A hamiltonian result on L_1 – graphs

Rao Li
School of Computer and Information Sciences
Georgia Southwestern State University
Americus, GA 31709
Email: rl@canes.gsw.edu

Abstract

A graph G is called an L_1 - graph if, for each triple of vertices x, y, and z with $d(x, y) = 2$ and $z \in N(x) \cap N(y)$, $d(x) + d(y) \geq |N(x) \cup N(y) \cup N(z)| - 1$. Let G be a 3 - connected L_1 - graph of order $n \geq 18$. If $\delta(G) \geq n/3$, then every pair of vertices u and v in G with $d(u, v) \geq 3$ is connected by a hamilton path of G.

1. Introduction

We consider only finite undirected graphs without loops and multiple edges. Notation and terminology not defined here follow that in [6]. If $S \subseteq V(G)$, then $N(S)$ denotes the neighbors of S, that is, the set of all vertices in G adjacent to at least one vertex in S. For a subgraph H of G and $S \subseteq V(G) - V(H)$, let $N_H(S) = N(S) \cap V(H)$ and $|N_H(S)| = d_H(S)$. If $S = \{ s \}$, then $N_H(S)$ and $|N_H(S)|$ are written as $N_H(s)$ and $d_H(s)$ respectively. For disjoint subsets A, B of the vertex set $V(G)$ of a graph G, let $e(A, B)$ be the number of the edges in G that join a vertex in A and a vertex in B. For any integer $p \geq 3$, let $(I_p + e)$ denote the graph with p vertices and one edge. We define $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{K}$ as $\{G : K_{p,p} \subseteq G \subseteq K_p + pK_1$ for some $p \geq 3\}$, $\{G : K_{p,p+1} \subseteq G \subseteq K_p + (I_{p+1} + e)$ for some $p \geq 3\}$, and $\mathcal{K} = \mathcal{M} \cup \mathcal{N}$ respectively. If P is a path of G, let $\overrightarrow{P}$ denote the path P with a given orientation. If vertices u, v are on P and u precedes v along the direction of P, then we use $\overrightarrow{P}[u, v]$ to denote the consecutive vertices on P from u to v in the direction specified by $\overrightarrow{P}$. The same set of vertices, in reverse order, is denoted by $\overleftarrow{P}[v, u]$. We use x^- and x^+ to denote the predecessor and successor of a vertex x on P along the orientation of P. If $A \subseteq V(P)$, then A^- and A^+ are defined as $\{v^- : v \in A\}$ and

JCMCC **40** (2002), pp. 79-95

$\{v^+ : v \in A\}$ respectively. A graph G is a claw – free graph if G has no induced subgraph isomorphic to $K_{1,3}$. For an integer i, a graph G is called an L_i – graph if $d(x) + d(y) \geq |N(x) \cup N(y) \cup N(z)| - i$, or equivalently $|N(x) \cap N(y)| \geq |N(z) - (N(x) \cup N(y))| - i$ for each triple of vertices x, y, and z with $d(x, y) = 2$ and $z \in N(x) \cap N(y)$. It can easily be verified that every claw – free graph is an L_1 – graph (see [2]).

The long time interest in claw – free graphs motivates our study of L_1 – graphs. Several authors have already obtained results on the hamiltonian properties of L_i – graphs. Asratian and Khachatrian [4] proved that all connected L_0 – graphs of order at least three are hamiltonian and Saito [11] showed that if a graph G is a 2 – connected L_1 – graph of diameter two then either G is hamiltonian or $G \in \mathcal{K}$. More results related to the hamiltonian properties of L_i – graphs can be found in [1], [2], [3] and [5].

Recently, Li and Schelp [8] extended Matthews and Sumner's theorems [9] on the hamiltonicity and traceability of claw – free graphs to L_1 – graphs.

Theorem 1 *[8] Let G be a 2 – connected L_1 – graph of order n. If $\delta(G) \geq (n-2)/3$, then G is hamiltonian or $G \in \mathcal{K}$.*

Theorem 2 *[8] Let G be a connected L_1 – graph of order n. If $\delta(G) \geq (n-2)/3$, then G is traceable.*

The objective of this paper is to present a result which is related to hamilton – connectedness of L_1 – graphs.

Theorem 3 *Let G be a 3 – connected L_1 – graph of order $n \geq 18$. If $\delta(G) \geq n/3$, then every pair of vertices u and v in G with $d(u, v) \geq 3$ is connected by a hamilton path of G.*

Notice that for L_1 – graphs Asratian et al. [2] presented another condition that gives the same conclusion as in Theorem 3.

Theorem 4 *[2] Let G be a connected L_1 – graph of order at least 3 such that $|N(u) \cup N(v)| \geq 2$ for every pair of vertices with $d(u, v) = 2$. Then every pair of vertices x and y with $d(x, y) \geq 3$ is connected by a hamilton path of G.*

2. Lemmas

Lemma 1 *[7] Suppose G is a 2 – connected graph of order n. Then through each edge of G there passes a cycle of length at least* $\min\{n, 2\delta(G) - 1\}$.

Lemma 2 *[10] Let G be a graph of order n. If for each pair of non-adjacent vertices u and v $d(u) + d(v) \geq n + 1$, then G is hamilton connected.*

Lemma 3 *If G is a 3 – connected L_1 – graph, then either $G \in \mathcal{K}$ or $\omega(G-S) \leq |S|-1$ for every subset S of vertex set $V(G)$ with $\omega(G-S) > 1$, where $\omega(G-S)$ denotes the number of components in the graph $G-S$.*

Lemma 3 is modified from Theorem 5 in [2]. It should be pointed out that the proof of Lemma 3 is similar to that of Theorem 5 in [2]. For the sake of completeness, we still include the proof of Lemma 3 here.

Proof of Lemma 3. In order to prove Lemma 3, we need a fact, which is identical to Lemma 13 in [2], on L_1 – graphs. The proof of this fact can be found in [2].

Fact. Let G be an L_1 – graph, v a vertex of G and $W = \{w_1, w_2, ..., w_k\}$ a subset of $N(v)$ of cardinality k. Assume G contains an independent set $U = \{u_1, u_2, ..., u_k\}$ of cardinality k such that $U \cap (N(v) \cup \{v\}) = \emptyset$ and, for $i = 1, 2, ..., k$, $u_i w_i \in E(G)$ and $N(u_i) \cap (N(v) - W) = \emptyset$. Then $N(w_i) - (N(v) \cup \{v\}) \subseteq N(u_i) \cup U$, $(i = 1, 2, ..., k)$.

