Blocking Sets for Paths of a Given Length

Paul Erdős
Hungarian Academy of Sciences

Ralph Faudree, Edward T. Ordman,
Cecil Rousseau, and Richard Schelp

Department of Mathematical Sciences
The University of Memphis
Memphis, Tennessee 38152-3240

Abstract

How many vertices must we delete from a graph so that it no longer contains a path P_k on k vertices? We explore this question for various special graphs (hypercubes, square lattice graphs) as well as for some general families.

1 Introduction

For basic definitions and notation, we refer the reader to standard texts on graph theory [3], [4], [7]. Given a graph G, let us say that $Z \subset V(G)$ is *k-blocking* if $G \setminus Z$ contains no path of order k. Given a graph G and an integer $k \geq 2$, we seek $\min|Z|$, where the minimum is taken over all k-blocking subsets $Z \subset V(G)$. The ratio of $\min|Z|$ to $|V(G)|$ will be called the *k-blocking ratio* of G.

The problem is suggested by various computer science applications. For the first, suppose that each vertex represents a state of a program (or finite state machine) and each edge a possible transition between states.

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It is desired to select a set of distinguished states (the set Z) so that the program will enter a distinguished state after at most k steps. (We assume, for this abstraction, that the program does not return to the same state during the k steps.)

For the second application, suppose that each vertex represents a computer and each edge a communication channel. We wish to record all "long distance" messages, specifically, all messages traveling at least k steps. If we could identify a subset Z of the computers representing all paths of order k, we could place recorders at only those processes.

Alon and Chung considered a problem of this type in connection with fault tolerant networks [1]. They obtained the following striking result.

Theorem (Alon, Chung). *For every $\epsilon > 0$ and every integer $k \geq 2$ there exists a graph G with (k/ϵ) vertices, maximum degree $\Delta = O(1/\epsilon^2)$, and k-blocking ratio at least $1 - \epsilon$.*

The proof of Alon and Chung uses the Ramanujan graphs studied by Lubotzky, Phillips, and Sarnak [8]. We shall study commonplace examples (hypercubes, grid graphs) and some general families of graphs, for example graphs with bounded degree. By presenting these comparatively simple results, we hope to stimulate interest in this subject. More extensive studies are needed on k-blockings for more general classes of graphs.

Among the networks that have been used extensively in parallel computing are the hypercubes. Let Q_n denote the n-dimensional hypercube. This is the graph with vertex set $V(Q_n) = \{0,1\}^n$ in which $uv \in E(Q_n)$ if and only if the binary n-tuples u and v differ in exactly one component. Recursively, $Q_n = K_2 \times Q_{n-1}$, that is Q_n is obtained by taking two disjoint copies of Q_{n-1} and adding the $n-1$ edges that join corresponding vertices. [In general, $G_1 \times G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ in which $uv \in E(G_1 \times G_2)$ if $u_1v_1 \in E(G_1)$ and $u_2 = v_2$ or $u_1 = v_1$ and

$u_2v_2 \in E(G_2)$.]

Proposition 1. *For a hypercube Q_n of dimension $n \geq 2$, the 2-blocking ratio is $1/2$, and so is the 3-blocking ratio.*

Proof. First, let us show that the 2-blocking ratio is no more than $1/2$. Let $w(v) = \sum_{i=1}^{n} v_i$ be the Hamming weight of vertex v. Note that Q_n is bipartite with bipartition $V(Q_n) = (W, Z)$ where $W = \{v | \ w(v) \text{ is even}\}$ and $Z = \{v | \ w(v) \text{ is odd}\}$. Then $|W| = |Z| = 2^{n-1}$ and Z (or W) is 2-blocking (and hence 3-blocking). In the other direction, first note that the recursive definition shows inductively that Q_n has a 2-factor consisting of 2^{n-2} C_4's. Any 3-blocking set must contain at least two vertices from each of these C_4's, and thus any 3-blocking set must contain at least 2^{n-1} vertices. □

2 Graphs of Large Degree

The following result uses the following well-known theorem of Erdős and Gallai [6]: a graph of order n that contains no P_k has at most $n(k-2)/2$ edges.

