u_2, \cdots, u_k$ are the vertices of C with $\deg u_i \ge 3$, $1 \le i \le k$, and T_i is the tree rooted at u_i. Then*

$$\sum_{i=1}^{k} f_{T_i}(u_i) \le \dim G \le 2 + \sum_{i=1}^{k} f_{T_i}(u_i). \tag{9}$$

Proof. We first show that $\sum_{i=1}^{k} f_{T_i}(u_i) \le \dim G$. If $\sum_{i=1}^{k} f_{T_i}(u_i) \le 2$, then $\sum_{i=1}^{k} f_{T_i}(u_i) \le 2 \le \dim G$ by (2). So we assume that $\sum_{i=1}^{k} f_{T_i}(u_i) \ge 3$. Assume, to the contrary, that $\dim G \le \left(\sum_{i=1}^{k} f_{T_i}(u_i)\right) - 1$. Let $r = \left(\sum_{i=1}^{k} f_{T_i}(u_i)\right) - 1 \ge 2$ and let W be a resolving set of G with $|W| = r$. Then there exists an integer i with $1 \le i \le k$ such that $|W \cap V(T_i)| = r_i \le f_{T_i}(u_i) - 1$. Let $W_i = W \cap V(T_i)$. By Lemma 2.1, $W_i \cup \{u_i\}$ is a resolving set for T_i. If $r_i \le f_{T_i}(u_i) - 2$, then $|W_i \cup \{u_i\}| \le f_{T_i}(u_i) - 1 \le \dim T_i - 1$, which is a contradiction. So we may assume that $r_i = f_{T_i}(u_i) - 1$. If $f_{T_i}(u_i) = \dim T_i$, then u_i is not a basis vertex of T_i. On the other hand, since $|W_i \cup \{u_i\}| = r_i + 1 = (f_{T_i}(u_i) - 1) + 1 = \dim T_i$, it follows that the set $W_i \cup \{u_i\}$ is a basis for T_i, implying that u_i is a basis vertex of

T_i, producing a contradiction. So we assume that $f_{T_i}(u_i) = \dim T_i - 1$. Then $|W_i \cup \{u_i\}| = r_i + 1 = f_{T_i}(u_i) = \dim T_i - 1$, which is impossible since $W_i \cup \{u_i\}$ is a resolving set. Therefore, $\sum_{i=1}^{k} f_{T_i}(u_i) \leq \dim G$.

We now verify the upper bound for $\dim G$. Let C be the unique cycle in G, where C has order n. For each integer i with $1 \leq i \leq k$, if $f_{T_i}(u_i) = \dim T_i$, then let B_i be any basis for T_i. If $f_{T_i}(u_i) = \dim T_i - 1$, then let B_i be a basis for T_i containing u_i. Hence $|B_i - \{u_i\}| = f_{T_i}(u_i)$ for each integer i with $1 \leq i \leq k$. We now consider two cases.

Case 1. *n is odd.* First assume that $k = 1$. Let U_1 be the unicyclic graph obtained from the cycle C and the tree T_1 by identifying a vertex in C and a vertex in T_1 and labeling it u_1. Let $W_0 = \{u, v\}$, where $u, v \in V(C)$ and $d(u, v) = \lfloor n/2 \rfloor$. Since $[v]_{W_0} = \{v\}$ for all $v \in V(C)$, it follows by Theorem 2.5 that $\dim U_1 \leq 2 + f_{T_1}(u_1)$.