Let G be a 3 – connected L_1 – graph and assume that $\omega(G-A) \geq |A|$ for some subset A of vertex set $V(G)$ with $\omega(G-A) > 1$. Let X be a subset of vertex set $V(G)$ of minimum cardinality for which $\omega(G-X) \geq |X|$ and $\omega(G-X) > 1$. Since G is 3 – connected, $l := |X| \geq 3$ and $m := \omega(G-X) \geq |X| = l \geq 3$. Let $H_1, H_2, ..., H_m$ be the components of $G-X$. To prove that $G \in \mathcal{K}$, we need the following claims.

Claim 1. For every nonempty proper subset S of X, $|\{i : N(S) \cap V(H_i) \neq \emptyset\}| \geq |S| + 2$.

Suppose $S \subseteq X$, $\emptyset \neq S \neq X$ and $|\{i : N(S) \cap V(H_i) \neq \emptyset\}| \leq |S| + 1$. Set $T = X - S$. Then $\omega(G-T) \geq m - (|S|+1) + 1 \geq l - |S| = |T|$. This contradiction to the choice of X proves Claim 1.

Choose any subset S of cardinality $l-1$ from X. Then $m \geq |\{i : N(S) \cap V(H_i) \neq \emptyset\}| \geq |S| + 2 = l + 1 \geq 4$.

Claim 2. If $v \notin X$ and $N(v) \cap X \neq \emptyset$, then $X \subseteq N(v)$.

Suppose $v \notin X$ and $N(v) \cap X \neq \emptyset$, but $X \not\subseteq N(v)$. Set $W = N(v) \cap X$ and $k = |W|$. Then $1 \leq k < l$. Let $w_1, w_2, ..., w_k$ be the vertices of W. By Claim 1 and Hall's Theorem (See Bondy and Murty [6, p. 72]), $N(W) - X$ contains a subset $U := \{u_1, u_2, ..., u_k\}$ of cardinality k such that no two vertices of $U \cup \{v\}$ are in the same componment of $G-X$ and $u_1w_1, u_2w_2, ..., u_kw_k \in E(G)$. By the above fact, we have

$N(w_i) - (N(v) \cup \{v\}) \subseteq N(u_i) \cup U$ for each i, $1 \le i \le k$. But then $|\{i : N(S) \cap V(H_i) \neq \emptyset\}| \le k+1 \le |W|+1$. This contradiction to Claim 1 proves Claim 2.

For each i, $1 \le i \le m$, let y_i be a vertex in H_i with $N(y_i) \cap X \neq \emptyset$. Set $Y = \{y_1, y_2, ..., y_m\}$. By Claim 2, $X \subseteq N(y_i)$ for each i, $1 \le i \le m$, implying $Y \subseteq N(x)$, where x is any vertex in X. Since G is an L_1 - graph, we have, for any vertex x in X and any pair of distinct vertices y_i and y_j in Y,

$$\begin{aligned} 0 &\le |N(y_i) \cap N(y_j)| - |N(x) - (N(y_i) \cup N(y_j))| + 1 \\ &= |X| - |N(x) - (N(y_i) \cup N(y_j))| + 1 \\ &\le |X| - |Y| + 1 = l - m + 1 \le 1. \end{aligned}$$

Therefore $|X| - |N(x) - (N(y_i) \cup N(y_j))| + 1 = 0$ or 1 for any vetex x in X and any pair of distinct vertices y_i and y_j in Y. The remainder of our proof is divided into several cases.

Case 1. $|X| - |N(x) - (N(y_i) \cup N(y_j))| + 1 = 1$ for some vertex x in X and some pair of distinct vertices y_i and y_j in Y.

Then $N(x) - (N(y_i) \cup N(y_j)) = Y$ and $m = l$. Therefore $N(x) \cap V(H_k) = \{y_k\}$ for each $k \in \{1, 2, ..., m\} - \{i, j\}$. Furthermore, for each $x_1 \in X - \{x\}$ and each $k \in \{1, 2, ..., m\} - \{i, j\}$, $N(x_1) \cap V(H_k) = \{y_k\}$; otherwise there exists a vertex $y_0 \notin X - (V(H_i) \cup V(H_j))$ and $N(y_0) \cap X \neq \emptyset$. Thus by Claim 2 $X \subseteq N(y_0)$, contradicting to $N(x) \cap V(H_k) = \{y_k\}$ for each $k \in \{1, 2, ..., m\} - \{i, j\}$. Since G is 3 - connected, $V(H_k) = \{y_k\}$ for each $k \in \{1, 2, ..., m\} - \{i, j\}$.

Case 1.1. $N(x) \cap [(V(H_i) - \{y_i\}) \cup (V(H_j) - \{y_j\})] = \emptyset$.

Then $N(x_1) \cap [(V(H_i) - \{y_i\}) \cup (V(H_j) - \{y_j\})] = \emptyset$ for each $x_1 \in X - \{x\}$; otherwise there exists a vertex $y_0 \in [(V(H_i) - \{y_i\}) \cup (V(H_j) - \{y_j\})]$ with $N(y_0) \cap X \neq \emptyset$, thus by Claim 2 $X \subseteq N(y_0)$, in particular, $y_0 \in N(x)$, contradicting to the assumption of this case. Since G is 3 - connected, $V(H_k) = \{y_k\}$ for $k = i$ and $k = j$. Therefore $G \in \mathcal{M} \subseteq \mathcal{K}$.

Case 1.2. $N(x) \cap [(V(H_i) - \{y_i\}) \cup (V(H_j) - \{y_j\})] \neq \emptyset$.