Theorem 1. *Suppose that for $i = 1, 2, 3, \ldots$ the graph G_i has order n_i and is regular of degree d_i, where $d_i \to \infty$ as $i \to \infty$. Given $k \geq 2$ and $\epsilon > 0$, there is an integer $N = N(\epsilon, k)$ such that for all $n_i > N$ the k-blocking ratio of G_i exceeds $\frac{1}{2} - \epsilon$.*

Proof. If G is a d-regular graph of order n, then G has $nd/2$ edges. Deleting any $(\frac{1}{2} - \epsilon)n$ vertices from G yields a graph with $(\frac{1}{2} + \epsilon)n$ vertices and at least $\frac{nd}{2} - (\frac{1}{2} - \epsilon)nd = nd\epsilon$ edges. If we assume that this graph contains no P_k, then the Erdős-Gallai theorem gives

$$d\epsilon < \left(\frac{1}{2} + \epsilon\right) \frac{k-2}{2},$$

which is clearly false provided d is sufficiently large. Since $d_i \to \infty$ for the given sequence (G_i), it follows that for all sufficiently large i the k-blocking ratio of G_i exceeds $\frac{1}{2} - \epsilon$. □

Corollary 1. *For any fixed $k \geq 2$ and any $\epsilon > 0$ the k-blocking ratio of Q_n is at least $(\frac{1}{2} - \epsilon)$ for all sufficiently large n.*

Proof. The hypercube Q_n is a regular graph of order 2^n and degree n. Hence Theorem 1 applies. □

3 Graphs of Bounded Degree.

With W and Z disjoint subsets of $V(G)$, we shall denote by $E(W, Z)$ the edge set $\{wz \in E(G) | \, w \in W, z \in Z\}$.

Theorem 2. *Suppose G is a graph of order n and maximum degree Δ.*

(a) Some set of $\lfloor n\Delta/(\Delta + 1)\rfloor$ vertices in G is 2-blocking.

(b) Some set of $\lfloor n\Delta/(\Delta + 2)\rfloor$ vertices in G is 3-blocking.

Both results are sharp. Thus, for the family of all graphs with maximum degree Δ, the largest possible 2-blocking ratio is $\Delta/(\Delta + 1)$ and the largest 3-blocking ratio is $\Delta/(\Delta + 2)$.

Proof. (a) Equivalently, $\alpha(G) \geq \lceil n/(\Delta + 1)\rceil$. This follows immediately from the following Ramsey result of Chvátal [5]: $r(T, K_m) = (m - 1)q + 1$ for any tree T with q edges (in particular, $T = K_{1,q}$). For another simple proof, note that if $W \subset V(G)$ is an independent set of order $\alpha(G)$ and $Z = V(G) \setminus W$, then Z is 2-blocking. Since W is a maximal independent set, each vertex in Z is adjacent to at least one vertex in W. Hence

$$|Z| \leq |E(W, Z)| \leq (n - |Z|)\,\Delta,$$

which gives $|Z| \leq \lfloor n\Delta/(\Delta+1) \rfloor$. To see that this bound is sharp, consider the example $G \cong mK_{\Delta+1}$ where $m = n/(\Delta+1)$. In order to obtain an independent set, at least Δ vertices must be deleted from each component. Hence, any 2-blocking set contains at least $n\Delta/(\Delta+1)$ vertices.

(b) Let $W \subset V(G)$ be such that $P_3 \not\subset \langle W \rangle$, and, subject to this condition, $|W|$ is as large as possible. Further, assume that of all such sets with the maximum possible cardinality, W has been chosen so as to minimize the number of edges of $\langle W \rangle$. We claim that each vertex in $Z = V(G) \setminus W$ is adjacent to at least two vertices of W. Clearly, each $z \in Z$ is adjacent to at least one vertex in W. If $\Gamma(z) \cap W = \{w\}$ where w is isolated in $\langle W \rangle$, then $W' = W \cup \{z\}$ satisfies $|W'| > |W|$ and $P_3 \not\subset \langle W' \rangle$, a contradiction. Similarly, if $\Gamma(z) \cap W = \{w_1\}$ where $w_1 w_2$ is an isolated edge in $\langle W \rangle$, then $W' = (W \setminus \{w_1\}) \cup \{z\}$ satisfies $|W'| = |W|$ and $P_3 \not\subset \langle W' \rangle$. However, $\langle W' \rangle$ has fewer edges than $\langle W \rangle$, a contradiction. Hence