Next, let U_2 be the graph obtained from U_1 and T_2 by identifying a vertex of C in U_1 and a vertex in T_2 and labeling it u_2. Let $W_1 = W_0 \cup (B_1 - \{u_1\})$. By Lemma 2.4, W_1 is a resolving set for U_1. If $v \in V(T_1)$, then $v \in [u_1]_{W_0}$ in U_1. This implies that $[u_2]_{W_0} = \{u_2\}$ in U_1 and so $[u_2]_{W_1} = \{u_2\}$ in U_1. Applying Theorem 2.5 to U_1 and T_2, we have

$$\dim U_2 \leq |W_1| + f_{T_2}(u_2) = [2 + f_{T_1}(u_1)] + f_{T_2}(u_2).$$

Repeating the procedure above, for $k \geq 3$, we let U_k be obtained from U_{k-1} and the tree T_k by identifying a vertex of C in U_{k-1} and a vertex in T_k and labeling it u_k. Let

$$\begin{aligned} W_{k-1} &= W_{k-2} \cup (B_{k-1} - \{u_{k-1}\}) \\ &= W_0 \cup (B_1 - \{u_1\}) \cup (B_2 - \{u_2\}) \cup \cdots \cup (B_{k-1} - \{u_{k-1}\}). \end{aligned}$$

Then $|W_{k-1}| = \sum_{i=1}^{k-1} f_{T_i}(u_i)$. Since $W_0 \subseteq W_{k-1}$ and $[u_k]_{W_0} = \{u_k\}$, it follows that $[u_k]_{W_{k-1}} = \{u_k\}$ in U_{k-1}. Again, we apply Theorem 2.5 to U_{k-1} and T_k and obtain

$$\dim U_k \leq |W_{k-1}| + f_{T_k}(u_k) = 2 + \sum_{i=1}^{k} f_{T_i}(u_i).$$

Case 2. *n is even.* Since G is a unicyclic graph of type 2, there exists a vertex u_i $(1 \leq i \leq k)$ such that either $\deg u_i \geq 4$ or T_i is not a path. Assume, without loss of generality, that $u_i = u_1$.

Again, let U_1 be the unicyclic graph obtained from C and T_1 by identifying a vertex in C and a vertex in T_1 and labeling it u_1. We claim that $\dim U_1 \leq 2 + f_{T_1}(u_1)$. Let $W_0 = \{u, v\}$, where $d(u, v) = n/2$ and $u, v \in V(C) - \{u_1\}$ and let $W_1 = W_0 \cup (B_1 - \{u_1\})$, where $|W_1| = 2 + f_{T_1}(u_1)$.

If T_1 is a path, then u_1 is not an end-vertex of T_1 and so u_1 does not belong to any basis of T_1 by Theorem 3.1. Otherwise, T_1 is not a path and $\dim T_1 \geq 2$. Hence $B_1 - \{u_1\} \neq \emptyset$, implying that $W_0 \neq W_1$. Next we show that W_1 is a resolving set of U_1. Let x and y be distinct vertices of $U_1 - W_1$. We show that $r(x|W_1) \neq r(y|W_1)$. There are three subcases.

Subcase 2.1 $x, y \in V(C)$. Let

$$S_1 = \{u, v, u_1\} \text{ and } S_2 = \{u, v, w\} \subseteq W_1, \tag{10}$$

where $w \in B_1 - \{u_1\}$. Since $d(x,w) = d(x,u_1) + d(u_1,w)$ and $d(y,w) = d(y,u_1) + d(u_1,w)$, it follows that $r(x|S_2) = r(x|S_1) + (0,0,d(u_1,w))$ and $r(y|S_2) = r(y|S_1) + (0,0,d(u_1,w))$. By Lemma 4.1, S_1 is a resolving set of C. Hence $r(x|S_1) \neq r(y|S_1)$ and so $r(x|S_2) \neq r(y|S_2)$, implying that $r(x|W_1) \neq r(y|W_1)$.