If there exists a vertex y_0 such that $y_0 \in N(x) \cap (V(H_i) - \{y_i\}) \neq \emptyset$ and $N(x) \cap (V(H_j) - \{y_j\}) = \emptyset$. Then by Claim 2 we have $X \subseteq N(y_0)$. Since G is an L_1 - graph, we have, for any $x_1 \in X - \{x\}$ and any distinct pair of vertices $y_{i_1}, y_{j_1} \in Y - \{y_i\}$

$$\begin{aligned}0 &\le |N(y_{i_1}) \cap N(y_{j_1})| - |N(x_1) - (N(y_{i_1}) \cup N(y_{j_1}))| + 1 \\ &= |X| - |N(x_1) - (N(y_{i_1}) \cup N(y_{j_1}))| + 1 \\ &\le |X| - |Y \cup \{ y_0 \}| + 1 = l - m \le 0.\end{aligned}$$

Thus $N(x_1) - (N(y_{i_1}) \cup N(y_{j_1})) = Y \cup \{ y_0 \}$ for any $x_1 \in X - \{ x \}$ and any distinct pair of vertices $y_{i_1}, y_{j_1} \in Y - \{ y_i \}$. Then $N(X - \{ x \}) \cap V(H_i) = \{y_0, y_i\}$ and $N(X - \{ x \}) \cap V(H_j) = \{ y_j \}$. Moreover, $N(x) \cap V(H_i) = \{y_0, y_i\}$ and $N(x) \cap V(H_j) = \{ y_j \}$; otherwise by Claim 2 there exists a vertex $y_{m+1} \in [(V(H_i) - \{y_0, y_i\}) \cup (V(H_j) - \{ y_j \})]$ such that $X \subseteq N(y_{m+1})$, contradicting to either $N(X - \{ x \}) \cap V(H_i) = \{y_0, y_i\}$ or $N(X - \{ x \}) \cap V(H_j) = \{ y_j \}$. Thus $N(X) \cap V(H_i) = \{y_0, y_i\}$ and $N(X) \cap V(H_j) = \{ y_j \}$. Since G is 3 - connected, $V(H_i) = \{y_0, y_i\}$ and $V(H_j) = \{ y_j \}$. Therefore $G \in \mathcal{N} \subseteq \mathcal{K}$.

If there exists a vertex y_0 such that $y_0 \in N(x) \cap (V(H_j) - \{ y_j \}) \neq \emptyset$ and $N(x) \cap (V(H_i) - \{ y_i \}) = \emptyset$, then by a similar argument as before we can prove that $G \in \mathcal{N} \subseteq \mathcal{K}$.

Next we will show that it is impossible for $N(x) \cap (V(H_i) - \{ y_i \}) \neq \emptyset$ and $N(x) \cap (V(H_j) - \{ y_j \}) \neq \emptyset$ hold simultaneously. Suppose not, then there exist vertices y_0 and y_{m+1} such that $y_0 \in N(x) \cap (V(H_i) - \{ y_i \}) \neq \emptyset$ and $y_{m+1} \in N(x) \cap (V(H_j) - \{ y_j \}) \neq \emptyset$. Thus by Claim 2 we have $X \subseteq N(y_0)$ and $X \subseteq N(y_{m+1})$. Let x_1 be a vertex in $X - \{ x \}$ and y_{i_1}, y_{j_1} two distinct vertices in $Y - \{ y_i, y_j \}$ (since $|Y| = m \ge 4$). Then we have following contradiction,

$$\begin{aligned}0 &\le |N(y_{i_1}) \cap N(y_{j_1})| - |N(x_1) - (N(y_{i_1}) \cup N(y_{j_1}))| + 1 \\ &= |X| - |N(x_1) - (N(y_{i_1}) \cup N(y_{j_1}))| + 1 \\ &\le |X| - |Y \cup \{ y_0, y_{m+1} \}| + 1 = l - m - 1 \le -1.\end{aligned}$$

Case 2. $|X| - |N(x) - (N(y_i) \cup N(y_j))| + 1 = 0$ for some vertex x in X and some pair of distinct vertices y_i and y_j in Y.

Case 2.1. $|X| - |Y| + 1 = 0$.

Then $N(x) - (N(y_i) \cup N(y_j)) = Y$ for vertex x in X and distinct vertices y_i and y_j in Y and $m = l + 1$. Using a similar argumnent as in Case 1, we can show that $V(H_k) = \{ y_k \}$ for each $k \in \{ 1, 2, ..., m \} - \{ i, j \}$.

Case 2.1.1. $N(x) \cap [(V(H_i) - \{ y_i \}) \cup (V(H_j) - \{ y_j \})] = \emptyset$.

Using a similar argument as in Case 1.1, we can prove that $V(H_k) = \{ y_k \}$ for $k = i$ and $k = j$. Therefore $G \in \mathcal{N} \subseteq \mathcal{K}$.

Case 2.1.2. $N(x) \cap [(V(H_i) - \{ y_i \}) \cup (V(H_j) - \{ y_j \})] \neq \emptyset$.

If there exists a vertex y_0 such that $y_0 \in N(x) \cap (V(H_i) - \{ y_i \}) \neq \emptyset$ and $N(x) \cap (V(H_j) - \{ y_j \}) = \emptyset$, then by Claim 2 we have $X \subseteq N(y_0)$. Since G is an L_1 - graph, we have, for any $x_1 \in X - \{ x \}$ and any distinct pair of vertices $y_{i_1}, y_{j_1} \in Y - \{ y_i \}$,

$$\begin{aligned} 0 &\leq |N(y_{i_1}) \cap N(y_{j_1})| - |N(x_1) - (N(y_{i_1}) \cup N(y_{j_1}))| + 1 \\ &= |X| - |N(x_1) - (N(y_{i_1}) \cup N(y_{j_1}))| + 1 \\ &\leq |X| - |Y \cup \{ y_0 \}| + 1 = l - m \leq 0. \end{aligned}$$

Thus $m = l$, contradicting to $m = l + 1$.

If there exists a vertex y_0 such that $y_0 \in N(x) \cap (V(H_j) - \{ y_j \}) \neq \emptyset$ and $N(x) \cap (V(H_i) - \{ y_i \}) = \emptyset$, then by a similar argument as before we can derive a contradiction.

If there exist vertices y_0 and y_{m+1} such that $y_0 \in N(x) \cap (V(H_i) - \{ y_i \}) \neq \emptyset$ and $y_{m+1} \in N(x) \cap (V(H_j) - \{ y_j \}) \neq \emptyset$, then by Claim 2 we have $X \subseteq N(y_0)$ and $X \subseteq N(y_{m+1})$. Let x_1 be a vertex in $X - \{ x \}$ and y_{i_1}, y_{j_1} two distinct vertices in $Y - \{ y_i, y_j \}$ (since $|Y| = m \geq 4$). Then we have following contradiction,

$$\begin{aligned} 0 &\leq |N(y_{i_1}) \cap N(y_{j_1})| - |N(x_1) - (N(y_{i_1}) \cup N(y_{j_1}))| + 1 \\ &= |X| - |N(x_1) - (N(y_{i_1}) \cup N(y_{j_1}))| + 1 \\ &\leq |X| - |Y \cup \{ y_0, y_{m+1} \}| + 1 = l - m - 1 \leq -1. \end{aligned}$$

Therefore Case 2.1.2 cannot occur.