$$2|Z| \leq |E(W, Z)| \leq (n - |Z|)\,\Delta,$$

which gives $|Z| \leq n\Delta/(\Delta+2)$. To see that this bound is sharp, suppose Δ is even, and consider $G \cong mCP(\Delta/2+1)$, where $m = n/(\Delta+2)$ and $CP(r) = \overline{rK_2}$ denotes the *cocktail-party* graph [2, p. 17]. If fewer than $m\Delta$ vertices are deleted from G, then some component retains at least three vertices, and the subgraph spanned by these three contains P_3. Hence any 3-blocking set contains at least $m\Delta = n\Delta/(\Delta+2)$ vertices. □

Next we prove that the examples used to show sharpness in Theorem 2 are unique. The following notation will be used: for $z \notin W$, write $\Gamma_W(z) = \Gamma(z) \cap W$.

Theorem 3. *Suppose G has order n and maximum degree Δ.*

(a) If G has no set of fewer than $n\Delta/(\Delta+1)$ vertices that is 2-blocking, then $G \cong mK_{\Delta+1}$.

(b) If G has no set of fewer than $n\Delta/(\Delta+2)$ vertices is 3-blocking, then Δ is even and $G \cong mCP(\Delta/2+1)$ where $m = n/(\Delta+2)$.

Proof. (a) A review of the above proof shows that Z is a 2-blocking set with $|Z| < (n-|Z|)\Delta$, and so $|Z| < n\Delta/(\Delta+1)$, unless each vertex in W has degree Δ and $|\Gamma_W(z)| = 1$ for each $z \in Z$. Since $|W| = \alpha(G)$, it follows that if $z_1, z_2 \in Z$ have a common neighbor in $w \in W$ then $z_1z_2 \in E(G)$; otherwise, $W \setminus \{w\}) \cup \{z_1, z_2\}$ is an independent set with more than $|W|$ vertices. Hence each component of G is isomorphic to $K_{\Delta+1}$, and it follows that $G \cong mK_{\Delta+1}$ where $m = n/(\Delta+1)$.

(b) A review of the above proof shows that Z is a 3-blocking set with $2|Z| < (n-|Z|)\Delta$, and so $|Z| < n\Delta/(\Delta+2)$, unless $W = V(G) \setminus Z$ is an independent set, each vertex $w \in W$ has degree Δ, and each vertex $z \in Z$ satisfies $|\Gamma_W(z)| = 2$. Let $w \in W$ be arbitrary, and consider $\langle\Gamma(w)\rangle$. We claim that Δ is even and $\langle\Gamma(w)\rangle \cong CP(\Delta/2)$. To prove this claim, we first note that if $z \in \Gamma(w)$ has degree $\Delta - 1$ in $\langle\Gamma(w)\rangle$ then it has degree $(\Delta-1)+2 > \Delta$ in G, a contradiction. Suppose $z \in \Gamma(w)$ has degree $\Delta - 3$ or less in $\langle\Gamma(w)\rangle$. Specifically, suppose that there are distinct vertices $z', z'' \in \Gamma(w)$ such that $zz' \notin E(G)$, and $zz'' \notin E(G)$. We may assume that $\Gamma_W(z) = \{w, w'\}$ where $w' \neq w$. If $\Gamma_W(z') = \{w, w''\}$ where $w'' \neq w'$, then $W' = (W \setminus \{w\}) \cup \{z, z'\}$ satisfies $|W'| > |W|$ and $P_3 \not\subset \langle W'\rangle$, a contradiction. Hence, we conclude that $\Gamma_W(z) = \Gamma_W(z') = \Gamma_W(z'') = \{w, w'\}$. But then $W'' = (W \setminus \{w, w'\}) \cup \{z, z', z''\}$ satisfies $|W''| > |W'|$ and $P_3 \not\subset \langle W''\rangle$, a contradiction. It follows that $\langle\Gamma(w)\rangle \cong CP(\Delta/2)$ as claimed. Since each $z \in Z$ satisfies $|\Gamma_W(z)| = 2$ and belongs to a subgraph of $\langle Z\rangle$ isomorphic to the cocktail-party graph $CP(\Delta/2)$, it follows that G is regular of degree Δ. Again suppose $zz' \notin E(G)$, so $\Gamma_W(z) = \Gamma_W(z') = \{w, w'\}$. Then $(W \setminus \{w, w'\}) \cup \{z, z'\}$ is an independent set. Thus z can play the role initially played by w. It follows that $\Gamma(w') = \Gamma(w)$ and the subgraph spanned by $\Gamma(w) \cup \{w, w'\}$ is isomorphic to $CP(\Delta/2+1)$. Clearly, such a subgraph is a component of G, for else some vertex has degree exceeding Δ. A repetition of this argument yields the fact that $G \cong mCP(\Delta/2+1)$ where $m = n/(\Delta+2)$. □