Subcase 2.2 $x, y \in V(T)$. If $u_1 \notin B_1$, then $B_1 - \{u_1\} = B_1$, which is a basis for U_1. So $r(x|B_1) \neq r(y|B_1)$, implying that $r(x|W_1) \neq r(y|W_1)$. So we assume that $u_1 \in B_1$. Let $S = (B_1 - \{u_1\}) \cup \{u\} \subseteq W_1$. Since $d(x,u) = d(x,u_1) + d(u_1,u)$ and $d(y,u) = d(y,u_1) + d(u_1,u)$, it follows that $r(x|S) = r(x|B_1) + (0,0,\cdots,0,d(u_1,u))$ and $r(y|S) = r(y|B_1) + (0,0,\cdots,0,d(u_1,u))$. Since B_1 is a basis for T_1, it follows that $r(x|S) \neq r(y|S)$ and so $r(x|W_1) \neq r(y|W_1)$.

Subcase 2.3 $x \in V(C)$ *and* $y \in V(T)$. If $y \neq u_1$, then either $d(x,u) \neq d(y,u)$ or $d(x,v) \neq d(y,v)$ and so $r(x|W_1) \neq r(y|W_1)$. Hence we may assume that $y = u_1$. Let $w \in B_1$. Then $d(x,w) = d(x,y) + d(y,w)$. Since $y \neq w$, it follows that $d(y,w) \neq d(x,w)$, implying that $r(x|W_1) \neq r(y|W_1)$. So W_1 is a resolving set of U_1 and $\dim U_1 \leq |W_1| = 2 + f_{T_1}(u_1)$, as claimed.

We now let U_2 be the graph obtained from U_1 and T_2 by identifying a vertex in C of U_1 and a vertex in T_2 and labeling it u_2. In order to apply Theorem 2.5 to U_1 and T_2, we need to show that $[x]_{W_1} = \{x\}$ for all $x \in V(C) - \{u_1\}$. Again, let S_1 and S_2 be the sets defined in (10). We show that $[x]_{S_2} = \{x\}$ for all $x \in V(C) - \{u_1\}$. Assume, to the contrary, that there exists $y \neq x$ such that $y \in [x]_{S_2}$, that is, $r(x|S_2) = r(y|S_2) + (a,a,\cdots,a)$ for some integer a. If $y \in V(T)$, then $y \in [u_1]_{W_0}$ and so $y \notin [x]_{W_0}$, implying that $y \notin [x]_{S_2}$. So we may assume that $y \in V(C)$. Since $d(x,w) = d(x,u_1) + d(u_1,w)$ and $d(y,w) = d(y,u_1) + d(u_1,w)$, it follows that $r(x|S_2) = r(x|S_1) + (0,0,d(u_1,w))$ and $r(y|S_2) = r(x|S_1) + (0,0,d(u_1,w))$. This implies that $r(x|S_1) - r(y|S_1) = r(x|S_2) - r(y|S_2) = (a,a,\cdots,a)$, which contradicts Corollary 4.1. Therefore, $[x]_{S_2} = \{x\}$ for all $x \in V(C) - \{u_1\}$. Since $S_2 \subseteq W_1$, it follows that $[x]_{W_1} = \{x\}$ for all $x \in V(C) - \{u_1\}$. In particular, $[u_2]_{W_1} = \{u_2\}$. Applying Theorem 2.5 to U_1 and T_2, we have

$$\dim U_2 \leq |W_1| + f_{T_2}(u_2) = 2 + f_{T_1}(u_1) + f_{T_2}(u_2).$$

Then an argument similar to that used in Case 1 shows that (9) is true for all U_k with $k \geq 3$. ∎

If G is a unicyclic graph of type 1, then $f_{T_i}(u_i) = 0$ for all i with $1 \leq i \leq k$. This implies that (9) is not true if G is a unicyclic graph of type 1 and G contains an even cycle. Also, for $k = 2$, Theorem 4.3 provides an improved lower bound for the dimension of $G = G[G_1, G_2, v_1, v_2]$ in Corollary 2.2.