Case 2.2. $|X| - |Y| + 1 = 1$.

Then $N(x) - (N(y_i) \cup N(y_j)) = Y \cup \{ y_0 \}$ for some vertex $y_0 \in N(x) - (N(y_i) \cup N(y_j) \cup Y) \subseteq V(G) - X$ and $m = l$.

Case 2.2.1. $y_0 \in V(H_k) - \{ y_k \}$ for some $k \neq i$ and $k \neq j$.

By Claim 2, we have $X \subseteq N(y_0)$. Let x_1 be any vertex in $X - \{ x \}$ and y_{i_1}, y_{j_1} any pair of distinct vertices in $Y - \{ y_k \}$. Since G is an L_1 - graph, we have

$$
\begin{aligned}
0 &\leq |N(y_{i_1}) \cap N(y_{j_1})| - |N(x_1) - (N(y_{i_1}) \cup N(y_{j_1}))| + 1 \\
&= |X| - |N(x_1) - (N(y_{i_1}) \cup N(y_{j_1}))| + 1 \\
&\leq |X| - |Y \cup \{ y_0 \}| + 1 = l - m \leq 0.
\end{aligned}
$$

Then $N(x_1) - (N(y_{i_1}) \cup N(y_{j_1})) = Y \cup \{ y_0 \}$ for any x_1 in $X - \{ x \}$ and any pair of distinct vertices y_{i_1}, y_{j_1} in $Y - \{ y_k \}$. Thus $N(X - \{ x \}) \cap V(H_k) = \{y_0, y_k\}$ and $N(X - \{ x \}) \cap V(H_i) = \{ y_i \}$ for $i \neq k$. Moreover, $N(x) \cap V(H_k) = \{y_0, y_k\}$ and $N(x) \cap V(H_i) = \{ y_i \}$ for $i \neq k$; otherwise there exists a vertex $y_{m+1} \in V(G) - (X \cup Y)$ such that $N(y_{m+1}) \cap X \neq \emptyset$ and Claim 2 implies that $X \subseteq N(y_{m+1})$, contradicting to either $N(X - \{ x \}) \cap V(H_k) = \{y_0, y_k\}$ or $N(X - \{ x \}) \cap V(H_i) = \{ y_i \}$ for $i \neq k$. Since G is 3 – connected, $V(H_k) = \{y_0, y_k\}$ and $V(H_i) = \{ y_i \}$ for $i \neq k$. Therefore $G \in \mathcal{N} \subseteq \mathcal{K}$.

Case 2.2.2. $y_0 \in V(H_i) - \{ y_i \}$ or $y_0 \in V(H_i) - \{ y_i \}$.

Without loss of generality, we assume that $y_0 \in V(H_i) - \{ y_i \}$. Then by Claim 2 we have $X \subseteq N(y_0)$. Since G is an L_1 – graph and $y_0 \notin N(y_i) \cup N(y_j) \cup Y$, we have, for any vertex x_1 in $X - \{ x \}$ and any pair of distinct vertices y_i and y_j in Y,

$$
\begin{aligned}
0 &\leq |N(y_i) \cap N(y_j)| - |N(x_1) - (N(y_i) \cup N(y_j))| + 1 \\
&= |X| - |N(x_1) - (N(y_i) \cup N(y_j))| + 1 \\
&\leq |X| - |Y \cup \{ y_0 \}| + 1 = l - m \leq 0.
\end{aligned}
$$

Then $N(x_1) - (N(y_i) \cup N(y_j)) = Y \cup \{ y_0 \}$ for any x_1 in $X - \{ x \}$ and any pair of distinct vertices y_i, y_j in Y. Using a similar argument as in the last part of Case 2.2.1, we have $G \in \mathcal{N} \subseteq \mathcal{K}$.

Case 3. $|X| - |N(x) - (N(y_i) \cup N(y_j))| + 1 = 0$ for some vertex x in X and any pair of distinct vertices y_i and y_j in Y.

Case 4. $|X| - |N(x) - (N(y_i) \cup N(y_j))| + 1 = 0$ for any vertex x in X and some pair of distinct vertices y_i and y_j in Y.

Case 5. $|X| - |N(x) - (N(y_i) \cup N(y_j))| + 1 = 0$ for any vertex x in X and any pair of distinct vertices y_i and y_j in Y.

Since the conditions in Cases 3, 4, and 5 all imply the conditions in Case 2, we have $G \in \mathcal{K}$.

Therefore we complete the proof of Lemma 3.