Question 1. *Is it true that for graphs maximum degree Δ the largest possible 4-blocking ratio is $\Delta/(\Delta+3)$?*

For $\Delta \geq 2$ the graph $G \cong mH$ where $m = n/(\Delta+3)$ and $H \cong \overline{C_{\Delta+3}}$ shows that there is a graph with maximum degree Δ and 4-blocking ratio at least $\Delta/(\Delta+3)$. (If fewer than $n\Delta/(\Delta+3)$ vertices are deleted from G, then there is a set of four vertices left from one of the original components. If X is such a set then $\langle X\rangle_{\overline{G}} \subseteq P_4$ and thus $\langle X\rangle_G \supseteq P_4$.)

4 Grid Graphs

Let $GP(n) = P_n \times P_n$ and $GC(n) = C_n \times C_n$. We shall refer to $GP(n)$ as the *square grid graph.* Specifically, we shall take $GP(n)$ to be the graph with vertex set $V = \{(x,y) |\ 0 \leq x, y < n\}$ in which two vertices are adjacent if their indices agree in one coordinate and differ by exactly one in the other. Note that $GC(n)$ is regular of degree four.

Proposition 2. *If n is even, then $GP(n)$ has 2-blocking ratio $1/2$. The 3-blocking ratio is $1/2$ as well.*

Proof. The proof is practically the same as that for hypercubes. Let $Z = \{(x,y) |\ x+y \equiv 1 \pmod 2\}$. Then $|Z| = n^2/2$ and every edge of $GP(n)$ is incident with a vertex in Z, so Z is 2-blocking. In the other direction, note that $GP(n)$ has a 2-factor consisting of $n^2/4$ C_4's. Any 3-blocking set Z must contain at least two vertices from each of these C_4's, so the 3-blocking ratio is at least $1/2$. □

Proposition 3. *(a) The 4-blocking ratio of $G = GP(n)$ is at most $3/8$.*

(b) The 4-blocking ratio of $GC(n)$ is at least $3/8$.

(c) The 4-blocking ratio of $GP(n)$ converges to $3/8$ as $n \to \infty$.

Proof. (a) Note that for $G \cong GP(n)$ the set

$$Z = \{(x, y) \mid 0 \le x, y < n, \;\; x \pm y \equiv 0 \pmod 4\}$$

is 4-blocking, since each connected component of $G \setminus Z$ is isomorphic to $K_{1,4}$ or some subgraph thereof. Figure 1 shows Z (the darkened vertices) for the case of $n = 8$. In this case $|Z| = (3/8) \cdot 64 = 24$.