We conclude this section by showing that the bounds in Theorem 4.3 are sharp. In fact, the three numbers $2 + \sum_{i=1}^{k} f_{T_i}(u_i)$, $1 + \sum_{i=1}^{k} f_{T_i}(u_i)$, and $\sum_{i=1}^{k} f_{T_i}(u_i)$ are all attainable as the dimensions of some unicyclic graphs. We will only give the unicyclic graphs with an odd cycle since the examples for the unicyclic graphs with an even cycle are similar. Consider the unicyclic graphs G_1, G_2, and G_3 of Figure 7. We describe a basis for each of those unicyclic graphs with the aid of the following lemma whose routine proof is omitted.

Lemma 4.4 *If S is a p-subset of $V(G)$, where $p \geq 2$, such that $d(u, x) = d(v, x)$ for all $u, v \in S$ and $x \neq u, v$, then every resolving set of G contains at least $p - 1$ elements in S.*

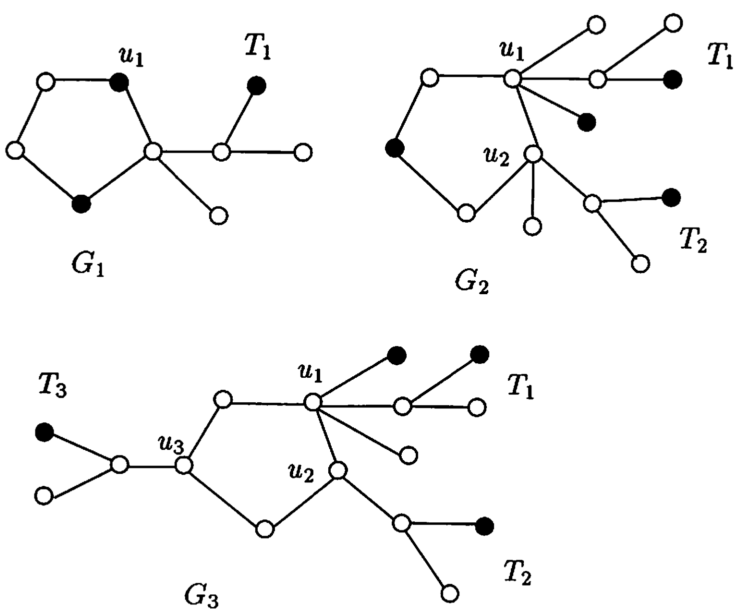

Figure 7: The unicyclic graphs G_1, G_2, and G_3

The unicyclic graph G_1 of Figure 7 has dimension 3 where the three solid vertices form a basis for G_1. The tree T_1 in G_1 has dimension 2 by Theorem A. Since $\deg_{T_1} u_1 = 2$, it follows that u_1 is a basis vertex of T_1 by Corollary 3.3 and so $f_{T_1}(u_1) = \dim T_1 - 1 = 1$. Therefore, $\dim G_1 = 3 = f_{T_1}(u_1) + 2$.

The unicyclic graph G_2 of Figure 7 has dimension 4. By Theorem A, $\dim T_1 = 2$ and $\dim T_2 = 2$. The vertex u_1 is an exterior major vertex of T_1 and so u_1 is not a basis vertex of T_1. Thus $f_{T_1}(u_1) = \dim T_1 = 2$. The vertex u_2 is an end-vertex of T_2 and so u_2 is a basis vertex of T_2. So $f_{T_2}(u_2) = \dim T_2 - 1 = 1$. Therefore, $\dim G_2 = 4 = 2 + 1 + 1 = f_{T_1}(u_1) + f_{T_2}(u_2) + 1$.

The unicyclic graph G_3 of Figure 7 has dimension 4. By Theorem A, $\dim T_1 = \dim T_2 = \dim T_2 = 2$. Similarly, $f_{T_1}(u_1) = \dim T_1 = 2$, $f_{T_2}(u_2) = \dim T_2 - 1$ and $f_{T_3}(u_3) = \dim T_3 - 1$. Therefore, $\dim G_3 = 4 = f_{T_1}(u_1) + f_{T_2}(u_2) + f_{T_3}(u_3)$.

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