3. Proof of Theorem 3

Proof of Theorem 3. Let G be a graph satisfying the conditions in Theorem 3. Suppose that there exists a pair of vertices u and v in G with $d(u,v) \geq 3$ and no hamilton path in G connects u and v. Construct a graph $G_1 = G + uv$. Clearly, $\delta(G_1) \geq \delta(G)$. By Lemma 1 uv is contained in a cycle of length at least $2\delta(G_1) - 1$ in G_1. Thus in G there exists a path from u and v such that the order of that path is at least $2\delta(G_1) - 1 \geq 2\delta(G) - 1$. Choose any longest path $P = P[u,v]$ between u and v in G and assume that it is orientated from u to v. Then $|V(P)| \geq 2\delta(G)-1$. Let H be any component of the subgraph $G[V(G)-V(P)]$. Then $|V(H)| \leq n - |V(P)| \leq 3\delta(G) - (2\delta(G) - 1) = \delta(G) + 1$. We further claim that there exists a vertex q in H such that $|N(q) \cap V(P)| \geq 3$. Suppose not, then for each vertex q in H, $d_P(q) \leq 2$. Thus $d_H(q) \geq d(q) - 2$ and $|V(H)| \geq 1+d_H(q) \geq d(q)-1 \geq \delta(G)-1$. Thus for any pair of non-adjacent vertices x and y in H $d_H(x)+d_H(y) \geq 2\delta(G)-4 \geq \delta(G)+2 \geq |V(H)|+1$. Hence by Lemma 2 H is hamilton connected. Since G is 3 – connected, there exist three distinct vertices x_1, x_2, and x_3 in H and three distinct vertices y_1, y_2, and y_3 on P such that x_1y_1, x_2y_2, and x_3y_3 are in E. Here we assume that y_1, y_2, and y_3 are ordered with increasing index in the direction of P. Since P is a longest path connecting u and v and H is hamilton connected, $|\overrightarrow{P}[y_1^+, y_2^-]| \geq |V(H)|$ and $|\overrightarrow{P}[y_2^+, y_3^-]| \geq |V(H)|$. Thus $n \geq |\overrightarrow{P}[u, y_1^-]|+|\overrightarrow{P}[y_1^+, y_2^-]|+|\overrightarrow{P}[y_2^+, y_3^-]|+|\overrightarrow{P}[y_3^+, v]|+|\{y_1, y_2, y_3\}|+|V(H)| \geq 3|V(H)| + 3 \geq 3(\delta(G) - 1) + 3 \geq n$. Therefore $y_1 = u$, $y_3 = v$, and $|V(H)| = \delta(G) - 1$. Moreover, $d_H(x_1) = d_H(x_3) = \delta(G) - 2$ and $d_P(x_1) = d_P(x_3) = 2$. Since $d(u,v) \geq 3$, $x_1v \notin E$. Also $x_1p \notin E$ for each vertex $p \in V(P) - \{y_1, y_2, y_3\}$; otherwise by the hamilton connectedness of H it can be derived that $|V(P)| \geq 2|V(H)| + 5$ when $p \in \overrightarrow{P}[y_1^+, y_2^-]$ or $|V(P) \geq 3|V(H)| + 4$ when $p \in \overrightarrow{P}[y_2^+, y_3^-]$, both imply a contradiction of $|V(P)| + |H| \geq n + 1$. Thus $x_1y_2 \in E$. Similarly, $x_3y_2 \in E$. Next we will show that $y_2^- y_2^+ \in E$. Suppose not. Since G is an L_1 – graph and $d(y_2^-, x_1) = 2$, $d(y_2^-) + d(x_1) \geq |N(y_2^-) \cup N(y_2) \cup N(x_1)| - 1$, i. e., $|N(y_2^-) \cap N(x_1)| \geq |N(y_2) - (N(y_2^-) \cup N(x_1))| - 1 \geq |\{y_2^-, y_2^+, x_1\}| - 1 = 2$. So $y_2^- u \in E$. Similarly, $|N(y_2^-) \cap N(x_3)| \geq 2$ and $y_2^- v \in E$. This contradicts the assumption of $d(u,v) \geq 3$. Thus $y_2^- y_2^+ \in E$. Again by the choice of P and hamilton connectedness of H, we have $n \geq |V(P)| + |V(H)| \geq |\overrightarrow{P}[y_1^+, y_2^{--}]| + |\overrightarrow{P}[y_2^{++}, y_3^-]| + |\{y_1, y_2^-, y_2, y_2^+, y_3\}| + |V(H)| \geq |V(H)| + |V(H)| + 5 + |V(H)| = 3\delta(G) + 2 \geq n + 2$, a contradiction.

Choose a vertex q in $V(H)$ such that $|N(q) \cap V(P)| \geq 3$. Let A be the set $N(q) \cap V(P) := \{a_1, a_2, ..., a_l\}$ with the a_i's ordered with increasing index in the direction of P. Let b_i and d_i be the predecessor and successor of a_i

along P respectively, $1 \le i \le l$. Set $B := \{b_1, b_2, ..., b_l\}$, $D := \{d_1, d_2, ..., d_l\}$.

Since $d(u, v) \ge 3$, uq and vq cannot be in E simultaneously. Thus the remainder of our proof can be divided into the following cases.

Case 1. $u \ne a_1$ and $v \ne a_l$.

Clearly, $B \cup \{q\}$ is independent and $N(b_i) \cap N(q) \cap (V(G) - V(P)) = \emptyset$ for each i, $1 \le i \le l$. Moreover, $d(q, b_i) = 2$ and $a_i \in N(q) \cap N(b_i)$ for each i, $1 \le i \le l$. Since G is an L_1 - graph, we have

$$|N(q) \cap N(b_i)| \ge |N(a_i) - (N(q) \cup N(b_i))| - 1.$$

Obviously, $N_B(a_i) \subseteq N(a_i) - (N(q) \cup N(b_i) \cup \{q\})$. Thus,

$|N_B(a_i)| \le |N(a_i) - (N(q) \cup N(b_i))| - 1$. Therefore,

$|N_B(a_i)| \le |N(q) \cap N(b_i)| = |N_A(b_i)|$. Hence,

$e(A, B) = \sum_{i=1}^{l} |N_B(a_i)| \le \sum_{i=1}^{l} |N_A(b_i)| = e(A, B)$.

It follows, for each i, $1 \le i \le l$, that

$$N(a_i) - (N(q) \cup N(b_i) \cup \{q\}) = N_B(a_i) \subseteq B. \quad (1)$$

Similarly, for each i, $1 \le i \le l$,

$$N(a_i) - (N(q) \cup N(d_i) \cup \{q\}) = N_D(a_i) \subseteq D. \quad (2)$$

We claim that for each i, $1 \le i \le l$, $b_i d_i \in E$ and for each i, $1 \le i \le l-1$, $b_{i+1} \ne d_i$. First of all, we show that $b_1 d_1 \in E$. Suppose not. Then $b_1 \in N(a_1) - (N(q) \cup N(d_1) \cup \{q\})$ and by (2) we have $b_1 \in D$, a contradiction. Since $b_1 d_1 \in E$, $b_2 \ne d_1$; otherwise P would not be a longest path connecting u and v. Using a similar argument as before, we have $b_2 d_2 \in E$ and $b_3 \ne d_2$. Repeating this process, we have for each i, $1 \le i \le l$, $b_i d_i \in E$ and for each i, $1 \le i \le l - 1$, $b_{i+1} \ne d_i$.

Let s be the first vertex on $\overrightarrow{P}[d_1, b_2]$ such that $b_1 s \notin E(G)$. The existence of s is guaranteed by the fact that $b_2 \notin N(b_1)$. Also observe that $s \notin N(a_1)$; otherwise by (1) we have $s \in N(a_1) - (N(q) \cup N(b_1) \cup \{q\}) = N_B(a_1) \subseteq B$, so $s = b_2$. Then G has a path

$$\overrightarrow{P}[u, b_1]\overrightarrow{P}[d_1, b_2]a_1 q \overrightarrow{P}[a_2, v]$$

between u and v which is longer than P, contradicting the choice of P. Similarly, there exists a vertex t which is the first vertex on $\overrightarrow{P}[d_2, b_3]$ such that $b_2t \notin E(G)$ and $t \notin N(a_2)$.