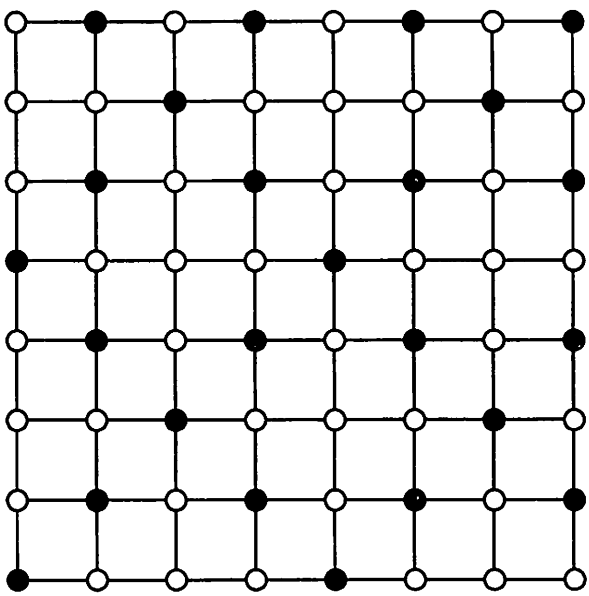


Figure 1. 4-Blocking Set for $GP(8)$

(b) Note that $GC(n)$ has n^2 vertices and is regular of degree 4 so it has $2n^2$ edges. Suppose Z is a 4-blocking set. Then $G \setminus Z$ has $n^2 - |Z|$ vertices and at least $2n^2 - 4|Z|$ edges, and it contains no P_4. Since $G \setminus Z$ contains no P_4, each nontrivial component is isomorphic to $P_2, P_3, K_{1,3}$ or $K_{1,4}$.

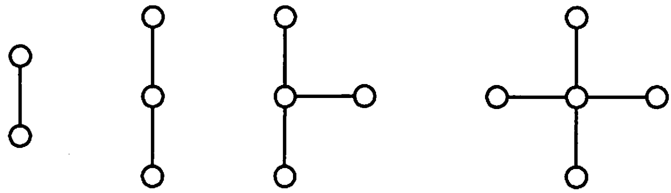


Figure 2. Components of $G \setminus Z$

These graphs have average degree 1, 4/3, 3/2, 8/5, respectively, and it follows that $G \setminus Z$ has average degree at most 8/5. Hence

$$\frac{2(2n^2 - 4|Z|)}{n^2 - |Z|} \leq \frac{8}{5},$$

which gives $|Z|/n^2 \geq 3/8$. (c) Clearly, as $n \to \infty$ the "edge effects" become negligible, so the fact that the 4-blocking ratio for $GC(n)$ is a least 3/8 implies that the 4-blocking ratio for $GP(n)$ is at least $3/8 - \epsilon$ for all sufficiently large n. □

In the same way, we can prove the following result on the 6-blocking ratio.

Proposition 4. *(a) The 6-blocking ratio of $G = GP(n)$ is at most* 1/3. *(b) The 6-blocking ratio of $GC(n)$ is at least* 1/3. *(c) The 6-blocking ratio of $GP(n)$ converges to* 1/3 *as $n \to \infty$.*

An example showing that the 6-blocking ratio of $GP(n)$ is at most 1/3 is

$$Z = \{(x,y)|\ 0 \leq x,y < n,\ \ x+y \equiv 0 \pmod 4 \text{ or } x-y \equiv 0 \pmod 6\}.$$

This set is illustrated for the case $n = 12$ in Fig. 3.