The proof for this case is completed by counting the degree sum of the vertices q, s, and t and arriving at a contrdiction. To accomplish this, we set

$$V_1 = \overrightarrow{P}[u, b_1],$$

$$V_2 = \overrightarrow{P}[a_1, s^-],$$

$$V_3 = \overrightarrow{P}[s, b_2],$$

$$V_4 = \overrightarrow{P}[a_2, t^-],$$

$$V_5 = \overrightarrow{P}[t, v],$$

$$V_6 = V(G) - V(P).$$

In the remainder of our proof, we, for each vertex w in G, simplify some notation letting $N_i(w)$ replace $N_{V_i}(w)$, $|N_i(w)| = d_i(w)$, $(N_i(w))^- = N_i^-(w)$, and $(N_i(w))^+ = N_i^+(w)$.

Clearly, $d_1(q) = 0$. Also $N_1^+(s) \cap N_1(t) = \emptyset$; otherwise in G there is a path between u and v which is longer than P. Therefore,

$$d_1(q) + d_1(s) + d_1(t) = |N_1^+(s)| + |N_1(t)| = |N_1^+(s) \cup N_1(t)| \leq |V_1|.$$

Notice that $N(q) \cap (V_2 - \{a_1\}) = \emptyset$ and $N(t) \cap V_2 = \emptyset$; otherwise G has paths between u and v which are longer than P. Therefore,

$$d_2(q) + d_2(s) + d_2(t) \leq 1 + |V_2| - 1 = |V_2|.$$

Clearly, $d_3(q) = 0$. Also $N_3(s) \cap N_3^+(t) = \emptyset$; otherwise in G there is a path between u and v which is longer than P. Therefore,

$$d_3(q)+d_3(s)+d_3(t) = |N_3(s)|+|N_3^+(t)| = |N_3(s) \cup N_3^+(t)| \leq |V_3 - \{s\}| = |V_3| - 1.$$

Notice that $N(q) \cap (V_4 - \{a_2\}) = \emptyset$ and $N(s) \cap V_4 = \emptyset$; otherwise G has paths between u and v which are longer than P. Therefore,

$d_4(q) + d_4(s) + d_4(t) \le 1 + |V_4| - 1 = |V_4|$.

We further notice that $N_5(q) \cap N_5(s) = \emptyset$, $N_5(s) \cap N_5^-(t) = \emptyset$, and $N_5^-(t) \cap N_5(q) = \emptyset$; otherwise in G there are again paths between u and v which are longer than P. Therefore,

$d_5(q) + d_5(s) + d_5(t) = |N_5(q)| + |N_5(s)| + |N_5^-(t)| = |N_5(q) \cup N_5(s) \cup N_5^-(t)| \le |V_5|$.

Obviously, $d_6(q) + d_6(s) + d_6(t) \le |V_6| - 1$.

Hence, $n \le d(q) + d(s) + d(t) \le \sum_{i=1}^{6}(d_i(q) + d_i(s) + d_i(t)) \le n - 2$, a contradiction.

Case 2. $u = a_1$ and $v \ne a_l$.

Notice that in this case we still have

$$N(a_i) - (N(q) \cup N(d_i) \cup \{q\}) = N_D(a_i) \subseteq D. \qquad (\star)$$

Subase 2.1. $d_i \ne b_{i+1}$ for each i, $1 \le i \le l-1$.

Because of $(\star)$, we can still prove that $b_i d_i \in E$ for each i, $2 \le i \le l$. Let s be the first vertex on $\overleftarrow{P}[b_2, d_1]$ such that $d_2 s \notin E(G)$ and t be the first vertex on $\overleftarrow{P}[b_3, d_2]$ such that $d_3 t \notin E(G)$. The existence of s and t is ensured by the facts of $d_2 d_1 \notin E$ and $d_3 d_2 \notin E$. As before, $s \notin N(a_2)$ and $t \notin N(a_3)$ (in fact, $s \notin N(a_i)$ and $t \notin N(a_i)$ for each i, $2 \le i \le l$). Next we again count the degree sum of vertices q, s, and t and derive a contradiction. Set

$$V_1 = \overrightarrow{P}[u, s],$$

$$V_2 = \overrightarrow{P}[s^+, a_2],$$

$$V_3 = \overrightarrow{P}[d_2, t],$$

$$V_4 = \overrightarrow{P}[t^+, a_3],$$

$$V_5 = \overrightarrow{P}[d_3, v],$$

$$V_6 = V(G) - V(P).$$

Clearly, $d_1(q) = 1$. Also $N_1^+(s) \cap N_1(t) = \emptyset$; otherwise in G there is a path between u and v which is longer than P. Therefore,

$d_1(q) + d_1(s) + d_1(t) = 1 + |N_1^+(s)| + |N_1(t)| = 1 + |N_1^+(s) \cup N_1(t)| \leq 1 + |V_1|$.

Notice that $N(q) \cap (V_2 - \{a_2\}) = \emptyset$ and $N(t) \cap V_2 = \emptyset$; otherwise G has paths between u and v which are longer than P. Therefore,

$d_2(q) + d_2(s) + d_2(t) \leq 1 + |V_2| - 1 = |V_2|$.

Clearly, $d_3(q) = 0$. Also $N_3(s) \cap N_3^+(t) = \emptyset$; otherwise in G there is a path between u and v which is longer than P. Therefore,

$d_3(q) + d_3(s) + d_3(t) = |N_3(s)| + |N_3^+(t)| = |N_3(s) \cup N_3^+(t)| \leq |V_3 - \{d_2\}| = |V_3| - 1$.

Notice that $N(q) \cap (V_4 - \{a_3\}) = \emptyset$ and $N(s) \cap V_4 = \emptyset$; otherwise G has paths between u and v which are longer than P. Therefore,

$d_4(q) + d_4(s) + d_4(t) \leq 1 + |V_4| - 1 = |V_4|$.

We further notice that $N_5^-(q) \cap N_5(s) = \emptyset$, $N_5(s) \cap N_5^-(t) = \emptyset$, and $N_5^-(t) \cap N_5^-(q) = \emptyset$; otherwise in G there are again paths between u and v which are longer than P. Therefore,

$d_5(q) + d_5(s) + d_5(t) = |N_5^-(q)| + |N_5(s)| + |N_5^-(t)| = |N_5^-(q) \cup N_5(s) \cup N_5^-(t)| \leq |V_5|$.

Obviously, $d_6(q) + d_6(s) + d_6(t) \leq |V_6| - 1$.

Hence, $n \leq d(q) + d(s) + d(t) \leq \sum_{i=1}^{6}(d_i(q) + d_i(s) + d_i(t)) \leq n - 1$, a contradiction.

Subcase 2.2. There exist i and j, $1 \leq i, j \leq l-1$, such that $d_i = b_{i+1}$ and $d_j \neq b_{j+1}$.

Let $k := \min\{j : d_j \neq b_{j+1}, 1 \leq j \leq l-1\}$. Then using $(\star)$ we can prove that $b_m d_m \in E$ and $b_m \neq d_{m-1}$ for each m, $k+1 \leq m \leq l$. Since the set $\{i : d_i = b_{i+1}, 1 \leq i \leq l-1\}$ is nonempty, we have $k \geq 2$ and $d_i = b_{i+1}$ for each i, $1 \leq i \leq k-1$. Let $s = u^+$ and $t = b_{k+1}$. Again, we derive a contradiction by finding a upper bound on the degree sum of vertices q, s, and t. Set

$$V_1 = \overrightarrow{P}[u, s],$$

$$V_2 = \overrightarrow{P}[s^+, a_k],$$

$$V_3 = \overrightarrow{P}[d_k, t],$$

$$V_4 = \overrightarrow{P}[a_{k+1}, a_l],$$

$$V_5 = \overrightarrow{P}[d_l, v],$$

$$V_6 = V(G) - V(P).$$

Clearly, $d_1(q) = d(s) = 1$ and $st \notin E$. Notice that $ut \notin E$; otherwise by $(\star)$ we have $t \in D$, a contradiction. Therefore,

$d_1(q) + d_1(s) + d_1(t) = 2 = |V_1|$.

Clearly, $sd_i \notin E$ and $td_i \notin E$ for each i, $2 \leq i \leq k-1$. Also $ta_i \notin E$ for each i, $2 \leq i \leq k-1$; otherwise by $(\star)$ we have $t \in D$, a contradiction. Therefore,

$d_2(q) + d_2(s) + d_2(t) \leq (|V_2| + 1)/2 + (|V_2| + 1)/2 + |\{a_k\}| = |V_2| + 2.$

Clearly, $d_3(q) = 0$. Also $N_3(s) \cap N_3^+(t) = \emptyset$; otherwise in G there is a path between u and v which is longer than P. Therefore,

$d_3(q) + d_3(s) + d_3(t) = |N_3(s)| + |N_3^+(t)| = |N_3(s) \cup N_3^+(t)| \leq |V_3 - \{d_k\}| = |V_3| - 1.$

Notice that $N_4(q) \cap N_4^+(s) = \emptyset$, $N_4^+(s) \cap N(t) = \emptyset$ and $N(t) \cap N(q) = \emptyset$; otherwise G has paths between u and v which are longer than P. Clearly, $a_l \notin N_4(s)$, $b_l \notin N_4(q)$, and $b_l \notin N_4(t)$. It is observed that $b_l \notin N_4^+(s)$; otherwise G has a path

$$us\overleftarrow{P}[b_l^-, s^+]qa_lb_l^-\overrightarrow{P}[b_l^+, v]$$

between u and v which is longer than P, a contradiction. Therefore,

$d_4(q) + d_4(s) + d_4(t) = |N_4(q)| + |N_4^+(s)| + |N(t)| = |N_4(q) \cup N_4^+(s) \cup N(t)| \leq |V_4 - \{b_l\}| = |V_4| - 1.$

Clearly, $d_5(q) = 0$. Notice that $N_5^+(s) \cap N_5(t) = \emptyset$; otherwise G has a path from u to v which is longer than P. Since $d(u,v) \geq 3$, $v \notin N(s)$. Thus $N_5^+(s) \subseteq V_5$. Therefore,

$$d_5(q) + d_5(s) + d_5(t) = |N_5^+(s)| + |N_5(t)| = |N_5^+(s) \cup N_5(t)| \leq |V_5|.$$

Obviously, $d_6(q) + d_6(s) + d_6(t) \leq |V_6| - 1$.

Hence, $n \leq d(q) + d(s) + d(t) \leq \sum_{i=1}^{6}(d_i(q) + d_i(s) + d_i(t)) \leq n - 1$, a contradiction.

Subcase 2.3. $d_i = b_{i+1}$ for each i, $1 \leq i \leq l-1$.

Firstly, we prove that $V(H) = \{q\}$. Suppose not, then for any vertex w in $V(H)$ it is obvious that $wd_i \notin E$ and $wa_i \notin E$ for each i, $1 \leq i \leq l$; otherwise G contains paths between u and v which are longer than P. Since G is 3 - connected, $G[V(G) - \{q, v\}]$ is connected. Thus there exist vertex $w \in V(H) - \{q\}$ and vertex $p \in \overrightarrow{P}[d_l^+, v^-]$ such that $wp \in E$. Let $s = d_l$ and $t = p^+$. Set

$$V_1 = \overrightarrow{P}[u, a_l],$$

$$V_2 = \overrightarrow{P}[d_l, p],$$

$$V_3 = \overrightarrow{P}[t, v],$$

$$V_4 = V(G) - V(P).$$

Clearly, $sd_i \notin E$ and $td_i \notin E$ for each i, $1 \leq i \leq l$. Also $ta_i \notin E$ for each i, $1 \leq i \leq l$; otherwise by $(\star)$ we have $t \in D$, a contradiction. Therefore,

$$d_1(q) + d_1(s) + d_1(t) \leq (|V_1| + 1)/2 + (|V_1| + 1)/2 = |V_1| + 1.$$

Clearly, $d_2(q) = 0$. Notice that $N_2^-(s) \cap N_2(t) = \emptyset$; otherwise G has a path between u and v which is longer than P. Therefore,

$$d_2(q) + d_2(s) + d_2(t) = |N_2^-(s)| + |N_2(t)| = |N_2^-(s) \cup N_2(t)| \leq |V_2|.$$

Clearly, $d_3(q) = 0$. Notice that $N_3(s) \cap N_2^-(t) = \emptyset$; otherwise G has a path between u and v which is longer than P. Since $d(u,v) \geq 3$, $v \notin N(s)$. Thus $N_3(s) \subseteq V_3 - \{v\}$. Therefore,

$d_3(q)+d_3(s)+d_3(t) = |N_3(s)|+|N_3^-(t)| = |N_3(s) \cup N_3^-(t)| \leq |V_3 - \{v\}| \leq |V_3| - 1$.