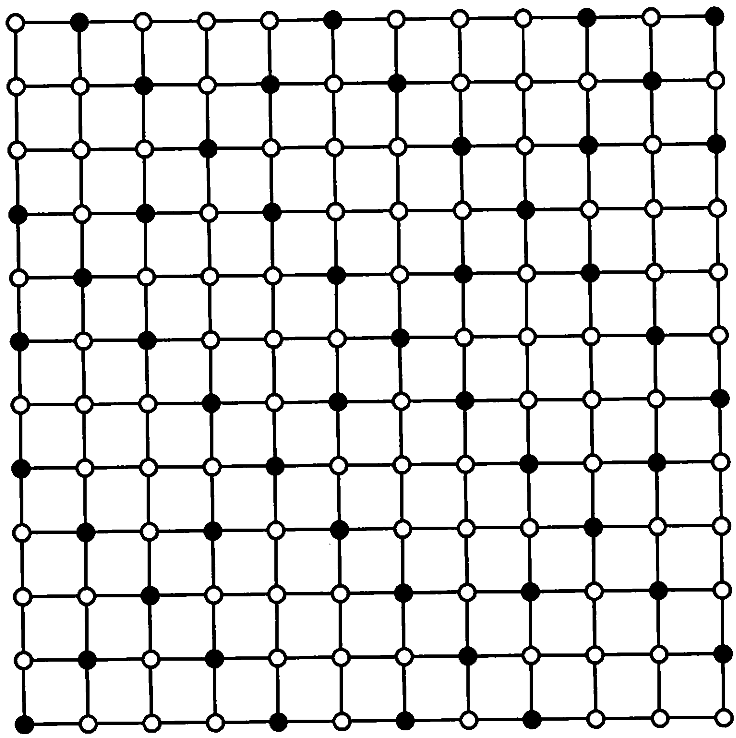


Figure 3. 6-Blocking Set for $GP(12)$

The components of $GP(n) \setminus Z$ are subgraphs of the graph shown in Fig. 4.

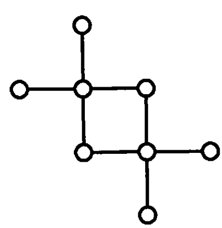


Figure 4.

To prove that the 6-blocking ratio of $G \cong GC(n)$ is at least 1/3, it suffices to check that if Z is any 6-blocking set then each component of $G \setminus Z$ has average degree at most 2. Indeed, a component of $G \setminus Z$ will either be a tree, and thus have average degree less than 2, or else it will contain a C_4, and then it will be a subgraph of the graph shown in Fig. 4. Inspection shows that each such subgraph has average degree at most 2. In view of the average degree condition, we have $2(2n^2 - 4|Z|)/(n^2 - |Z|) \leq 2$, so $|Z|/n^2 \geq 1/3$.

Lemma 1. *If $p < r$, and p vertices are deleted from $GP(r)$, then the resulting graph contains a path of order $(r-p)^2 + p$.*

Proof. The vertices of $GP(r)$ fall into r rows and r columns. The deletion of any $p < r$ vertices leaves $r - p$ rows and $r - p$ columns intact. Then there is an obvious zig-zag path that uses each of the $r - p$ intact rows, and follows the leftmost and rightmost intact columns to go between these rows, as illustrated below. This gives a path with $(r-p)^2 + p$ vertices. □

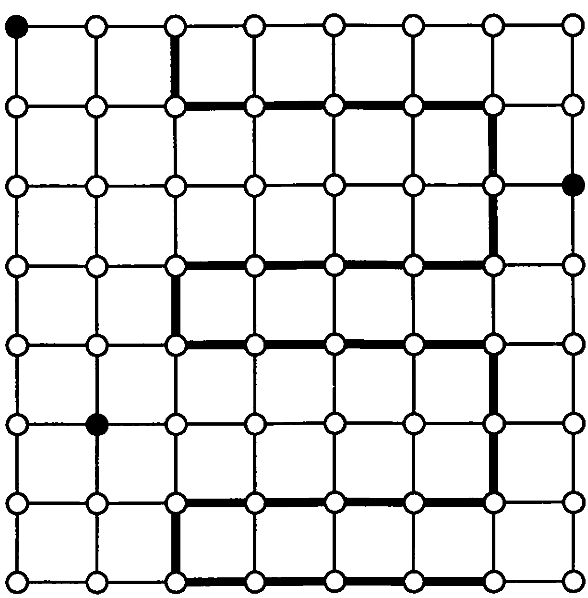


Figure 5. Illustration of the Lemma

Theorem 4. *For $k \geq 3$ and for all sufficiently large values of n, the k-blocking ratio of $GP(n)$ is between $1/(4\sqrt{k})$ and $\sqrt{2/k}$.*

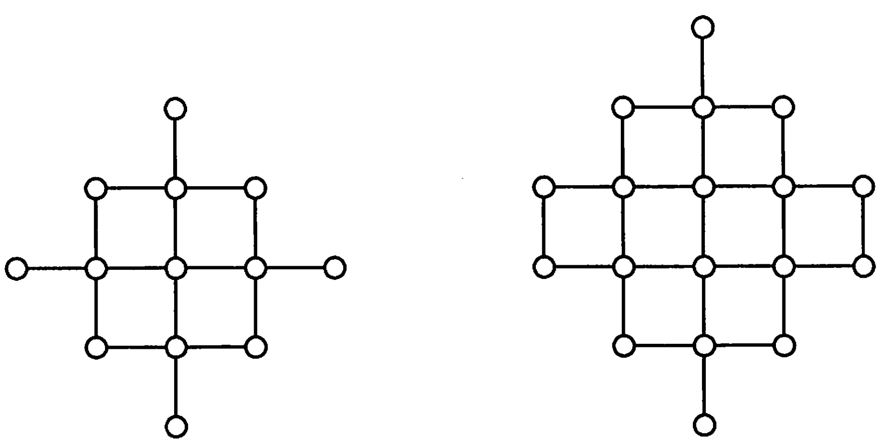
Proof. First we prove that the k-blocking ratio is less than $\sqrt{2/k}$. For this purpose, we use the blocking set

$$Z = \{(x,y) |\ 0 \leq x, y < n,\ \ x \pm y \equiv 0 \pmod{m}\}, \qquad m = \lceil \sqrt{2k} \rceil.$$

For simplicity, assume first that m divides n. Then

$$\frac{|Z|}{n^2} = \begin{cases} \dfrac{2m-2}{m^2}, & m \text{ even,} \\ \\ \dfrac{2m-1}{m^2}, & m \text{ odd,} \end{cases}$$

and a maximal component of $GP(n) \setminus Z$ has $m^2/2 - m + 1$ vertices if m is even and $(m-1)^2/2$ vertices if m is odd. Each component is a subgraph of one of the maximal ones. The maximal components are illustrated in Fig. 6 for for $m = 6$ and $m = 7$.



$m = 6$ $\qquad$ $m = 7$

Figure 6. Components of $GP(n) \setminus Z$

With this choice, $|Z|/n^2 < 2/m < \sqrt{2/k}$ and $P_k \not\subset GP(n) \setminus Z$ since no component of $GP(n) \setminus Z$ has more than k vertices, and the largest component does not have a hamiltonian path. The same conclusion holds in case n is not divisible by m, since $|Z|/n^2 < 2/m$ still holds.

To prove the lower bound, we shall use Lemma 1. Set $d = \lceil \sqrt{k} \rceil = \sqrt{k} + \epsilon$ where $0 \leq \epsilon < 1$. For simplicity, first assume that $2d$ divides n. Then there are $(n/2d)^2$ copies of $GP(2d)$ in $G = GP(n)$. If $|Z| \leq n^2/(4\sqrt{k})$, then a simple averaging argument shows that in $GP(n)$ there is a copy of $GP(2d)$ having at most $(2d)^2/(4\sqrt{k}) = d^2/\sqrt{k}$ vertices in common with Z. By Lemma 1, in $GP(n) \setminus Z$ such a copy contains a path with at least

$$\begin{aligned}\left(2d - \frac{d^2}{\sqrt{k}}\right)^2 + \frac{d^2}{\sqrt{k}} &= d^2\left(2 - \frac{d}{\sqrt{k}}\right)^2 + \frac{d^2}{\sqrt{k}} \\ &= \frac{(k-\epsilon^2)^2}{k} + \sqrt{k} + 2\epsilon + \frac{\epsilon^2}{\sqrt{k}} \\ &\geq k + \sqrt{k}\end{aligned}$$

vertices. Now it is easy to see that the condition $(2d)|n$ can be removed provided n is sufficiently large. By continuity, we can choose $\delta > 0$ so that with $d^2/\sqrt{k}$ replaced by $d^2/\sqrt{k} + \delta$ in the above calculation, the final value is at least k. For all sufficiently large n, there exists a copy of $GP(2d)$ having at most $d^2/\sqrt{k} + \delta$ vertices in common with Z, and this gives the desired result. □

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