Obviously, $d_4(q) + d_4(s) + d_4(t) \leq |V_4| - 1$.

Hence, $n \leq d(q) + d(s) + d(t) \leq \sum_{i=1}^{4}(d_i(q) + d_i(s) + d_i(t)) \leq n - 1$, a contradiction.

Secondly, we prove that $G[V(G) - V(P)]$ has only one component. Suppose not, let H_1 be any component of $G[V(G) - V(P) - V(H)]$ and q_1 a vertex in H_1 such that $|N(q_1) \cap V(P)| \geq 3$. Clearly, $uq_1 \notin E$; otherwise $(\star)$ implies that $q_1 \in D$, which is impossible. Moreover, we have $a_i q_1 \notin E$ for each i, $1 \leq i \leq l$. Let C be the set $N(q_1) \cap V(P) := \{c_1, c_2, ..., c_{l_1}\}$ with the c_i's ordered with increasing index in the reverse direction of P.

If $c_1 \neq v$, then using a similar argument as in Case 1, we can derive a contradiction. If $c_1 = v$, then $c_i^- = c_{i+1}^+$ for each i, $1 \leq i \leq l_1 - 1$; otherwise using similar arguments as in Subcase 2.1 and Subcase 2.2, we can derive contradictions. Furthermore, using a similar argument as in the first part of Subcase 2.3, we can show that $H_1 = \{ q_1 \}$. Since $d(q) \geq n/3$, $d(q_1) \geq n/3$, and $a_l q_1 \notin E$, $b_l q_1 \in E$, $d_l q_1 \in E$. Thus G has a path

$$\overrightarrow{P}[u, a_{l-1}]qa_l b_l q_1 \overrightarrow{P}[d_l, v]$$

between u and v which is longer than P, a contradiction.

Finally, we derive the last contradiction. Since G is a 3 – connected L_1 – graph and any graph in the family $\mathcal{K}$ contains no pair of vertices of distance at least three, by Lemma 3 we have $\omega(G - S) \leq |S| - 1$ for every subset S of vertex set $V(G)$ with $\omega(G - S) > 1$. Clearly, $|\overrightarrow{P}[d_l, v]| \neq 1$; otherwise $l - 1 \geq \omega(G - \{a_1, a_2, ..., a_l\}) = l + 1$. Also $|\overrightarrow{P}[d_l, v]| \neq 2$; otherwise $l \geq \omega(G - \{a_1, a_2, ..., a_l, v\}) = l + 1$. Thus $|\overrightarrow{P}[d_l, v]| \geq 3$. Since $\omega(G - \{a_1, a_2, ..., a_l, v\}) \leq l$, there exists a vertex in $\{ d_1, d_2, ..., d_{l-1} \}$, say d_r, $1 \leq r \leq l-1$, and a vertex w in $|\overrightarrow{P}[d_l, v^-]|$ such that $d_r w \in E(G)$. Now consider the vertex w^+. We will show that $N(w^+) \cap \overrightarrow{P}[u, d_l] = \emptyset$. If there exists some vertex, say d_m, in $\{ d_1, d_2, ..., d_r \}$ that is also in $N(w^+)$, then G has a path

$$\overrightarrow{P}[u, a_m]q\overrightarrow{P}[a_{r+1}, w]\overleftarrow{P}[d_r, d_m]\overrightarrow{P}[w^+, v]$$

between u and v which is longer than P, a contradiction. If there exist some vertex, say d_m, in $\{ d_{r+1}, d_{r+2}, ..., d_l \}$ that is also in $N(w^+)$, then G has a path

$$\overrightarrow{P}[u, a_r]q\overleftarrow{P}[a_m, d_r]\overleftarrow{P}[w, d_m]\overrightarrow{P}[w^+, v]$$

between u and v which is longer than P, a contradiction. If some vertex a_m, $1 \leq m \leq l$, is in $N(w^+)$, then by $(\star)$ we have $w^+ \in D$, which is impossible. Hence, $n = |\{q\}| + |V(P)| = 1 + |\overrightarrow{P}[u, d_l]| + |\overrightarrow{P}[d_l^+, v]| \geq 1 + 2d(q) + d(w^+) + 1 \geq 3\delta(G) + 2 \geq n + 2$, a contradiction.

Case 3. $u \neq a_1$ and $v = a_l$.

This case is symmetric to Case 2. The arguments in Case 2 can be symmetrically applied to this case and we can arrive at a contradiction.

The combination of proofs for Cases 1, 2, and 3 completes the proof of Theorem 3.

References

[1] A.S. Asratian, Some properties of graphs with local Ore condition, *Ars Combinatoria* **41** (1995), 97 - 106.

[2] A.S. Asratian, H.J. Broersma, J. van den Heuvel, and H.J. Veldman, On graphs satisfying a local Ore - type condition, *J. Graph Theory* **21** (1996), 1 - 10.

[3] A.S. Asratian, R. Häggkvist, and G.V. Sarkisian, A characterization of panconnected graphs satisfying a local Ore - type condition, *J. Graph Theory* **22**(1996), 95 - 103.

[4] A.S. Asratian, N.H. Khachatrian, Some localization theorems on hamiltonian circuits, *J. Combin. Theory Ser. B* **49**(1990), 287 - 294.

[5] A.S. Asratian and G.V. Sarkisian, Some panconnected and pancyclic properties of graphs with a local Ore - type condition, *Graphs and Combinatorics* **12**(1996), 209 - 219.

[6] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, Macmillan, London and Elsevier, New York (1976).

[7] H. Enomoto, K. Hirohata and K. Ota, Long cycles passing through a specified edge in a 3 - connected graph, *J. Graph Theory* **24** (1997), 273 - 279.

[8] R. Li and R. H. Schelp, Some hamiltonian properties of L_1 - graphs, *to appear in Discrete Mathematics.*

[9] M.M. Matthews and D.P. Sumner, Longest paths and cycles in $K_{1,3}$ - free graphs, *J. Graph Theory* **9** (1985), 269 - 277.

[10] O. Ore, Hamiltonian connected graphs, *J. Math. Pure Appl.* **42** (1963), 21 - 27.

[11] A. Saito, A local Ore-type conditions for graphs of diameter two to be Hamiltonian, *J. Comb. Math. Comb. Comput.*2(1996), 155 - 159.